Fixed Point Theorems for Geraghty Type Contraction Mappings in Complete Partial $b_v(s)$-Metric Spaces

Ebru Altiparmak and Ibrahim Karahan

Sahand Communications in Mathematical Analysis

Print ISSN: 2322-5807
Online ISSN: 2423-3900
Volume: 18
Number: 2
Pages: 45-62

Sahand Commun. Math. Anal.
DOI: 10.22130/scma.2020.127414.799

Sahand Communications in Mathematical Analysis (SCMA) Vol. 18 No. 2 (2021), 45-62
http://scma.maragheh.ac.ir
DOI: 10.22130/scma.2020.127414.799

Fixed Point Theorems for Geraghty Type Contraction Mappings in Complete Partial $b_v(s)$-Metric Spaces

Ebru Altiparmak[1*] and Ibrahim Karahan[2]

Abstract. In this paper, necessary and sufficient conditions for the existence and uniqueness of fixed points of generalized Geraghty type contraction mappings are given in complete partial $b_v(s)$-metric spaces. The results are more general than several results that exist in the literature because of the considered space. A numerical example is given to support the obtained results. Also, the existence and uniqueness of the solutions of an integral equation has been verified considered as an application.

1. Introduction and Preliminaries

Generalizations of the notions and carrying theorems to the more general situations are in the nature of mathematics. Since the fixed point theorems for special classes of mappings defined in metric spaces are very important in pure and applied sicences, many researchers have tried to give some different kind of generalizations of metric spaces and mappings. From this point of view, many authors introduced some generalized metric spaces such as b-metric, rectangular, v-generalized, $b_v(s)$, partial, partial b-metric and so on. Also, Geraghty, Ciric, cyclic, Meir-Keeler and F-contraction mappings are just the small part of the special classes of generalized contraction mappings. Among all these mappings, Geraghty type mappings have a great importance in fixed point theory.

2020 *Mathematics Subject Classification.* 54H25, 47H09, 47H10.
Key words and phrases. Generalized Geraghty contraction, Fixed point, Partial $b_v(s)$ metric spaces, Generalized metric space.
Received: 11 May 2020, Accepted: 20 September 2020.
* Corresponding author.

Let (X,d) be a metric space. A mapping $T : X \to X$ is called Geraghty contraction if it satisfies

$$d(Tx,Ty) \le \beta(d(x,y))\, d(x,y), \tag{1.1}$$

for all $x, y \in X$ and for a function $\beta \in \mathscr{F}$ where $\mathscr{F}$ denote the family of all functions $\beta : [0,\infty) \to [0,1)$ satisfying

$$\lim_{n\to\infty} \beta(t_n) = 1 \text{ implies } \lim_{n\to\infty} t_n = 0.$$

Geraghty [15] also proved that a mapping which satisfies the inequality (1.1) has a unique fixed point in a complete metric space. Then, many authors made an effort to generalize and extend his results. Therefore, Dukic et al. [11] extended his results to partial metric spaces, ordered partial metric spaces and metric-type spaces. In 2010, Gordji [16] proved similar theorems for special multivalued mappings. In 2015, Kadelburg and Kumam [20] proved a common coupled fixed point theorems in metric spaces. Also, Faraji et al. [14], Erhan [13], Aydi et al. [8] and Piao [29] proved some fixed point theorems for Geraghty type contraction mappings in b-metric space, Branciari b-metric space, metric like space and 2-metric space, respectively. For more details, please see also the other references. On the other hand, some authors generalized the class of Geraghty type mappings. In recent years, Aydi et al. [8], Shahkoohi and Razani [32], Karapınar and Samet [24], Chandok [9], Altun and Sadarangani [6], Acar and Altun [2] and Alqahtani et al. [5] published some papers about the existence and uniqueness of fixed points of different kind of generalizations of Geraghty type contraction mappings, see also [3, 4, 7, 8, 10, 12, 17–19, 22, 23, 25–28, 30, 31, 33]

In this paper, we extend most of these theorems by using generalized Geraghty contraction mappings defined on partial $b_v(s)$-metric spaces which is introduced by Karahan and Isik [21] and Abdullahi and Kumam [1] individually.

Definition 1.1. Let X be a nonempty set, $d : X \times X \to [0,\infty)$ be a mapping and $v \in \mathbb{N}$. Then, (X,d) is called a partial $b_v(s)$-metric space if there exists a real number $s \ge 1$ such that following conditions hold for all $x, y, z_1, z_2, \dots, z_v \in X$:

(i) $x = y \quad \Leftrightarrow \quad d(x,x) = d(x,y) = d(y,y)$,
(ii) $d(x,x) \le d(x,y)$,
(iii) $d(x,y) = d(y,x)$,
(iv) $d(x,y) \le s\left[d(x,z_1) + d(z_1,z_2) + \cdots + d(z_{v-1},y)\right] - \sum_{i=1}^{v} d(z_i,z_i)$.

It is easy to see that every $b_v(s)$-metric space is a partial $b_v(s)$-metric space. However, the converse is not true in general. So, a partial $b_v(s)$-metric space is a generalized version of usual metric space, b-metric space, rectangular metric space, v-generalized metric space, $b_v(s)$-metric space, partial metric space, partial b-metric space, partial rectangular metric space and partial v-generalized metric space.

Now, we give definitions of a convergent sequence, Cauchy sequence and complete partial $b_v(s)$-metric spaces in the following manner.

Definition 1.2 ([21])**.** Let (X,d) be a partial $b_v(s)$-metric space and $\{x_n\}$ be any sequence in X. Then,

(i) the sequence $\{x_n\}$ is said to be convergent to x, if

$$\lim_{n\to\infty} d(x_n,x) = d(x,x).$$

(ii) the sequence $\{x_n\}$ is said to be Cauchy sequence in (X,d) if $\lim_{n,m\to\infty} d(x_n,x_m)$ exist and is finite.

