Joint Continuity of Bi-multiplicative Functionals

Abbas Zivari-Kazempour and Mohammad Valaei

Sahand Communications in Mathematical Analysis

Print ISSN: 2322-5807
Online ISSN: 2423-3900
Volume: 18
Number: 2
Pages: 33-44

Sahand Commun. Math. Anal.
DOI: 10.22130/scma.2020.127223.795

Sahand Communications in Mathematical Analysis (SCMA) Vol. 18 No. 2 (2021), 33-44
http://scma.maragheh.ac.ir
DOI: 10.22130/scma.2020.127223.795

Joint Continuity of Bi-multiplicative Functionals

Abbas Zivari-Kazempour[1] * and Mohammad Valaei[2]

Abstract. For Banach algebras $\mathcal{A}$ and $\mathcal{B}$, we show that if $\mathfrak{A} = \mathcal{A} \times \mathcal{B}$ is unital, then each bi-multiplicative mapping from $\mathfrak{A}$ into a semisimple commutative Banach algebra $\mathcal{D}$ is jointly continuous. This conclusion generalizes a famous result due to Šilov, concerning the automatic continuity of homomorphisms between Banach algebras. We also prove that every n-bi-multiplicative functionals on $\mathfrak{A}$ is continuous if and only if it is continuous for the case $n = 2$.

---

1. Introduction

Let $\mathcal{A}$ and $\mathcal{B}$ be complex Banach algebras and $f : \mathcal{A} \longrightarrow \mathcal{B}$ be a linear map. Then, f is called an *n-homomorphism* if for all $a_1, a_2, \ldots, a_n \in \mathcal{A}$,

$$f(a_1 a_2 \cdots a_n) = f(a_1) f(a_2) \cdots f(a_n).$$

The concept of an n-homomorphism was studied for complex algebras in [3, 6]. A 2-homomorphism is then just a homomorphism, in the usual sense. One may refer to [3, 12], for certain properties of 3-homomorphisms.

In 1967, a classical result of Johnson shows that if $f : \mathcal{A} \longrightarrow \mathcal{B}$ is a surjective homomorphism between a Banach algebra $\mathcal{A}$ and a semisimple Banach algebra $\mathcal{B}$, then f is automatically continuous. Then the Johnson's result was extended to n-homomorphism in [7] with the extra condition that the Banach algebra $\mathcal{B}$ is factorizable, and then it is generalized to non factorizable Banach algebras in [5].

2010 *Mathematics Subject Classification.* 47B48, 46H40, 46H25.

Key words and phrases. Jointly continuous, Bi-multiplicative functional, Almost bi-multiplicative.

Received: 07 May 2020, Accepted: 15 August 2020.

* Corresponding author.

Šilov in [4] has given a well-known result concerning the automatic continuity of homomorphisms between Banach algebras, that we mention in the following (see also [2]).

Theorem 1.1. *Let $\mathcal{A}$ and $\mathcal{B}$ be Banach algebras that $\mathcal{B}$ is commutative and semisimple. Then, every homomorphism $f : \mathcal{A} \longrightarrow \mathcal{B}$ is automatically continuous.*

A linear map f between Banach algebras $\mathcal{A}$ and $\mathcal{B}$ is called *almost multiplicative* if there exists $\varepsilon > 0$ such that for all $a, b \in \mathcal{A}$,

$$\|f(ab) - f(a)f(b)\| \leq \varepsilon \|a\|\|b\|.$$

K. Jarosz in [8] introduced the concept of almost multiplicative function and investigated the automatic continuity of almost multiplicative linear maps between Banach algebras.

After that, Johnson obtained some results on continuity of almost multiplicative functionals [9], and then he generalized his result to almost multiplicative maps between Banach algebras [10].

Homomorphisms and their automatic continuity of Banach algebras, have been widely studied by many authors. One may refer to the monographs of Dales [4] and Jarosz [8].

