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Some Properties of Complete Boolean Algebras

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Abstract. The main result of this paper is a characterization of the strongly algebraically closed algebras in the lattice of all real-valued continuous functions and the equivalence classes of λ-measurable. We shall provide conditions which strongly algebraically closed algebras carry a strictly positive Maharam submeasure. Particularly, it is proved that if B is a strongly algebraically closed lattice and (B, σ) is a Hausdorff space and B satisfies the G_σ property, then B carries a strictly positive Maharam submeasure.

1. Introduction

The problem of finding an algebraically closed algebraic structure can be formulated for any class of algebraic structures of an arbitrary language. In [12] it is shown that algebras carry a continuous submeasure. In the present note we study the existence of a Maharam submeasure in strongly algebraically closed algebras. We borrow from [3, 8–10, 13, 17] for basic results and definitions of strongly algebraically closed algebras and Maharam submeasure.

Schmid in [16] characterized algebraic structures, when a distributive lattice is an algebraically closed lattice and he proved that any strongly algebraically closed lattice is a complete Boolean lattice. Later, the author in [13] proved that if a complete Boolean lattice is q'-compact, then it is a strongly algebraically closed lattice. By [13], if every set of equations (finite or infinite) with coefficients in an algebra A, which is solvable in some algebras of the class of algebras containing A, already has a solution in A, then A is a strongly algebraically closed algebra.

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As shown in [8] a complemented distributive lattice is called a Boolean lattice and a Boolean lattice together with the unary operation of complementation is said to be a Boolean algebra. For the concept of Boolean algebras, the reader is referred to [6, 19, 20].

Let us remark that an arbitrary set of equations is called a system. We say that two Boolean algebras B_1, B_2 are geometrically equivalent if any system of equations over B_1 and B_2 are isomorphic [18]. The problem of B. Plotkin in [15] asks "When two geometrically equivalent extensions L_1 and L_2 of a field F have different elementary theories in the logic?". This problem for equationally Noetherian groups was solved in [14]. Following this, in our paper we define q'-compactness for the set of all bands of a lattice of all real-valued continuous functions, the set of equivalence classes of μ-measurable almost everywhere finite functions on a nonempty set, the set of all L-projections and the inclusion-ordered system of all regular open sets in topological space in order to obtain necessary conditions to be a strongly algebraically closed algebra. Also, we prove that if B is a strongly algebraically closed lattice and (B, σ) is a Hausdorff space and B satisfies the G_σ property, then B carries a strictly positive Maharam submeasure.

2. Main Results

In this section, strongly algebraically closed lattices are related to L^p-projection, the set of the equivalence classes of μ-measurable on a nonempty set, the lattice of all real-continues function and regular open sets (see [3, 4, 8]).

We recall from [11] that a vector lattice is a real vector space equipped with some partial order making this space into a lattice so that the usual axioms hold:

(a) $x + z < y + z$ if $x < y$, for all z,
(b) $\mu x < \mu y$ if $x < y$, for all real $\mu > 0$,

for all x, y, z of the vector lattice.

Assume $\mathbb{R}_0$ is a subspace of $\mathbb{R}$ and $a \in \mathbb{R}_0$, $b \in \mathbb{R}$. Then $\mathbb{R}_0$ is caaled solid if $|b| \leq a$ implies $b \in \mathbb{R}_0$ (see, [4]). Then two elements x, y of a vector lattice are called disjoint if $|x| \wedge |y| = 0$. The disjointness of a and b is denoted by $a \perp b$. The totality of all $a \in X$ disjoint from E (i.e. disjoint from every $b \in E$) is said the disjoint complement of E and we denote it by $E^\perp$ and we set $E^\perp = \{a \in X \mid \forall b \in E,\ a \perp b\}$. Recall that the disjoint complement has the following properties:

$$E \subseteq E^{\perp\perp}, \qquad E^\perp = E^{\perp\perp\perp}, \qquad E^\perp \cap E^{\perp\perp} = 0.$$

For simplicity we write $E^{\perp\perp}$ instead of $(E^\perp)^\perp$ and a set E is called a band whenever $E = E^{\perp\perp}$.