(iii) (X,d) is said to be a complete partial $b_v(s)$-metric space if for every Cauchy sequence $\{x_n\}$ in X there exists $x\in X$ such that

$$\begin{aligned}\lim_{n,m\to\infty} d(x_n,x_m) &= \lim_{n\to\infty} d(x_n,x)\\ &= d(x,x).\end{aligned}$$

Note that the limit of a convergent sequence may not be unique in a partial $b_v(s)$ metric space.

In partial $b_v(s)$-metric spaces, the mapping $\beta:[0,\infty)\to\left[0,\frac{1}{s}\right)$ used in the definition of Geraghty type contractions has been modified in a way to satisfy the following condition:

$$\limsup_{n\to\infty}\beta(t_n)=\frac{1}{s} \text{ implies } \lim_{n\to\infty} t_n = 0.$$

Then, the set of such mappings is denoted by $\mathscr{F}_s$.

2. Main Results

In this section, we present some fixed point theorems for generalized Geraghty type contraction mappings in complete partial $b_v(s)$-metric spaces.

Theorem 2.1. *Let (X,d) be a complete partial $b_v(s)$-metric space with a parameter $s\geq 1$ and T be a mapping on X satisfying,*

$$d(Tx,Ty)\leq\beta(M(x,y))\,M(x,y), \tag{2.1}$$

for all $x,y\in X$, where

$$M(x,y)=\max\{d(x,y),d(x,Tx),d(y,Ty)\},$$

and $\beta\in\mathscr{F}_s$. Then T has a unique fixed point u and $d(u,u)=0$.

Proof. Let $x_0 \in X$ be an arbitrary initial point. Let define the sequence $\{x_n\}$ by using Picard iterative method, that is, $x_n = Tx_{n-1} = T^n x_0$. If there exists $n \in \mathbb{N}$ such that $x_n = x_{n+1}$, then x_n becomes a fixed point of T. So, we can assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$. By (2.1), we have

$$\begin{aligned} d(x_n, x_{n+1}) &= d(Tx_{n-1}, Tx_n) \\ &\leq \beta(M(x_{n-1}, x_n)) M(x_{n-1}, x_n) \end{aligned} \tag{2.2}$$

where

$$\begin{aligned} M(x_{n-1}, x_n) &= \max\{d(x_{n-1}, x_n), d(x_{n-1}, Tx_{n-1}), d(x_n, Tx_n)\} \\ &= \max\{d(x_{n-1}, x_n), d(x_{n-1,} x_n), d(x_n, x_{n+1})\} \\ &= \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\}. \end{aligned}$$

If $d(x_{n-1}, x_n) \leq d(x_n, x_{n+1})$, then, we get $M(x_{n-1}, x_n) = d(x_n, x_{n+1})$. From (2.2) we have,

$$\begin{aligned} d(x_n, x_{n+1}) &\leq \beta(M(x_{n-1}, x_n)) M(x_{n-1}, x_n) \\ &< \frac{1}{s} d(x_{n+1}, x_n). \end{aligned}$$

Since the last inequality is a contradiction, we obtain that

$$d(x_n, x_{n+1}) \leq d(x_{n-1}, x_n), \quad \forall n \in \mathbb{N}.$$

By the same way, we can show that

$$\begin{aligned} d(x_n, x_{n+1}) &\leq \beta(M(x_{n-1}, x_n)) M(x_{n-1}, x_n) \\ &< \frac{1}{s} d(x_{n-1}, x_n). \end{aligned} \tag{2.3}$$

This means that $\{d(x_{n-1}, x_n)\}$ is a decreasing sequence. Therefore, there exists $d \geq 0$ such that $\lim_{n\to\infty} d(x_{n-1}, x_n) = d$. We assert that $d = 0$. Suppose on the contrary that $d > 0$. Then, from (2.3), we have

$$d \leq \limsup_{n\to\infty} \beta(M(x_{n-1}, x_n)) d.$$

Then, it is clear that

$$\begin{aligned} \frac{1}{s} &\leq 1 \\ &\leq \limsup_{n\to\infty} \beta(M(x_{n-1}, x_n)) \\ &\leq \frac{1}{s}. \end{aligned}$$

Since $\beta \in \mathscr{F}_s$, we get $\lim_{n\to\infty} M(x_{n-1}, x_n) = 0$ and so $\lim_{n\to\infty} d(x_{n-1}, x_n) = 0$ which is a contradiction, namely $d = 0$. Now, we show that $\{x_n\}$ is a Cauchy sequence. So, we need to show that $\lim_{n,m\to\infty} d(x_n, x_m)$ exists and is finite. Particularly, we show that $\lim_{n,m\to\infty} d(x_n, x_m) = 0$. Assume on the contrary that $\lim_{n,m\to\infty} d(x_n, x_m) \neq 0$. Then, there exists $\varepsilon > 0$ such that there exist subsequences $\left\{x_{m(k)}\right\}$ and $\left\{x_{n(k)}\right\}$ of $\{x_n\}$ for which $n(k) > m(k) > k$,

$$d\left(x_{m(k)}, x_{n(k)}\right) \geq \varepsilon \tag{2.4}$$

and

$$d\left(x_{m(k)+v-1}, x_{n(k)-1}\right) < \varepsilon. \tag{2.5}$$

It follows from (2.4) and triangular inequality that

$$\begin{aligned}
\varepsilon &\leq d\left(x_{m(k)}, x_{n(k)}\right) \\
&\leq s\left[d\left(x_{m(k)}, x_{m(k)+1}\right) + d\left(x_{m(k)+1}, x_{m(k)+2}\right) + \cdots \right.\\
&\quad \left. + d\left(x_{m(k)+v-1}, x_{m(k)+v}\right) + d\left(x_{m(k)+v}, x_{n(k)}\right)\right] \\
&\quad - \sum_{i=1}^{v} d\left(x_{m(k)+i}, x_{m(k)+i}\right) \\
&\leq sd\left(x_{m(k)}, x_{m(k)+1}\right) + sd\left(x_{m(k)+1}, x_{m(k)+2}\right) + \cdots \\
&\quad + sd\left(x_{m(k)+v-1}, x_{m(k)+v}\right) + sd\left(x_{m(k)+v}, x_{n(k)}\right).
\end{aligned}$$