Throughout the paper, let $\mathfrak{A} = \mathcal{A} \times \mathcal{B}$ for complex Banach algebras $\mathcal{A}$ and $\mathcal{B}$. Then $\mathfrak{A}$ is a Banach algebra for the multiplication

$$(a, b)(x, y) = (ax, by), \qquad (a, b), (x, y) \in \mathfrak{A},$$

and with norm

$$\|(a, b)\| := \|a\| + \|b\|.$$

Let $\mathcal{D}$ be a complex Banach algebra. A bilinear map is a function $\varphi : \mathfrak{A} \longrightarrow \mathcal{D}$ such that φ is linear with respect to both variable. A bilinear map $\varphi : \mathfrak{A} \longrightarrow \mathcal{D}$ is called bounded if there is a real number M such that for all $(a, b) \in \mathfrak{A}$,

$$\|\varphi(a, b)\| \leq M\|a\|\|b\|.$$

Obviously, φ is bounded if and only if it is jointly continuous.

A bilinear map φ is called n-bi-multiplicative if for all $(a_i, b_i) \in \mathfrak{A}$,

$$\varphi(a_1 \cdots a_n, b_1 \cdots b_n) = \varphi(a_1, b_1) \cdots \varphi(a_n, b_n).$$

The concept of an n-bi-multiplicative mapping introduced by the first author in [13]. A 2-bi-multiplicative is called simply bi-multiplicative.

Clearly, each bi-multiplicative functional is an n-bi-multiplicative for every $n \geq 2$, but the converse does not hold in general. For instance, if $\varphi : \mathfrak{A} \longrightarrow \mathbb{C}$ is a bi-multiplicative functional, then $\psi := -\varphi$ is a 3-bi-multiplicative which is not bi-multiplicative.

The aim of this paper is to investigate the joint continuity of bi-multiplicative mappings and the general cace n-bi-multiplicative on the product of Banach algebras as well as automatic continuity of homomorphisms between Banach algebras.

In Section 2, we show that every separately continuous action $\varphi : \mathcal{A} \times X \longrightarrow X$ of a unital Banach algebra $\mathcal{A}$ on a compact Hausdorff space X is jointly continuous.

In Section 3, we prove that each bi-multiplicative maps φ from unital Banach algebra $\mathfrak{A}$ into a semisimple commutative Banach algebra $\mathcal{D}$ is jointly continuous.

We also show that the joint continuity of n-bi-multiplicative functionals, passes from the case $n = 2$ to all $n \in \mathbb{N}$. Finally, the joint continuity of almost bi-multiplicative functionals on $\mathfrak{A}$ is presented.

2. Joint Continuity of Bilinear Maps

Let $\mathcal{A}$ be a unital Banach algebra and let $G = Inv(\mathcal{A})$, where $Inv(\mathcal{A})$ is the set of all invertible elements of $\mathcal{A}$. Then G is called the *group of invertible elements* of $\mathcal{A}$.

Since with the topology induced from the norm topology of $\mathcal{A}$, the mapping $a \longmapsto a^{-1}$ is a homeomorphism of G onto G, hence G is a topological group and it is an open subset of $\mathcal{A}$ by [2, Theorem 11, § 2].

We commence with the following result which prove that every unital Banach algebra is the linear span of its invertible elements.

Proposition 2.1. *Suppose that $\mathcal{A}$ is a unital Banach algebra. Then $\mathcal{A} = lin(G)$, where $G = Inv(\mathcal{A})$.*

Proof. Let e be a unit element of $\mathcal{A}$, and let $a \in \mathcal{A}$ such that $\|a\| < 1$. Then $e - a \in G$ by Theorem 9 § 2 of [2], and therefore the equality $a = -(e - a) + e$ implies that $a \in lin(G)$. Hence $\mathcal{A} = lin(G)$, for all $a \in \mathcal{A}$ with $\|a\| < 1$. Now let $\|a\| \geq 1$. Put $b = \frac{a}{2\|a\|}$ then $\|b\| < 1$, hence by the above argument $b \in lin(G)$, which implies that $a \in lin(G)$ for all $a \in \mathcal{A}$. Thus, $\mathcal{A} = lin(G)$. $\square$

Definition 2.2. Let $\mathcal{A}$ be a Banach algebra and X be a topological space. We say that $\mathcal{A}$ acts on X if the map $\varphi : \mathcal{A} \times X \longrightarrow X$ satisfies in the following conditions.

(i) $\varphi(a, .) : X \longrightarrow X$ is continuous for all $a \in \mathcal{A}$,
(ii) $\varphi(ab, x) = \varphi(a, \varphi(b, x))$, for every $a, b \in \mathcal{A}$ and $x \in X$.

Example 2.3. (i) It is clear that the multiplication operator on a Banach algebra $\mathcal{A}$ is an action of $\mathcal{A}$ on itself.