In [18], two algebras A and B are meant geometrically equivalence, if $Rad_A(S) = Rad_B(S)$ for any system S, where $Rad_A(S)$ and $Rad_B(S)$ are the sets of solutions of S in A and B, respectively. Now, we define q'-compactness for the set of all bands of a lattice of all real-valued continuous functions.

Definition 2.1. The set of all bands of a lattice of all real-valued continuous functions on a topological space is called q'-compact if it is a geometrically equivalence to any of its elementary extensions.

Theorem 2.2. *Suppose that X is a topological space and A is the lattice of all real-valued continuous functions on X with the order induced by that of $\mathbb{R}$. If the set B of all bands of A is q'-compact, then B is a strongly algebraically closed lattice.*

Proof. Since B is the set of all bands endowed with the natural order, we immediately get that each subset of a partially order set B has a supremum and an infimum. Also, we know that the intersection of each class of bands is a band (nonempty class) and there exists a least band of all subsets of B. So, there exist the greatest lower bound and the least upper bound of the set of bands, which they are intersection and union of this set of bands, respectively. On the other hand, the zero of B is the band consisting only the zero and the unit of B is the entire B. Therefore, B possesses the distinct zero and unity and it is a lattice.
Now, we prove that every band has complement. We claim that the disjoint complement of every band is its complement. Suppose that F is an arbitrary band. We prove that the disjoint complement $F^{\perp}$ is complement of F. It is clear that $F \cap F^{\perp}$ is a sole zero and thus $F \cap F^{\perp} = F \wedge F^{\perp} = 0_B$. Indeed, the system $\{F, F^{\perp}\}$ is complete and $F \vee F^{\perp} = 1_B$. In order to establish the theorem it suffices to show that B has distributivity law. Let F_1, F_2, F_3 be bands. We must prove that:

$$F_1 \wedge (F_2 \vee F_3) = (F_1 \wedge F_2) \vee (F_1 \wedge F_3).$$

In general, we have in lattices

$$F_1 \geq (F_1 \wedge F_2), \qquad (F_2 \vee F_3) \geq F_3 \geq (F_1 \wedge F_3),$$

and then

$$F_1 \wedge (F_2 \vee F_3) \geq (F_1 \wedge F_2) \vee (F_1 \wedge F_3).$$

It suffices to show

$$F_1 \wedge (F_2 \vee F_3) \leq (F_1 \wedge F_2) \vee (F_1 \wedge F_3).$$

If $F_1 \wedge (F_2 \vee F_3) \leq (F_1 \wedge F_2) \vee (F_1 \wedge F_3)$ is not true, then there is an element

$$x \in [F_1 \wedge (F_2 \vee F_3)] - [(F_1 \wedge F_2) \vee (F_1 \wedge F_3)].$$

In this case there is a nonzero element $y \in [(F_1 \wedge F_2) \vee (F_1 \wedge F_3)]^{\perp}$ such that $y \leq x$. Otherwise, we know that the disjoint complement is solid i.e. $y \in [(F_1 \wedge F_2) \vee (F_1 \wedge F_3)]^{\perp}$ and $x \leq y$ imply $x \in [(F_1 \wedge F_2) \vee (F_1 \wedge F_3)]^{\perp}$, thus all elements of the form $z \wedge x$ and $z \in [(F_1 \wedge F_2) \vee (F_1 \wedge F_3)]^{\perp}$ must be equal to zero. As a result, $x \in [(F_1 \wedge F_2) \vee (F_1 \wedge F_3)]^{\perp\perp} = B$. On the other hand, by assumption, we have $[(F_1 \wedge F_2) \vee (F_1 \wedge F_3)]^{\perp\perp} = [(F_1 \wedge F_2) \vee (F_1 \wedge F_3)]$, although $x \notin [(F_1 \wedge F_2) \vee (F_1 \wedge F_3)]$. So, the required element y exists. Using of the interpretation of suprema in B we have

$$(F_1 \wedge F_2) \vee (F_1 \wedge F_3) = [(F_1 \cap F_2) \cup (F_1 \cap F_3)]^{\perp\perp}.$$