By taking limsup for $k \to \infty$, we obtain

$$\frac{\varepsilon}{s} \leq \limsup_{k\to\infty} d\left(x_{m(k)+v}, x_{n(k)}\right). \tag{2.6}$$

Thus, we get

$$\begin{aligned}
&\limsup_{k\to\infty} M\left(x_{m(k)+v-1}, x_{n(k)-1}\right) \\
&\quad = \limsup_{k\to\infty} \max\left\{d\left(x_{m(k)+v-1}, x_{n(k)-1}\right),\right. \\
&\qquad \left. d\left(x_{m(k)+v-1}, Tx_{m(k)+v-1}\right), d\left(x_{n(k)-1}, Tx_{n(k)-1}\right)\right\} \\
&\quad = \limsup_{k\to\infty} \max\left\{d\left(x_{m(k)+v-1}, x_{n(k)-1}\right),\right. \\
&\qquad \left. d\left(x_{m(k)+v-1}, x_{m(k)+v}\right), d\left(x_{n(k)-1}, x_{n(k)}\right)\right\} \\
&\quad < \varepsilon.
\end{aligned}$$

From (2.6) and (2.1), we have

$$\begin{aligned}
\frac{\varepsilon}{s} &\leq \limsup_{k\to\infty} d\left(x_{m(k)+v}, x_{n(k)}\right) \\
&\leq \limsup_{k\to\infty} \beta\left(M\left(x_{m(k)+v-1}, x_{n(k)-1}\right)\right) M\left(x_{m(k)+v-1}, x_{n(k)-1}\right) \\
&\leq \limsup_{k\to\infty} \beta\left(M\left(x_{m(k)+v-1}, x_{n(k)-1}\right)\right) \limsup_{k\to\infty} M\left(x_{m(k)+v-1}, x_{n(k)-1}\right)
\end{aligned}$$

$$\leq \varepsilon \limsup_{k\to\infty} \beta\left(M\left(x_{m(k)+v-1}, x_{n(k)-1}\right)\right).$$

Then, we obtain $\frac{1}{s} \leq \limsup_{k\to\infty} \beta\left(M\left(x_{m(k)+v-1}, x_{n(k)-1}\right)\right) \leq \frac{1}{s}$. Since $\beta \in \mathscr{F}_s$, thus $\lim_{k\to\infty} M\left(x_{m(k)+v-1}, x_{n(k)-1}\right) = 0$. As a result, we get that the sequence $\left\{d\left(x_{m(k)+v-1}, x_{n(k)-1}\right)\right\}$ converges to 0 as $k \to \infty$. From (2.4) and using the triangular inequality, we have

$$\begin{aligned}
\varepsilon \leq d\left(x_{m(k)}, x_{n(k)}\right) &\leq s\left[d\left(x_{m(k)}, x_{m(k)+1}\right) + d\left(x_{m(k)+1}, x_{m(k)+2}\right)\right. \\
&\quad \left.+ \cdots + d\left(x_{m(k)+v-1}, x_{n(k)-1}\right) + d\left(x_{n(k)-1}, x_{n(k)}\right)\right] \\
&\quad - \sum_{i=1}^{v-1} d\left(x_{m(k)+i}, x_{m(k)+i}\right) - d\left(x_{n(k)-1}, x_{n(k)-1}\right) \\
&\leq sd\left(x_{m(k)}, x_{m(k)+1}\right) + sd\left(x_{m(k)+1}, x_{m(k)+2}\right) + \cdots \\
&\quad + sd\left(x_{m(k)+v-1}, x_{n(k)-1}\right) + sd\left(x_{n(k)-1}, x_{n(k)}\right).
\end{aligned}$$

Therefore, we have $\lim_{k\to\infty} d\left(x_{m(k)}, x_{n(k)}\right) = 0$. This contradicts condition (2.4). Hence $\{x_n\}$ is a Cauchy sequence in X. Because of the completeness of X, there exists a point u in X such that

$$\begin{aligned}
\lim_{n\to\infty} d(x_n, u) &= \lim_{n,m\to\infty} d(x_n, x_m) \\
&= d(u, u) = 0.
\end{aligned}$$

Now, we show that u is a fixed point of T. It follows from triangular inequality and (2.1) that

$$\begin{aligned}
d(u, Tu) &\leq s\left[d(u, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{n+v-1}, x_{n+v})\right. \\
&\quad \left.+ d(x_{n+v}, Tu)\right] - \sum_{i=1}^{v} d(x_{n+i}, x_{n+i}) \\
&\leq sd(u, x_{n+1}) + sd(x_{n+1}, x_{n+2}) + \\
&\quad + \cdots + sd(x_{n+v-1}, x_{n+v}) + sd(x_{n+v}, Tu).
\end{aligned}$$

Taking limitsup as $n \to \infty$, we obtain

$$\begin{aligned}
d(u, Tu) &\leq s \limsup_{n\to\infty} d(u, x_{n+1}) + s \limsup_{n\to\infty} d(x_{n+1}, x_{n+2}) \\
&\quad + \cdots + \limsup_{n\to\infty} d(x_{n+v-1}, x_{n+v}) \\
&\quad + s \limsup_{n\to\infty} \beta(M(x_{n+v-1}, u)) \limsup_{n\to\infty} M(x_{n+v-1}, u),
\end{aligned} \tag{2.7}$$

where

$$\begin{aligned}
&\limsup_{n\to\infty} M(x_{n+v-1}, u) \\
&\qquad = \limsup_{n\to\infty} \max\left\{d(x_{n+v-1}, u), d(x_{n+v-1}, Tx_{n+v-1}), d(u, Tu)\right\}
\end{aligned}$$

$$= \limsup_{n\to\infty} \max\{d(x_{n+v-1},u), d(x_{n+v-1},x_{n+v}), d(u,Tu)\}$$
$$= d(u,Tu).$$