(ii) Let $\mathcal{A} = M_n(\mathbb{R})$, and define the mapping $\varphi : \mathcal{A} \times \mathbb{R} \longrightarrow \mathbb{R}$ by $\varphi(A, x) = det(A)x$, for all $A \in \mathcal{A}$. Then φ is an action of $\mathcal{A}$ on $\mathbb{R}$.

(iii) Let $\mathcal{A}$ be a unital Banach algebra, $G = Inv(\mathcal{A})$ and

$$B = \{a \in \mathcal{A} : \|a\| < 1\} .$$

Consider the quasi-product $a, b \in \mathcal{A}$ defined by $a \diamond b = a + b - ab$. Then the restriction $\varphi : \mathcal{A} \times \mathcal{A} \longrightarrow \mathcal{A}$ to $B \times G$ is an action of B on G with the operation defined by $\varphi(a, x) = (e - a)x$ for all $a \in B$ and $x \in G$. Indeed,

$$\begin{aligned} \varphi(a, \varphi(b, x)) &= (e - a)(e - b)x \\ &= (e - (a \diamond b))x \\ &= \varphi(a \diamond b, x). \end{aligned}$$

The following general theorem for the joint continuity is given in [1, Page 293].

Theorem 2.4. *Let X be a complete metric space, Y be a compact Hausdorff space, and let Z be a pseudometric space. If $\varphi : X \times Y \longrightarrow Z$ is a separately continuous function, then there is a dense G_δ subset E of X such that φ is jointly continuous on $E \times Y$.*

Lemma 2.5. *Let $\mathcal{A}$ be a unital Banach algebra with unit e, and $\varphi : \mathcal{A} \times X \longrightarrow X$ be a separately continuous action of $\mathcal{A}$ on a compact Hausdorff space X, and let $\varphi(e, .)$ be the identity map on X. Then for distinct element x_1, x_2 of X, there are neighborhoods U of e, V of x_1 and W of x_2 such that*

$$\varphi(U \times V) \cap W = \varnothing.$$

Proof. By Urysohn's Lemma there exists a function $f \in C(X)$, such that $f(x_1) = 0$, $f(x_2) = 1$ and $-1 \le f(t) \le 1$, for all $t \in X$. Take $\psi := f \circ \varphi$. Then by Theorem 2.4, ψ is continuous at each point of $E \times X$, where E is a dense G_δ subset of $\mathcal{A}$. Let

$$K = \{a \in \mathcal{A} : \psi(a, x_1) \neq \psi(a, x_2)\} .$$

Therefore K is nonempty, because $\varphi(e, .)$ is the identity map. Continuity of ψ yields that K is open and so it contains some point $a_0 \in E$. Take $\varepsilon = |\psi(a_0, x_1) - \psi(a_0, x_2)|$, from the continuity of ψ at (a_0, x_1) there exist neighborhoods U_1 of a_0 and V of x_1 such that

$$|\psi(p, q) - \psi(a_0, x_1)| < \frac{\varepsilon}{2}, \quad (p, q) \in U_1 \times V.$$

By the continuity of the left operator L_{a_0} on $\mathcal{A}$, there exists a neighborhood U of e such that $a_0 U \subseteq U_1$. Set

$$W = \left\{t \in X : |\psi(a_0, t) - \psi(a_0, x_2)| < \frac{\varepsilon}{2}\right\} .$$

Now let $(s,t) \in U \times V$ and $\varphi(s,t) \in W$. Clearly,

$$\psi(a_0 s, t) = \psi(a_0, \varphi(s,t)),$$

hence

$$\begin{aligned}\varepsilon &= |\psi(a_0,x_1) - \psi(a_0,x_2)| \\ &\leq |\psi(a_0,x_1) - \psi(a_0 s,t)| + |\psi(a_0,\varphi(s,t)) - \psi(a_0,x_2)| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon,\end{aligned}$$

which is a contradiction. Thus, $\varphi(U \times V) \cap W = \varnothing$, as claimed. □

Now we state and prove the main theorem of this section.

Theorem 2.6. *Let $\mathcal{A}$ be a unital Banach algebra with unit e, and let $\varphi : \mathcal{A} \times X \longrightarrow X$ be a separately continuous action of $\mathcal{A}$ on a compact Hausdorff space X. Then φ is jointly continuous on $G \times X$, where $G = Inv(\mathcal{A})$.*

Proof. We prove the result in the following steps.