Now, we conclude $y \in [(F_1 \cap F_2) \cup (F_1 \cap F_3)]^{\perp} = [F_1 \cap (F_2 \cup F_3)]$. Since we have $y \in [F_1 \wedge (F_2 \vee F_3)]$, we will obtain $y \in F_1$ and $y \in (F_2 \vee F_3) = (F_2 \cup F_3)^{\perp\perp}$. We know $(F_2 \cup F_3)^{\perp\perp} \cap (F_2 \cup F_3)^{\perp} = 0$. This shows that $y \notin (F_2 \cup F_3)^{\perp}$. It is clear that the set $F_2 \cup F_3$ is solid, thus there is a nonzero $t \in (F_2 \cup F_3)$ such that $t \leq y$ and $t \in (F_2 \cup F_3) \cap F_1$. Similarly, $t \in [(F_2 \cup F_3) \cap F_1]^{\perp}$ and it is a contradiction, since we have $t > 0$. Until this part of the proof, we proved that B satisfies the distributivity low and it is a complete Boolean algebra. By assumption, B is q'-compact and then by applying Theorem 3.5 of [13], B is a strongly algebraically closed lattice. $\square$

Let X be a nonempty set and B be a Boolean algebra. A positive countably additive function λ:

$$\lambda : B \longrightarrow \mathbb{R},$$

with

$$\lambda\left(\bigvee_{n=1}^{\infty} x_n\right) = \sum_{n=1}^{\infty} \lambda(x_n),$$

for every disjoint sequence (x_n) of B is called a measure. We denote by $M(\lambda)$ the set of equivalence classes of λ-measurable almost everywhere finite functions on X. We know that measurable functions can differ only on a measure zero set but they are equivalent. A partial order $\leq$ on $M(\lambda)$ is defined by the following rule

$$\overline{f} \leq \overline{g} \quad \Longleftrightarrow \quad f(x) \leq g(x)$$

which defines a lattice, for almost all $x \in X$. Therefore, $(M(\lambda), \leq)$ is a lattice. Here $\overline{f}$ is the coset of f. We denote by 1 the coset of the identically one function on X. Put $B(M) := \{e \in M(\lambda) \mid e \wedge (1-e) = 0\}$. It is clear that the set $B(M)$ is a complete Boolean algebra equipped with the operations:

$$e^* = 1 - e, \qquad c \vee e = c + e - c \cdot e, \qquad c \wedge e = c \cdot e,$$

where $+$, $\cdot$ and $-$ denote for the addition, multiplication and complementation of $B(M)$, for all c, $e \in B(M)$. We have the following definition:

Definition 2.3. The set $B(M)$ is called q'-compact if it is a geometrically equivalence to any of its elementary extensions.

So, with the above contents, we arrive at the following corollary:

Corollary 2.4. *Suppose that B is a Boolean algebra and $\lambda : B \longrightarrow \mathbb{R}$ is a measure. If $B(M)$ is q'-compact, then $B(M)$ is a strongly algebraically closed lattice.*

We recall from [11] that, each finite set $\{x_1, \ldots, x_n\}$ of a vector lattice has the join or $\sup\{x_1, \ldots, x_n\} = x_1 \vee \ldots \vee x_n$, the meet or $\inf\{x_1, \ldots, x_n\} = x_1 \wedge \ldots \wedge x_n$ and each element x of a vector lattice has the modulus $|x| = x \vee (-x)$. A norm on a vector lattice X is said a lattice norm if $|x| \leq |y|$ implies $\| x \| \leq \| y \|$ for all x, $y \in X$. A normed lattice is a vector lattice equipped with a lattice norm. A normed lattice which is complete with respect to the norm is called a Banach lattice. Assume now that Λ is a Banach lattice and X is a lattice normed space over Λ. A Banach space with mixed norm over Λ is a pair $(X, |.|)$ such that $|.|$ is a vector norm on X with values in the Banach lattice Λ.