Thus, from (2.7) we have,

$$d(u,Tu) \le s \limsup_{n\to\infty} \beta(M(x_{n+v-1},u))\, d(u,Tu).$$

Consequently,

$$\frac{1}{s} \le \limsup_{n\to\infty} \beta(M(x_{n+v-1},u)) \le \frac{1}{s}.$$

Since $\beta \in \mathscr{F}_s$, we conclude that $\lim_{n\to\infty} M(x_{n+v-1},u) = 0$. Therefore we obtain $Tu = u$, that is, u is a fixed point. Now, we need to show that u is a unique fixed point. Suppose to the contrary that there exists a distinct fixed point v. From (2.1) we have,

$$\begin{aligned} d(u,v) &= d(Tu,Tv) \\ &\le \beta(M(u,v))\, M(u,v), \end{aligned}$$

where

$$\begin{aligned} M(u,v) &= \max\{d(u,v), d(u,Tu), d(v,Tv)\}, \\ &= d(u,v). \end{aligned}$$

Therefore, we have

$$d(u,v) < \frac{1}{s} d(u,v).$$

This is a contradiction. So $u = v$, that is, u is a unique fixed point of T. This completes the proof. □

Corollary 2.2. *Let (X,d) be a complete $b_v(s)$-metric space with a constant $s \ge 1$ and let $\beta \in \mathscr{F}_s$ be a given function. Let T be a mapping on X satisfying,*

$$d(Tx,Ty) \le \beta(M(x,y))\, M(x,y), \tag{2.8}$$

for all $x, y \in X$, where

$$M(x,y) = \max\{d(x,y), d(x,Tx), d(y,Ty)\}.$$

Then T has a unique fixed point u and $d(u,u) = 0$.

Remark 2.3. In Theorem 2.1, if we take the constant $v = 2$, then we derive Corollary 3.4 of [13] in Branciari b metric spaces.

Now, we prove a fixed point theorem for Geraghty type contraction mappings in a complete partial $b_v(s)$-metric space. Precisely, if $d(x,y)$ is taken instead of (2.1) in Theorem 2.1, the following theorem is obtained.

Theorem 2.4. *Let (X,d) be a complete partial $b_v(s)$-metric space with parameter $s \geq 1$ and $T : X \to X$ be a mapping. Assume that there exists $\beta \in \mathscr{F}_s$ such that*

$$d(Tx,Ty) \leq \beta(d(x,y))\, d(x,y)\,, \tag{2.9}$$

for all $x,y \in X$. Then T has a unique fixed point $z \in X$ and for each $x \in X$ the Picard sequence $\{T^n x\}$ converges to z in (X,d).

Proof. Let $x_0 \in X$ be an arbitrary initial point and $Tx_{n-1} = x_n$ for $n \in \mathbb{N}$. If $x_{n_0} = x_{n_0+1}$ for some n_0, then it is obtained that $x_n = x_{n_0}$ for all $n \geq n_0$ and the proof is completed. So, we can suppose that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$. Then, using (2.9) we obtain

$$\begin{aligned} d(x_{n+1},x_n) &= d(Tx_n, Tx_{n-1}) \\ &\leq \beta(d(x_n,x_{n-1}))\, d(x_n,x_{n-1}) \\ &< \frac{1}{s} d(x_n,x_{n-1})\,. \end{aligned}$$

Similiarly to the proof of Theorem 2.1, it can be proved that $\lim_{n\to\infty} d(x_{n-1},x_n) = 0$ and $\{x_n\}$ is a Cauchy sequence in the partial $b_v(s)$-metric space X. Because of the completeness of X, $\{x_n\}$ converges to a point z in X. Now, we want to show that $\{x_n\}$ has a unique limit. Suppose on the contrary that the sequence $\{x_n\}$ converges to y and z. We need to show that $y = z$. From the third condition of the definition of partial $b_v(s)$-metric space, we get

$$\begin{aligned} d(y,z) &\leq s\left[d(y,x_n) + d(x_n,x_{n+1}) + \cdots + d(x_{n+v-2},x_{n+v-1})\right. \\ &\quad \left. + d(x_{n+v-1},z)\right] - \sum_{k=1}^{v-1} d(x_{n+i},x_{n+i}) \\ &\leq s\left[d(y,x_n) + d(x_n,x_{n+1}) + \cdots \right. \\ &\quad \left. + d(x_{n+v-2},x_{n+v-1}) + d(x_{n+v-1},z)\right]. \end{aligned}$$

By taking limit from both side, we obtain that $y = z$. So, the sequence $\{x_n\}$ has a unique limit. Now, we show that the mapping T which satisfies the condition (2.9) is a continuous mapping in the sense that $y_n \to y$ implies that $Ty_n \to Ty$. Let $y_n \to y$ as $n \to \infty$. We get from (2.9) that

$$\begin{aligned} d(Ty_n,Ty) &\leq \beta(d(y_n,y))\, d(y_n,y) \\ &< \frac{1}{s} d(y_n,y)\,. \end{aligned}$$

Obviously, since $y_n \to y$, we have $Ty_n \to Ty$. This means that T is a continuous mapping. So, it is clear that $Tz = z$, that is, z is a unique fixed point of T. □

Corollary 2.5. *Let (X,d) be a complete $b_v(s)$-metric space with parameter $s \geq 1$ and let $T: X \to X$ be a mapping. Assume that there exists $\beta \in \mathscr{F}_s$ such that*

$$d(Tx,Ty) \leq \beta(d(x,y))\, d(x,y),$$

for all $x,y \in X$. Then T has a unique fixed point $z \in X$ and for each $x \in X$ the Picard sequence $\{T^n x\}$ converges to z in (X,d).

Remark 2.6. (i) In Corollary 2.5, if $v = s = 1$, then we derive Theorem 1.3 of [15] in metric spaces
(ii) In Corollary 2.5, if we take the constant $v = 1$, then we derive Theorem 3.8 of [11] in metric type spaces.
(iii) In Corollary 2.5, if $v = 2$, then we obtain Corollary 3.6 of [13] in Branciari b metric spaces.
(iv) In Theorem 2.4, if $v = s = 1$, then we derive Theorem 3.1 of [11] in partial metric spaces.