Step 1: Without loss of generality we may assume that $\varphi(e,.)$ is the identity map: Because, suppose the conclusion of the result holds with this additional condition. Then for all $x \in X$,

$$\begin{aligned}\varphi(e,\varphi(e,x)) &= \varphi(e^2,x) \\ &= \varphi(e,x).\end{aligned}$$

Thus, $\varphi(e,.) : \varphi(e,X) \longrightarrow \varphi(e,X)$ is the identity map. So in the general case, the restricted form of the theorem implies that φ is continuous at each point of $\mathcal{A} \times \varphi(e,X)$. Now the continuity of φ on $\mathcal{A} \times X$ follows from the continuity on $\mathcal{A} \times \varphi(e,X)$. Indeed, let (a_α, x_α) converging to $(a,x) \in \mathcal{A} \times X$. Then

$$\varphi(a_\alpha, x_\alpha) = \varphi(a_\alpha e, x_\alpha) = \varphi(a_\alpha, \varphi(e,x_\alpha)) \longrightarrow \varphi(a,\varphi(e,x)) = \varphi(a,x).$$

Thus, we may assume at first that $\varphi(e,.)$ is the identity map.

Step 2: We prove that φ is continuous at (e,x), for all $x \in X$. Let W be a neighborhood of $\varphi(e,x) = x$. Then by Lemma 2.5, for every $y \in X \setminus W$, there exist neighborhoods U_y of e, V_y of x and W_y of y such that $\varphi(U_y \times V_y) \cap W_y = \varnothing$. Since $X \setminus W$ is compact, so there exists a finite subset $F \subseteq X \setminus W$ such that

$$X \setminus W \subseteq \bigcup_{y \in F} W_y.$$

Now set

$$U = \bigcap_{y \in F} U_y, \qquad V = \bigcap_{y \in F} V_y.$$

Thus, $U \times V$ is a neighborhood of (e, x) and $\varphi(U \times V) \subseteq W$. Therefore φ is continuous at each point of $\{e\} \times X$.

Step 3: We show that φ is continuous at each point of $G \times X$, where $G = Inv(\mathcal{A})$. Let $(a, x) \in G \times X$, and $(a_\alpha, x_\alpha) \longrightarrow (a, x)$.

Therefore $(a^{-1}a_\alpha, x_\alpha) \longrightarrow (e, x)$. Using preceding step, we conclude that $\lim_\alpha \varphi(a^{-1}a_\alpha, x_\alpha) = \varphi(e, x) = x$. Hence

$$\varphi(a_\alpha, x_\alpha) = \varphi(a, \varphi(a^{-1}a_\alpha, x_\alpha)) \longrightarrow \varphi(a, x),$$

which yields that φ is continuous on $G \times X$. This completes the proof. $\square$

Example 2.7. Let $\mathcal{A}$ be a unital commutative Banach algebra, $B_{[1]}$, $B'_{[1]}$ be the closed unit balls of $\mathcal{A}$ and $\mathcal{A}'$, respevtively. By Theorm 3.15 of [11] $B'_{[1]}$ is compact in the w^*-topology. Consider φ as an action of $B_{[1]}$ on $B'_{[1]}$, for all $n \in \mathbb{N}$ defined by

$$\varphi(a, f) = a^n \cdot f, \quad a \in B_{[1]}, f \in B'_{[1]},$$

where $(a^n \cdot f)b = f(ba^n)$. Then φ is separately continuous action, and so by above theorem it is jointly continuous on $G \times B'_{[1]}$.

3. Joint Continuity of Bi-multiplicative Functionals

Every multiplicative linear functional f on a Banach algebra $\mathcal{A}$ is continuous and $\|f\| \leq 1$, see [2, Proposition 3, § 16] for example. In the next theorem we prove the same result for bi-multiplicative functionals.

Theorem 3.1. *If $\mathfrak{A}$ is unital, then every bi-multiplicative functional $\varphi : \mathfrak{A} \longrightarrow \mathbb{C}$ is jointly continuous.*

Proof. Suppose that (e_1, e_2) is a unit element of $\mathfrak{A}$. Since φ is bi-multiplicative, we get $\varphi(e_1, e_2) = 1$. For $(x, y) \in G = Inv(\mathfrak{A})$,

$$\begin{aligned} 1 &= \varphi(e_1, e_2) \\ &= \varphi(xx^{-1}, yy^{-1}) \\ &= \varphi(x, y)\varphi(x^{-1}, y^{-1}), \end{aligned}$$

so $\varphi(x, y)^{-1} = \varphi(x^{-1}, y^{-1})$.