Suppose X is a Banach lattice. An M-projection is a band projection π in X if $\| x \| = \max\{\| \pi x \|, \| \pi^{\perp} x \|\}$ for all $x \in X$, where $\pi^{\perp} = I_X - \pi$ and I_X is the identity operator on X. The set of all M-projections forms the subalgebra of the Boolean algebra of all band projections in X. A natural generalization of the concept of M-projection is the concept of L^p-projection, $p \geq 1$, introduced by Beherends [2]. A linear projection on X is said an L^p-projection if

$$\| x \|^p = \| \pi x \|^p + \| (I - \pi)x \|^p,$$

for all $x \in X$. An L^1-projection is referred to as L-projection.

Definition 2.5. The set of all L-projections is called q'-compact if it is a geometrically equivalence to any of its elementary extensions.

Theorem 2.6. *Suppose that B is the set of all L-projections. If B is q'-compact, then B is a strongly algebraically closed lattice.*

Proof. We know that the set of all M-projections forms a (generally not complete) Boolean algebra and the set of all L-projections forms a complete Boolean algebra (see [4]). By using ([13, Theorem 3.5]), B is a strongly algebraically closed lattice. $\square$

Assume now that X is a topological space and $U \subseteq X$. We set $U^{\perp} = X \backslash \overline{U}$, the exterior of U. It is said that U is regular if $U^{\perp\perp} = U$. As usual $B(X)$ denote the set of all regular open sets.

Definition 2.7. The inclusion-ordered system of all regular open sets in a topological space is called q'-compact if it is a geometrically equivalence to any of its elementary extensions.

Therefore, we have the following theorem:

Theorem 2.8. *Assume $B(X)$ is the inclusion-ordered system of all regular open sets in a topological space X. If $B(X)$ is q'-compact, then $B(X)$ is a strongly algebraically closed lattice.*

Proof. Note that $B(X)$ is a Boolean algebra with $\neg U = U^{\perp}$, $U \wedge V = U \cap V$, $U \vee V = (U \cup V)^{\perp\perp}$, for all U, $V \in B(X)$. We know that for every topological space X, the inclusion-ordered system of all regular open sets in X is a complete Boolean algebra isomorphic to the Boolean algebra of all bands in the lattice of open sets. By [13], $B(X)$ is a strongly algebraically closed lattice. □

Suppose $\mathcal{B}$ is the set of all metric bands ordered by inclusion. A straightforward verification shows that if every band of the vector lattice contains the norm of some nonzero element of $\mathcal{B}$, then $\mathcal{B}$ is a complete Boolean algebra. Similarly, we define that the set of all metric bands ordered by inclusion is called q'-compact if it is a geometrically equivalence to any of its elementary extensions. So, we have the following corollary:

Corollary 2.9. *Suppose that the set of all metric bands ordered by inclusion $\mathcal{B}$ is q'-compact. If every band of the vector lattice contains the norm of some nonzero element of $\mathcal{B}$, then $\mathcal{B}$ is a strongly algebraically closed lattice.*

Definition 2.10. A weakly distributive B is a complete Boolean algebra if for every sequence $(C_n)_{n\in\omega}$ of countable maximal antichains such that there exists a maximal antichain D with the property that each d in D meets only finitely many elements of each C_n.