Now, we give an example which satisfies the conditions of Theorem 2.1.

Example 2.7. Let $X = \{1,2,3,4,5\}$ and $d : X \times X \to [0,\infty)$ be a mapping defined by,

$$d(x,y) = \begin{cases} 0, & \text{if } x = y = 3, \\ \frac{9}{10}, & \text{if } x \text{ or } y \in \{1,2\},\ x \neq y, \\ \frac{1}{10}, & \text{otherwise,} \end{cases}$$

for all $x,y \in X$. Then (X,d) is a complete partial $b_v(s)$-metric space with $v = 3$ and $s = \frac{3}{2}$. Let $T : X \times X \to [0,\infty)$ be a mapping defined by

$$T(x) = \begin{cases} 4, & \text{if } x = 1, \\ 3, & \text{if } x \neq 1, \end{cases}$$

for all $x,y \in X$ and $\beta(t) = \frac{2}{3}e^{-t}$. Then it is easy to see that T is a generalized Geraghty type contraction on X. So, T has a unique fixed point $u = 3$ and $d(3,3) = 0$.

The below theorem is an enlargement of [[14], Theorem 4] from the notion b-metric spaces to the case of partial $b_v(s)$-metric spaces. Although there exists a continuity condition in Theorem 4 of [14], we prove the next theorem without this condition.

Theorem 2.8. *Let (X,d) be a complete partial $b_v(s)$-metric space with a constant $s > 1$. If T, S are mappings on X satisfying,*

$$sd(Tx,Sy) \leq \beta(M(x,y))\, M(x,y), \quad x,y \in X, \tag{2.10}$$

where

$$M(x,y) = \max\{d(x,y), d(x,Tx), d(y,Sy)\},$$

and $\beta \in \mathscr{F}_s$, *then* T *and* S *have a unique common fixed point* x^* *and* $d(x^*, x^*) = 0$.

Proof. Let $x_0 \in X$ be an arbitrary initial point. Define a sequence $\{x_n\}$ in X by $x_{2n+1} = Tx_{2n}$ and $x_{2n+2} = Sx_{2n+1}$ for all $n \in \mathbb{N}$. Then we have

$$sd(x_{2n+1}, x_{2n+2}) = sd(Tx_{2n}, Sx_{2n+1}) \leq \beta(M(x_{2n}, x_{2n+1})) M(x_{2n}, x_{2n+1}), \tag{2.11}$$

where

$$\begin{aligned} M(x_{2n}, x_{2n+1}) &= \max\{d(x_{2n}, x_{2n+1}), d(x_{2n}, Tx_{2n}), d(x_{2n+1}, Sx_{2n+1})\} \\ &= \max\{d(x_{2n}, x_{2n+1}), d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2})\} \\ &= \max\{d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2})\}. \end{aligned}$$

If $d(x_{2n}, x_{2n+1}) \leq d(x_{2n+1}, x_{2n+2})$, then we get $M(x_{2n}, x_{2n+1}) = d(x_{2n+1}, x_{2n+2})$. So, we obtain

$$\begin{aligned} sd(x_{2n+1}, x_{2n+2}) &\leq \beta(M(x_{2n}, x_{2n+1})) M(x_{2n}, x_{2n+1}) \\ &< \frac{1}{s} d(x_{2n+1}, x_{2n+2}), \end{aligned}$$

which is a contradiction. Hence, we have $M(x_{2n}, x_{2n+1}) = d(x_{2n}, x_{2n+1})$. From (2.11), we get

$$\begin{aligned} sd(x_{2n+1}, x_{2n+2}) &\leq \beta(M(x_{2n}, x_{2n+1})) M(x_{2n}, x_{2n+1}) \\ &< \frac{1}{s} d(x_{2n}, x_{2n+1}). \end{aligned} \tag{2.12}$$

Then, we obtain $d(x_{2n+1}, x_{2n+2}) \leq \frac{1}{s} d(x_{2n}, x_{2n+1})$. In a similar way, we can easily show that $d(x_{2n+3}, x_{2n+2}) \leq \frac{1}{s} d(x_{2n+2}, x_{2n+1})$. So, we have

$$d(x_n, x_{n+1}) \leq \frac{1}{s} d(x_{n-1}, x_n). \tag{2.13}$$

This means that $\{d(x_n, x_{n+1})\}$ is a decreasing sequence. Therefore, there exists $e \geq 0$ such that $d(x_{n-1}, x_n) \to e$ as $n \to \infty$. We assert that $e = 0$. Suppose on the contrary that $e > 0$. By taking limitsup in (2.12) as n tends to infinity, we obtain

$$e \leq \limsup_{n \to \infty} \beta(M(x_{2n}, x_{2n+1})) e.$$

Then it is clear that

$$\begin{aligned} \frac{1}{s} &\leq 1 \\ &\leq \limsup_{n \to \infty} \beta(M(x_{2n}, x_{2n+1})) \\ &\leq \frac{1}{s}. \end{aligned}$$

Since $\beta \in \mathscr{F}_s$, we conclude that

$$\lim_{n\to\infty} M(x_{2n}, x_{2n+1}) = 0.$$

Thus, it follows that

$$\lim_{n\to\infty} d(x_{2n}, x_{2n+1}) = 0,$$

which is a contradiction. This means that $e = 0$. On the other hand, we observe that repeated application of (2.13) leads to

$$\begin{aligned} d(x_n, x_{n+1}) &\leq \frac{1}{s} d(x_{n-1}, x_n) \\ &\leq \frac{1}{s^2} d(x_{n-3}, x_{n-2}) \\ &\ \ \vdots \\ &\leq \frac{1}{s^n} d(x_0, x_1). \end{aligned} \tag{2.14}$$