Step 1: Let $(x, e_2) \in \mathfrak{A}$ with $\|x\| < 1$. Thus, $(e_1 - x, e_2) \in G$, and so $\varphi(e_1 - x, e_2) \neq 0$. Therefore $1 = \varphi(e_1, e_2) \neq \varphi(x, e_2)$. Hence for all $(x, e_2) \in \mathfrak{A}$ with $\|x\| < 1$ we have $\varphi(x, e_2) \neq 1$. Let $|\varphi(x, e_2)| > 1$, and take

$$a = \frac{x}{\varphi(x, e_2)}.$$

Then $\|a\| < 1$ and $\varphi(a, e_2) = 1$, which is a contradiction. So $|\varphi(x, e_2)| \leq 1$, for all $\|x\| < 1$. If we replace x by $\frac{x}{2\|x\|}$, we arrive at

$$|\varphi(x, e_2)| \leq 2\|x\|,$$

for all $(x, e_2) \in \mathfrak{A}$. Thus, φ is continuous at (x, e_2). Similarly, φ is continuous at (e_1, y).

Step 2: Let $G_1 = Inv(\mathcal{A})$ and $G_2 = Inv(\mathcal{B})$. Suppose that (x_n, y_n) is a sequence in $\mathfrak{A}$ converging to $(x, y) \in G_1 \times \mathcal{B}$. Then $(x^{-1}x_n, y_n)$ tends to (e_1, y), and hence by the continuity of φ at (e_1, y),

$$\varphi(x^{-1}x_n, y_n) \longrightarrow \varphi(e_1, y).$$

Therefore

$$\varphi(x, e_2)^{-1}\varphi(x_n, y_n) = \varphi(x^{-1}, e_2)\varphi(x_n, y_n) = \varphi(x^{-1}x_n, y_n) \longrightarrow \varphi(e_1, y).$$

Consequently,

$$\varphi(x_n, y_n) \longrightarrow \varphi(x, e_2)\varphi(e_1, y) = \varphi(x, y),$$

which proves that φ is continuous at each point of $G_1 \times \mathcal{B}$. Similarly, φ is continuous at each point of $\mathcal{A} \times G_2$.

Step 3: Suppose that $(a, b) \in \mathcal{A} \times \mathcal{B}$ and $(a_n, b_n) \longrightarrow (a, b)$. Applying Proposition 2.1, we get $a = \sum\limits_{i=1}^{k} \lambda_i a_i$ for some $a_i \in G_1$ and $\lambda_i \in \mathbb{C}$. Since $a_n - \sum\limits_{i=1}^{k-1} \lambda_i a_i \longrightarrow \lambda_k a_k \in G_1$ and $(\lambda_k a_k, b) \in G_1 \times \mathcal{B}$, therefore it follows from step 2 that

$$\varphi(a_n, b_n) - \sum_{i=1}^{k-1} \varphi(\lambda_i a_i, b_n) = \varphi\left(a_n - \sum_{i=1}^{k-1} \lambda_i a_i, b_n\right) \longrightarrow \varphi(\lambda_k a_k, b).$$

If we take

$$u_n = \sum_{i=1}^{k-1} \varphi(\lambda_i a_i, b_n),$$

then

$$u_n \longrightarrow u = \sum_{i=1}^{k-1} \varphi(\lambda_i a_i, b),$$

because $(\lambda_i a_i, b) \in G_1 \times \mathcal{B}$. Hence $\varphi(a_n, b_n) \longrightarrow \varphi\left(\sum\limits_{i=1}^{k} \lambda_i a_i, b\right) = \varphi(a, b)$. Consequently, φ is jointly continuous on $\mathfrak{A}$. $\square$

Corollary 3.2. *Every bi-multiplicative map φ from unital Banach algebra $\mathfrak{A}$ into a semisimple commutative Banach algebra $\mathcal{D}$ is jointly continuous.*

Proof. Suppose that $\mathcal{D}$ is semisimple and commutative. Let $\mathfrak{M}(\mathcal{D})$ be the maximal ideal space of $\mathcal{D}$. We associate with each $f \in \mathfrak{M}(\mathcal{D})$ a function $\varphi_f : \mathfrak{A} \longrightarrow \mathbb{C}$ defined by

$$\varphi_f(a,b) := f(\varphi(a,b)), \quad (a,b) \in \mathfrak{A}.$$

Then φ_f is a bi-multiplicative functional, so by Theorem 3.1 it is continuous.