Definition 2.11. A non-negative real-valued function μ on B is a submeasure if, for all x, $y \in B$, the following conditions hold:

(1) $\mu(0) = 0$,
(2) $\mu(a \vee b) \leq \mu(a) + \mu(b)$,
(3) $\mu(x) \leq \mu(y)$ if $x \leq y$.

If $a = 0$, then strictly positive submeasure has $\lambda(0) = 0$ and we know that a Maharam submeasure is continuous. Recall that an antichain in Boolean algebra B is a set $A \subseteq B$ such that for distinct elements x, $y \in A$, $x \wedge y = 0$.

Definition 2.12. Let every antichain in A_n be finite and B be a Boolean algebra. Then B satisfies the σ-finite cc if $B = \cup_{n\in\omega} A_n$.

Recall that a complete Boolean algebra B has the G_σ-property if 0 is a G_σ-set in topological space $(B,\, \sigma)$ i.e., if there exist open neighborhoods U_n of 0 such that $\cap U_n = \{0\}$.

Theorem 2.13. *Let B be strongly algebraically closed. If $(B,\, \sigma)$ is a Hausdorff space and B satisfies the G_σ-property, then B carries a strictly positive Maharam submeasure.*

Proof. First we prove that B is weakly distributive. Suppose that $b_0, \ldots, b_n$ is an independent system of regular elements such that $b_0^n \leq b_1^n \leq \cdots$ with condition $\bigvee_k b_k^n = 1$, for $n = 0,\, 1,\, \ldots$. Since B is a Hausdorff space it is sufficient to show that there exists a function $g : \omega \longrightarrow \omega$ such that $b \wedge \vee_n b_{g(n)}^n \neq 0$ and $b \neq 0$. Assume that V is an open neighborhood of b which is not in $cl(V)$. On the other hand, we have $\lim_k (b \wedge b_k^0) = b$ and then there exists some $k = g(0)$ and $c_0 = b \wedge b_k^0 \in V$. We prove the claim by induction $c_= b \wedge b_{g(0)}^0 \wedge \ldots \wedge b_{g(n)}^n \in V$. By assumption $\lim_k (c_n \wedge b_k^{n+1}) = c_n$. Therefore, we can find $k = g(n+1)$ and $b_{n+1} \wedge b_k^{n+1} \in V$. Since 0 is not in $cl(V)$ we have $b \wedge \vee_n b_{g(n)}^n = \lim_n c_n \neq 0$. Now, we know that every complete Boolean algebra with the G_σ-property has the σ-finite cc. Thus B is weakly distributive and the σ-finite cc. By [1], a complete Boolean algebra B carries a strictly positive Maharam submeasure if and only if B is weakly distributive and B has the G_σ-property. It means that B carries a strictly positive Maharam submeasure.

□

Theorem 2.14. *Let B be strongly algebraically closed. If B carries a strictly positive Maharam submeasure, then B is weakly distributive and B satisfies the σ-finite cc.*

Proof. Since B is strongly algebraically closed we have B is a complete Boolean algebra. By [1], a complete Boolean algebra B carries a strictly positive Maharam submeasure if and only if B is weakly distributive and B has the G_σ-property. On the other hand, every complete Boolean algebra with the G_σ-property has the σ-finite cc. This completes the proof. □

Example 2.15. Let $(S,\, o)$ be a group and C be the set of relations on S which are compatible with o. Then $(B,\, \cap,\, \cup,\, \subseteq)$ is a compete Boolean algebra. If B is q'-compact and $(B,\, \sigma)$ is a Hausdorff space and B satisfies the G_σ property, then B carries a strictly positive Maharam submeasure.

Example 2.16. Let B be a complete Boolean algebra carrying a strictly positive σ-additive measure λ. For any $x,\, y \in B$, let

$$\theta(x,\, y) = \lambda(a \triangle b),$$

θ is a metric on B and the topology given by θ coincides with the sequential topology. Then B is weakly distributive and B satisfies the σ-finite cc.

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