So, we have that $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$. Now, we show that $\{x_{2n}\}$ is a Cauchy sequence. Namely, we need to show that $\lim\limits_{n,m\to\infty} d(x_{2n}, x_{2m})$ exists and is finite. Particularly, we will show that $\lim\limits_{n,m\to\infty} d(x_{2n}, x_{2m}) = 0$. For $m > n$ by using inequality (2.14), we obtain

$$\begin{aligned} d(x_{2n}, x_{2m}) &\leq s\,[d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2}) + \cdots \\ &\quad + d(x_{2n+v-3}, x_{2n+v-2}) + d(x_{2n+v-2}, x_{2n+n_0}) \\ &\quad + d(x_{2n+n_0}, x_{2m+n_0}) + d(x_{2m+n_0}, x_{2m})] \\ &\quad - \sum_{k=1}^{v-2} d(x_{2n+k}, x_{2n+k}) - d(x_{2n+n_0}, x_{2m+n_0}) \\ &\quad - d(x_{2n+n_0}, x_{2m+n_0}) \\ &\leq s\,[d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2}) + \cdots \\ &\quad + d(x_{2n+v-3}, x_{2n+v-2}) + d(x_{2n+v-2}, x_{2n+n_0}) \\ &\quad + d(x_{2n+n_0}, x_{2m+n_0}) + d(x_{2m+n_0}, x_{2m})] \\ &\leq s\left[\frac{1}{s^{2n}} d(x_0, x_1) + \frac{1}{s^{2n+1}} d(x_0, x_1) + \cdots \right. \\ &\quad + \frac{1}{s^{2n+v-3}} d(x_0, x_1) + \frac{1}{s^{2n}} d(x_{v-2}, x_{n_0}) \\ &\quad \left. + \frac{1}{s^{n_0}} d(x_{2n}, x_{2m}) + \frac{1}{s^{2m}} d(x_{n_0}, x_0)\right]. \end{aligned}$$

So, we get

$$\left(1-\frac{1}{s^{n_0-1}}\right)d\left(x_{2n},x_{2m}\right)\leq s\left(\frac{1}{s^{2n}}+\frac{1}{s^{2n+1}}+\cdots+\frac{1}{s^{2n+v-3}}\right)d\left(x_0,x_1\right)$$
$$+s\frac{1}{s^{2n}}d\left(x_{v-2},x_{n_0}\right)+\frac{1}{s^{2m}}d\left(x_{n_0},x_0\right).$$

By taking limit from both side

$$\lim_{n\to\infty}d\left(x_{2n},x_{2m}\right)=0.$$

This implies that $\{x_{2n}\}$ is a Cauchy sequence and so is $\{x_n\}$. Since X is a complete space, there exists $x^*\in X$ such that

$$\begin{aligned}\lim_{n\to\infty}d\left(x_n,x^*\right)&=\lim_{n,m\to\infty}d\left(x_n,x_m\right)\\&=d\left(x^*,x^*\right)\\&=0.\end{aligned}$$

Now, we will show that x^* is a fixed point of T. Then, for $Tx^*\neq x^*$, we would write

$$\begin{aligned}d\left(Tx^*,x^*\right)&\leq s\left[d\left(Tx^*,x_{2n+2}\right)+d\left(x_{2n+2},x_{2n+3}\right)+\cdots\right.\\&\quad\left.+d\left(x_{2n+v},x_{2n+v+1}\right)+d\left(x_{2n+v+1},x^*\right)\right]\\&\quad-\sum_{i=2}^{v+1}d\left(x_{2n+i},x_{2n+i}\right)\\&\leq s\left[d\left(Tx^*,x_{2n+2}\right)+d\left(x_{2n+2},x_{2n+3}\right)+\cdots\right.\\&\quad\left.+d\left(x_{2n+v},x_{2n+v+1}\right)+d\left(x_{2n+v+1},x^*\right)\right].\end{aligned}$$

If we take the limit from the both side in the above last inequality, we get

$$\begin{aligned}d\left(Tx^*,x^*\right)&\leq s\lim_{n\to\infty}d\left(Tx^*,x_{2n+2}\right)\\&=s\lim_{n\to\infty}d\left(Tx^*,Sx_{2n+1}\right)\\&\leq\lim_{n\to\infty}\beta\left(M\left(x^*,x_{2n+1}\right)\right)M\left(x^*,x_{2n+1}\right)\\&<\frac{1}{s}\lim_{n\to\infty}M\left(x^*,x_{2n+1}\right),\end{aligned}$$

where

$$\begin{aligned}\lim_{n\to\infty}M\left(x^*,x_{2n+1}\right)&=\lim_{n\to\infty}\max\left\{d\left(x^*,x^*\right),d\left(x^*,Tx^*\right),d\left(x_{2n+1},Sx_{2n+1}\right)\right\}\\&=\lim_{n\to\infty}\max\left\{d\left(x^*,x^*\right),d\left(x^*,Tx^*\right),d\left(x_{2n+1},x_{2n+2}\right)\right\}\\&=d\left(x^*,Tx^*\right).\end{aligned}$$

So, we can write

$$\begin{aligned} d(Tx^*, x^*) &\leq \beta(M(x^*, x_{2n+1}))\, M(x^*, x_{2n+1}) \\ &< \frac{1}{s} d(x^*, Tx^*), \end{aligned}$$

which is a contradiction. Thus, we obtain $Tx^* = x^*$. By (2.10), we have

$$\begin{aligned} sd(x^*, Sx^*) &= sd(Tx^*, Sx^*) \\ &\leq \beta(M(x^*, x^*))\, M(x^*, x^*), \end{aligned}$$

where

$$\begin{aligned} M(x^*, x^*) &= \max\{d(x^*, x^*), d(x^*, Tx^*), d(x^*, Sx^*)\} \\ &= d(x^*, Sx^*). \end{aligned}$$