Now, suppose that $(a_n, b_n) \longrightarrow (0,0)$ and $\varphi(a_n, b_n) \longrightarrow x$ in $\mathcal{D}$. Then, by the continuity of φ_f, we have $\varphi_f(a_n, b_n) \longrightarrow 0$. On the other hand, by the continuity of $f : \mathcal{D} \longrightarrow \mathbb{C}$,

$$(f \circ \varphi)(a_n, b_n) = f(\varphi(a_n, b_n)) \longrightarrow f(x).$$

Consequently, $f(x) = 0$ and since $f \in \mathfrak{M}(\mathcal{D})$ was arbitrary, we get $x = 0$. Therefore φ is continuous by the Closed Graph Theorem. □

Since each bi-multiplicative functional is an n-bi-multiplicative, so every bi-multiplicative functional on $\mathfrak{A}$ is continuous whenever every n-bi-multiplicative functional on $\mathfrak{A}$ is continuous. The converse is also true which we present in the next result.

Theorem 3.3. *If every bi-multiplicative functional on $\mathfrak{A}$ is continuous, then every n-bi-multiplicative functional on $\mathfrak{A}$ is also continuous.*

Proof. Suppose that $\varphi : \mathfrak{A} \longrightarrow \mathbb{C}$ is an n-bi-multiplicative functional. Since $\varphi \neq 0$, there axists $(u,v) \in \mathfrak{A}$ such that $\varphi(u,v) = 1$. For each $(x,y) \in \mathfrak{A}$, define $\psi : \mathfrak{A} \longrightarrow \mathbb{C}$ by $\psi(x,y) = \varphi(ux, vy)$. Then for all $(a,b), (x,y) \in \mathfrak{A}$,

$$\begin{aligned}
\psi(ax, by) &= \varphi(uax, vby) \\
&= \varphi(uax, vby)\varphi(u,v)^{n-1} \\
&= \varphi(uaxu^{n-1}, vbyv^{n-1}) \\
&= \varphi(ua, vb)\varphi(xu, yv)\varphi(u,v)^{n-2} \\
&= \varphi(ua, vb)\varphi(xu, yv),
\end{aligned}$$

and

$$\begin{aligned}
\varphi(xu, yv) &= \varphi(u,v)^{n-1}\varphi(xu, yv) \\
&= \varphi(u,v)^{n-2}\varphi(ux, vy)\varphi(u,v) \\
&= \varphi(ux, vy).
\end{aligned}$$

Therefore we get

$$\begin{aligned}
\psi(ax, by) &= \varphi(ua, vb)\varphi(ux, vy) \\
&= \psi(a,b)\psi(x,y),
\end{aligned}$$

and hence ψ is bi-multiplicative functional on $\mathfrak{A}$. For all $(x, y) \in \mathfrak{A}$, we have

$$\begin{aligned}\psi(x,y) &= \varphi(ux, vy)\\ &= \varphi(u,v)^{n-1}\varphi(ux,vy)\\ &= \varphi(u^n x, v^n y)\\ &= \varphi(u^2,v^2)\varphi(u,v)^{n-2}\varphi(x,y)\\ &= \varphi(u^2,v^2)\varphi(x,y),\end{aligned}$$

which proves that $\varphi(u^2, v^2) \neq 0$. Let (x_k, y_k) be a sequence in $\mathfrak{A}$ converging to $(x, y) \in \mathfrak{A}$. Then by the continuity of ψ we have

$$\varphi(u^2,v^2)\varphi(x_k,y_k) = \psi(x_k,y_k) \longrightarrow \psi(x,y) = \varphi(u^2,v^2)\varphi(x,y).$$

Thus, $\varphi(x_k, y_k) \longrightarrow \varphi(x, y)$ and so φ is continuous. $\square$

Combining Theorems 3.1 and 3.3 we have the following result.

Corollary 3.4. *If $\mathfrak{A}$ is unital, then every n-bi-multiplicative functional $\varphi : \mathfrak{A} \longrightarrow \mathbb{C}$ is jointly continuous.*

It is known that the second dual of every C^*-algebra is unital [2], thus we have the following result.

Corollary 3.5. *Suppose that $\mathfrak{A}$ is a C^*-algebra, then each bi-multiplicative functional $\varphi : \mathfrak{A}'' \longrightarrow \mathbb{C}$ is jointly continuous.*

Example 3.6. Let $X = C([0, 1])$, the Banach space of all continuous real-valued polynomials on $[0, 1]$, equipped with the norm

$$\|f\| = \int_0^1 |f(t)|dt, \quad f \in X.$$

For all $(f, g) \in X \times X$, define $\varphi : X \times X \longrightarrow \mathbb{R}$ by

$$\varphi(f,g) = \int_0^1 f(t)g(t)dt.$$

Then φ is bilinear and separately continuous, but it is not jointly continuous. Note that neither X is a Banach algebra nor φ is bi-multiplicative.

The following result concerning the automatic continuity of almost multiplicative linear functionals obtained by Jarosz in [8].

Theorem 3.7. *Let $\mathcal{A}$ be a Banach algebra and $f : \mathcal{A} \longrightarrow \mathbb{C}$ be an almost multiplicative linear functional. Then $\|f\| \leq 1 + \varepsilon$, and hence f is continuous.*

A bilinear map $\varphi : \mathfrak{A} \longrightarrow \mathbb{C}$ is called *almost bi-multiplicative* if there exists $\varepsilon \geq 0$ such that for all $(a, b), (x, y) \in \mathfrak{A}$,

$$|\varphi(ax, by) - \varphi(a, b)\varphi(x, y)| \leq \varepsilon \ \|(a, b)\|\|(x, y)\|.$$

Now we prove an analogous result of Jaroz's theorem for almost bi-multiplicative.

Theorem 3.8. *Let $\mathfrak{A}$ be unital and $\varphi : \mathfrak{A} \longrightarrow \mathbb{C}$ be an almost bi-multiplicative. Then φ is jointly continuous.*

Proof. Let $\varphi : \mathfrak{A} \longrightarrow \mathbb{C}$ be an almost bi-multiplicative functional. Then, there exists $\varepsilon \geq 0$ such that

$$|\varphi(ax, by) - \varphi(a, b)\varphi(x, y)| \leq \varepsilon\|(a, b)\|\|(x, y)\|, \tag{3.1}$$

for every $(a, b), (x, y) \in \mathfrak{A}$. Set $\xi = \frac{1+\sqrt{1+4\varepsilon}}{2}$. If for all $(a, b) \in \mathfrak{A}$,

$$|\varphi(a, b)| \leq 4\xi\|a\|\|b\|, \tag{3.2}$$

then φ is bounded and hence it is continuous. If (3.2) does not hold for some $(s, t) \in \mathfrak{A}$, then we have

$$|\varphi(s, t)| > 4\xi\|s\|\|t\|. \tag{3.3}$$

We may assume without loss of generality that $\|s\| = \|t\| = \frac{1}{2}$ and $|\varphi(s, t)| > \xi$. Therefore, we can write $|\varphi(s, t)| = \xi + p$, for some $p > 0$. Then we have

$$\begin{aligned} |\varphi(s^2, t^2)| &\geq |\varphi(s, t)^2 - \big(\varphi(s, t)^2 - \varphi(s^2, t^2)\big)| \\ &\geq |\varphi(s, t)|^2 - |\varphi(s, t)^2 - \varphi(s^2, t^2)|, \end{aligned}$$

hence from (3.1),

$$\begin{aligned} |\varphi(s^2, t^2)| &\geq |\varphi(s, t)|^2 - \varepsilon \\ &= (\xi + p)^2 - \varepsilon \\ &= 2\xi p + p^2 + \xi \\ &> \xi + 2p. \end{aligned}$$

It can be shown, by induction, that

$$|\varphi(s^{2^n}, t^{2^n})| > \xi + (n + 1)p. \tag{3.4}$$

Given $(a, b), (x, y), (u, v) \in \mathfrak{A}$, take

$$I_1 = |\varphi(ax, by)\varphi(u, v) - \varphi(a, b)\varphi(xu, yv)|,$$

and

$$I_2 = |\varphi(a, b)\varphi(xu, yv) - \varphi(a, b)\varphi(x, y)\varphi(u, v)|.$$