Hence, we get

$$sd(x^*, Sx^*) < \frac{1}{s} d(x^*, Sx^*).$$

Therefore, we obtain $Sx^* = x^*$. So, it is concluded that x^* is a common fixed point of T and S. Now, we need to show that x^* is a unique fixed point. Let y be another common fixed point of T and S. By (2.10) we obtain

$$\begin{aligned} sd(x^*, y) &= sd(Tx^*, Sy) \\ &\leq \beta(M(x^*, y))\, M(x^*, y), \end{aligned}$$

where

$$\begin{aligned} M(x^*, y) &= \max\{d(x^*, y), d(x^*, Tx^*), d(y, Sy)\} \\ &= d(x^*, y). \end{aligned}$$

So, we get

$$sd(x^*, y) < \frac{1}{s} d(x^*, y),$$

which is a contradiction. Therefore, it follows that $x^* = y$. This completes the proof. □

Corollary 2.9. *Let (X, d) be a complete $b_v(s)$-metric space with a constant $s > 1$. If T, S are mappings on X satisfying,*

$$sd(Tx, Sy) \leq \beta(M(x, y))\, M(x, y), \quad x, y \in X,$$

where

$$M(x, y) = \max\{d(x, y), d(x, Tx), d(y, Sy)\}$$

and $\beta \in \mathscr{F}_s$, then T and S have a unique common fixed point x^* and $d(x^*, x^*) = 0$.

3. Applications to Nonlinear Integral Equations

In this section, as an application to the fixed point theorems proved in the previous section, we consider the existence of the solutions of nonlinear integral equations.

Let $X = C[0,r]$ with

$$d(x,y) = \max_{t\in[0,r]} \{|x(t) - y(t)|^p\}, \quad p > 1, \ x, y \in X.$$

Clearly, from the convexity of function $f(x) = x^p$ for $x \geq 0$ and Jensen inequality, we have

$$(x_1 + x_2 + \cdots + x_v)^p \leq v^{p-1}(x_1^p + x_2^p + \cdots + x_v^p),$$

for nonnegative real numbers $x_1, x_2, \ldots, x_v$. So, we get that (X,d) is a complete partial $b_v(s)$-metric space for $v \geq 3$ and $s = v^{p-1}$. Let consider the integral equation,

$$x(t) = h(t) + \int_0^r G(t,s)\, k(t,s,x(s))\, ds, \tag{3.1}$$

where $h : [0,r] \to \mathbb{R}$, $G : [0,r] \times [0,r] \to \mathbb{R}$ and $k : [0,r] \times [0,r] \times \mathbb{R} \to \mathbb{R}$ are continuous functions. If the following conditions are satisfied, then the integral equation (3.1) has a unique solution:

(i) (a) $|k(t,s,x(s)) - k(t,s,y(s))| \leq \left(\frac{e^{-M(x,y)}M(x,y)}{v^{p-1}}\right)^{1/p}$ for all t,$s \in [0,r]$ and $x, y \in X$,

(b) $\max \int_0^r G(t,s)^q\, ds \leq \frac{1}{r^{q/p}}$ for all t,$s \in [0,r]$ where $\frac{1}{p} + \frac{1}{q} = 1$.

Indeed, if we define the mapping $T : X \to X$ by

$$Tx(t) = h(t) + \int_0^r G(t,s)\, k(t,s,x(s))\, ds, \quad x \in X, \ t, s \in [0,r],$$

then we can write the following from the conditions (a) and (b) :

$$\begin{aligned} d(Tx,Ty) &= \max_{t\in[0,r]} \{|Tx(t) - Ty(t)|^p\} \\ &= \max_{t\in[0,r]} \left\{ \left| h(t) + \int_0^r G(t,s)\, k(t,s,x(s))\, ds \right.\right. \\ &\qquad \left.\left. -h(t) - \int_0^r G(t,s)\, k(t,s,y(s))\, ds \right|^p \right\} \end{aligned}$$

$$= \max_{t\in[0,r]} \left\{ \left| \int_0^r G(t,s)\left(k(t,s,x(s)) - k(t,s,y(s))\right) ds \right|^p \right\}$$
$$\leq \max_{t\in[0,r]} \left\{ \int_0^r \left| G(t,s)\left(k(t,s,x(s)) - k(t,s,y(s))\right) ds \right|^p \right\}$$
$$\leq \max_{t\in[0,r]} \left\{ \left(\int_0^r |G(t,s)|^q \, ds \right)^{1/q} \right.$$
$$\left. \times \left(\int_0^r |k(t,s,x(s)) - k(t,s,y(s))|^p \, ds \right)^{1/p} \right\}^p$$
$$= \max_{t\in[0,r]} \left\{ \left(\int_0^r |G(t,s)|^q \, ds \right)^{p/q} \right.$$
$$\left. \times \left(\int_0^r |k(t,s,x(s)) - k(t,s,y(s))|^p \, ds \right) \right\}$$
$$\leq \left(\frac{1}{r^{q/p}} \right)^{p/q} \int_0^r \left| \frac{e^{-M(x,y)} M(x,y)}{v^{p-1}} \right| ds$$
$$= \frac{e^{-M(x,y)}}{v^{p-1}} M(x,y).$$

So, we obtain

$$d(Tx,Ty) \leq \beta(M(x,y)) M(x,y).$$

Then, all conditions in Theorem (2.1) are satisfied for $\beta(t) = \frac{e^{-t}}{v^{p-1}}$, $t > 0$ and the integral equation (3.1) has a unique solution.

References

1. M.S. Abdullahi and P. Kumam, *Partial $b_v(s)$-metric spaces and fixed point theorems*, J. Fixed Point Theory Appl., 20 (2018), 13 pages.
2. Ö. Acar and I. Altun, *A fixed point theorem for F-Geraghty contraction on metric-like spaces*, Fasc. Math., 59 (2017), pp. 5-12.
3. H. Afshari, H. Alsulami and E. Karapınar, *On the extended multi-valued Geraghty type contractions*, J. Nonlinear Sci. Appl., 9 (2016), pp. 4695-4706.