Then

$$I_1 \leq |\varphi(axu, byv) - \varphi(a, b)\varphi(xu, yv)|$$

$$\begin{aligned}&+ |\varphi(axu, byv) - \varphi(ax, by)\varphi(u, v)| \\ &\leq 2\varepsilon\|(a, b)\|\|(x, y)\|\|(u, v)\|.\end{aligned}$$

Thus,

$$\begin{aligned}&|\varphi(ax, by)\varphi(u, v) - \varphi(a, b)\varphi(x, y)\varphi(u, v)| \\ &\qquad \leq I_1 + I_2 \\ &\qquad \leq \varepsilon\big(2\|(a, b)\| + |\varphi(a, b)|\big)\|(x, y)\|\|(u, v)\|.\end{aligned}$$

Therefore

$$|\varphi(u, v)||\varphi(ax, by) - \varphi(a, b)\varphi(x, y)| \leq \varepsilon\delta\|(u, v)\|, \tag{3.5}$$

where $\delta = \big(2\|(a, b)\| + |\varphi(a, b)|\big)\|(x, y)\|$. Replacing (u, v) by (s^{2^n}, t^{2^n}) in (3.5), and applying (3.4), to get

$$|\varphi(ax, by) - \varphi(a, b)\varphi(x, y)| \leq \frac{\varepsilon\delta}{\varepsilon + (n + 1)p},$$

for all $n \in \mathbb{N}$. Letting $n \to \infty$, we get $\varphi(ax, by) = \varphi(a, b)\varphi(x, y)$ for all $(a, b), (x, y) \in \mathfrak{A}$. Consequently, φ is a bi-multiplicative and so it is continuous by Theorem 3.1. □

Acknowledgment. The authors gratefully acknowledges the helpful comments of the anonymous referees.

References

1. J.F. Berglund, H.D. Junghenn and P. Milnes, *Analysis on Semigroups*, John-Wiley, New York, 1989.
2. F.F. Bonsall and J. Duncan, *Complete normed algebra*, Springer-Verlag, New York, 1973.
3. J. Bračič and M. S. Moslehian, *On automatic continuity of* 3-*homomorphisms on Banach algebras*, Bull. Malay. Math. Sci. Soc., 30(2) (2007), pp. 195-200.
4. H.G. Dales, *Banach Algebras and Automatic Continuity*, Vol. 24 London Mathematical Society Monographs, Clarendon Press, Oxford, 2000.
5. M. Eshaghi Gordji, A. Jabbari and E. Karapinar, *Automatic continuity of surjective n-homomorphisms on Banach algebras*, Bull. Iranian Math. Soc., 41(5) (2015), pp. 1207-1211.
6. Sh. Hejazian, M. Mirzavaziri and M.S. Moslehian, *n-homomorphisms*, Bull. Iranian Math. Soc., 31(1) (2005), pp. 13-23.
7. T.G. Honari and H. Shayanpour, *Automatic continuity of n-homomorphisms between Banach algebras*, Q. Math., 33(2) (2010), pp. 189-196.

8. K. Jarosz, *Perturbation of Banach algebras*, Lecture Notes in Mathematics, Springer-verlag, 1985.
9. B.E. Johnson, *Approximately multiplicative functionals*, J. London Math. Soc., 34(2) (1986), pp. 489-510.
10. B.E. Johnson, *Approximately multiplicative maps between Banach algebras*, J. London Math. Soc., 37(2) (1988), pp. 294-316.
11. W. Rudin, *Functional Analysis*, McGraw-Hill, New York, 1973.
12. A. Zivari-kazempour, *A characterization of* 3*-Jordan homomorphisms on Banach algebras*, Bull. Aust. Math. Soc., 93(2) (2016), pp. 301-306.
13. A. Zivari-Kazempour, *When is a bi-Jordan homomorphisms bi-homomorphisms?*, Kragujevac J. Math., 42(4) (2018), pp. 485-493.

---

[1] Department of Mathematics, Ayatollah Borujerdi University, Borujerd, Iran.

E-mail address: zivari@abru.ac.ir, zivari6526@gmail.com

[2] Department of Mathematics, Ayatollah Borujerdi University, Borujerd, Iran.

E-mail address: mohamad.valaei@abru.ac.ir