4. S. Aleksic, Z.D. Mitrovic and S. Radenovic, *A fixed point theorem of Jungck in $b_v(s)$-metric spaces,* Period. Math. Hungar., 77 (2018), pp. 224-231.
5. B. Alqahtani, A. Fulga and E. Karapınar, *A short note on the common fixed points of the Geraghty contraction of type $E_{S,T}$*, Demonstr. Math., 51 (2018), pp. 233-240.
6. I. Altun and K. Sadarangani, *Generalized Geraghty type mappings on partial metric spaces and fixed point results*, Arab J. Math., 2 (2013), pp. 247-253.
7. M. Arshad and A. Hussain, *Fixed point results for generalized rational α-Geragty contraction*, Miskolc Math. Notes, 18 (2017), pp. 611-621.
8. H. Aydi, A. Felhi and H. Afshari, *New Geraghty type contractions on metric-like spaces*, J. Nonlinear Sci. Appl., 10 (2017), pp. 780-788.
9. S. Chandok, *Some fixed point theorems for (α, β)-admissible Geraghty type contractive mappings and related results*, Math. Sci., 9 (2015), pp. 127-135.
10. A.K. Dubey, U. Mishra and W.H. Lim, *Some new fixed point theorems for generalized contractions involving rational expressions in complex valued b-metric spaces,* Nonlinear Funct. Anal. Appl., 24 (2019), pp. 477-483.
11. D. Dukic, Z. Kadelburg and S. Radenovic, *Fixed Points of Geraghty-Type Mappings in Various Generalized Metric Spaces*, Abstr. Appl. Anal., 2011 (2011), 13 pages.
12. O. Ege, *Complex valued rectangular b-metric spaces and an application to linear equations*, J. Nonlinear Sci. Appl., 8 (2015), pp. 1014-1021 .
13. I.M. Erhan, *Geraghty type contraction mappings on Branciari b-metric spaces*, Adv. Theory Analysis Appl., 1 (2017), pp. 147-160.
14. H. Faraji, D. Savic and S. Radenovic, *Fixed point theorems for Geraghty type mappings in b-metric spaces and Applications*, Axioms, 8 (2019), 12 pages.
15. M.A. Geraghty, *On contractive mappings*, Proc. Amer. Math. Soc., 40 (1973), pp. 604-608.
16. M.E. Gordji, H. Baghani, H. Khodaei and M. Ramezani, *Geraghty's fixed point theorem for special Multi-valued mappings*, Thai J. Math., 10 (2010), pp. 225-231.
17. H. Huang, L. Paunovic and S. Radenovic, *On some new fixed point results for rational Geraghty contractive mappings in ordered b-metric spaces*, J. Inequal. Appl., 8 (2015), pp. 800-807.

18. A. Hussain, *Modified Geraghty contraction involving fixed point theorems*, Jordan J. Math. Stat., 10 (2017), pp. 95-112.
19. M. Jovanovic, Z. Kadelburg and S. Radenovic, *Common fixed point results in metric-type spaces*, Fixed Point Theory Appl., 2010 (2010), 15 pages.
20. Z. Kadelburg and P. Kumam, S. Radenovic and W. Sintunavarat *Common coupled fixed point theorems for Geraghty-type contraction mappings using monotone property*, Fixed Point Theory Appl., 2015 (2015), 14 pages.
21. I. Karahan and I. Isik, *Partial $b_v(s)$, Partial v-generalized and $b_v(\theta)$ metric spaces and related fixed point theorems*, Facta Univ. Ser. Math. Inform., (In Press).
22. E. Karapınar, *On best proximity point of ψ-Geraghty contractions*, Fixed Point Theory Appl., 2013 (2013), 9 pages.
23. E. Karapınar, *α-ψ-Geraghty contraction type mappings and some related fixed point results*, Filomat, 28 (2014), pp. 37-48.
24. E. Karapınar and B. Samet, *A note on 'ψ-Geraghty type contractions'*, Fixed Point Theory Appl., 2014 (2014), 5 pages.
25. Z.D. Mitrovic, H. Aydi, Z. Kadelburg and G.S. Rad, *On some rational contractions in $b_v(s)$-metric spaces,* Rend. Circ. Mat. Palermo 2 (2019), 11 pages.
26. Z.D. Mitrovic, H. Aydi and S. Radenovic, *On Banach and Kannan type results in cone $b_v(s)$-metric spaces over Banach algebra,* Acta Math. Univ. Comenian., LXXXIX (2020), pp. 143-152.
27. Z.D. Mitrovic and S. Radenovic, *The Banach and Reich contractions in $b_v(s)$-metric spaces,* J. Fixed Point Theory Appl., 19 (2017), pp. 3087-3095.
28. Z. Mostefaoui, M. Bousselsal and J.K. Kim, *Some results in fixed point theory concerning rectangular b-metric spaces*, Nonlinear Funct. Anal. Appl., 24 (2019), pp. 49-59.
29. Y.J. Piao, *Fixed point theorems for contractive and expansive mappings of Geraghty type on 2-metric spaces*, Adv. Fixed Point Theory, 6 (2016), pp. 123-135.
30. K.N.V.V. Vara Prasad and A.K. Singh, *Fixed point results for rational α-Geraghty contractive mappings*, Adv. Inequal. Appl., 2019 (2019), 15 pages.
31. V.L. Rosa and P. Vetro, *Fixed points for Geraghty contractions in partial metric spaces*, J. Nonlinear Sci. Appl., 7 (2014), pp. 1-10.
32. R.J. Shahkoohi and A. Razani, *Some fixed point theorems for rational Geraghty contractive mappings in ordered b-metric spaces*, J. Inequal. Appl., 2014 (2014), 23 pages.

33. A.Wiriyapongsanon and N. Phudolsitthiphat, *Coincidence point theorems for Geraghty-type contraction mappings in generalized metric spaces*, Thai J. Math., (2018), pp. 145-158.

---

[1] Department of Mathematics, Faculty of Science, Erzurum Technical University, P.O.Box 25050, Erzurum, Turkey.
E-mail address: ebru.altiparmak@erzurum.edu.tr

[2] Department of Mathematics, Faculty of Science, Erzurum Technical University, P.O.Box 25050, Erzurum, Turkey.
E-mail address: ibrahimkarahan@erzurum.edu.tr