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Second Module Cohomology Group of Induced Semigroup Algebras

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ABSTRACT. For a discrete semigroup S and a left multiplier operator T on S, there is a new induced semigroup S_T, related to S and T. In this paper, we show that if T is multiplier and bijective, then the second module cohomology groups $\mathcal{H}^2_{\ell^1(E)}(\ell^1(S),\ell^\infty(S))$ and $\mathcal{H}^2_{\ell^1(E_T)}(\ell^1(S_T),\ell^\infty(S_T))$ are equal, where E and E_T are subsemigroups of idempotent elements in S and S_T, respectively. Finally, we show thet, for every odd $n\in\mathbb{N}$, $\mathcal{H}^2_{\ell^1(E_T)}(\ell^1(S_T),\ell^1(S_T)^{(n)})$ is a Banach space, when S is a commutative inverse semigroup.

1. Introduction

Amini in [1], introduced the concept of module amenability for a class of Banach algebras. He showed that, inverse semigroup S with subsemigroup E of idempotent elements is amenable if and only if semigroup algebra $\ell^1(S)$ is $\ell^1(E)$-module amenable, when $\ell^1(E)$ acts on $\ell^1(S)$ by multiplication from right and trivially from left. Indeed, module actions $\ell^1(E)$ on $\ell^1(S)$ are

$$\delta_e\cdot\delta_s=\delta_s,\qquad \delta_s\cdot\delta_e=\delta_{se},\quad (e\in E, s\in S), \tag{1.1}$$

where δ_s and δ_e are the point masses at $s\in S$ and $e\in E$, respectively.

After that, Amini and Bagha in [2], introduced the concept of weak module amenability and showed that, for every commutative inverse semigroup S with idempodent set E, semigroup algebra $\ell^1(S)$ is always weakly $\ell^1(E)$-module amenable, where module actions $\ell^1(E)$ on $\ell^1(S)$ is

$$\delta_e\cdot\delta_s=\delta_s\cdot\delta_e=\delta_{es},\quad (e\in E, s\in S). \tag{1.2}$$

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Indeed, they studied the first $\ell^1(E)$-module cohomology group of semigroup algebra $\ell^1(S)$ with coefficients in the dual space $(\ell^1(S))^* = \ell^\infty(S)$. Then this sentence has been expanded by second author of the current paper along with Pourabbas. They in [7] and [8], after introducing the concept of module cohomology group for Banach algebras extended this result and showed that the first and second $\ell^1(E)$-module cohomology groups of $\ell^1(S)$ with coefficients in $\ell^1(S)^{(2n-1)}$ $(n \in \mathbb{N})$, are zero and Banach space, respectively, when $\ell^1(S)$ is a Banach $\ell^1(E)$-bimodule with actions (1.2). Also, the second author of the current paper in [6], studied the first and second $\ell^1(E)$-module cohomology group of Clifford semigroup algebra $\ell^1(S)$ with coefficients in it's dual.

Let S be a semigroup and S_T be induced semigroup dependent on left multiplier $T : S \to S$, where E and E_T are sets of idempotent elements in S and S_T, respectively.

In this paper, we will show that if T is multiplier and bijective, then the second $\ell^1(E)$-module cohomology group $\ell^1(S)$ with coefficents in $\ell^\infty(S)$ is eqvalence with the second $\ell^1(E_T)$-module cohomology group $\ell^1(S_T)$ with coefficents in $\ell^\infty(S_T)$, when $\ell^1(S)$ and $\ell^1(S_T)$ are Banach $\ell^1(E)$-bimodule and Banach $\ell^1(E_T)$-bimodule, respectively, with convolution module actions. Indeed, we prove

$$\mathcal{H}^2_{\ell^1(E)}(\ell^1(S), \ell^\infty(S)) \simeq \mathcal{H}^2_{\ell^1(E_T)}(\ell^1(S_T), \ell^\infty(S_T)).$$

2. Preliminary

Let $\mathfrak{A}$ and A be Banach algebras such that A is a $\mathfrak{A}$-bimodule with compatible actions (for more details see, [1, 2, 7, 8] and especially Definition 2.4. of [10]).

Let X be a Banach A-$\mathfrak{A}$-module with compatible actions. If X is a (commutative) Banach A-$\mathfrak{A}$-module, then so is X^* (for more details see, [1, 2, 7, 8]).

In particular, if A is a commutative Banach $\mathfrak{A}$-module, then it is a commutative Banach A-$\mathfrak{A}$-module. In this case, the dual space A^* is also a commutative Banach A-$\mathfrak{A}$-module.

Let $\mathfrak{A}$ and A be Banach algebras such that A is a Banach $\mathfrak{A}$-module and let X be a Banach A-$\mathfrak{A}$-module with compatible actions. An n-$\mathfrak{A}$-module map is a mapping $\phi : A^n \longrightarrow X$ with the following properties;

$$\begin{aligned}\phi(a_1, a_2, \ldots, a_{i-1}, b \pm c, a_{i+1}, \ldots, a_n) &= \phi(a_1, a_2, \ldots, a_{i-1}, b, a_{i+1}, \ldots, a_n) \\ &\pm\ \phi(a_1, a_2, \ldots, a_{i-1}, c, a_{i+1}, \ldots, a_n),\end{aligned} \tag{2.1}$$

$$\phi(\alpha \cdot a_1, a_2, \ldots, a_n) = \alpha \cdot \phi(a_1, a_2, \ldots, a_n), \tag{2.2}$$

$$\phi(a_1, a_2, \ldots, a_n \cdot \alpha) = \phi(a_1, a_2, \ldots, a_n) \cdot \alpha, \tag{2.3}$$

and

$$\phi(a_1, a_2, \ldots, a_{i-1}, a_i \cdot \alpha, a_{i+1}, \ldots, a_n) = \phi(a_1, a_2, \ldots, a_{i-1}, a_i, \alpha \cdot a_{i+1}, \ldots, a_n), \tag{2.4}$$

where $a_1, \ldots, a_n, b, c \in A$ and $\alpha \in \mathfrak{A}$. Note that, ϕ is not necessarily n-linear.

The $\mathfrak{A}$-module complex is

$$0 \longrightarrow X \xrightarrow{\delta^0} \mathcal{C}^1_{\mathfrak{A}}(A,X) \xrightarrow{\delta^1} \mathcal{C}^2_{\mathfrak{A}}(A,X) \xrightarrow{\delta^2} \cdots,$$

where the map $\delta^0 : X \longrightarrow \mathcal{C}^1_{\mathfrak{A}}(A,X)$ is given by $\delta^0(x)(a) = a \cdot x - x \cdot a$ and for $n \in \mathbb{Z}^+$, $\delta^n : \mathcal{C}^n_{\mathfrak{A}}(A,X) \longrightarrow \mathcal{C}^{n+1}_{\mathfrak{A}}(A,X)$ is given by

$$\begin{aligned}[\delta^n T](a_1, \ldots, a_{n+1}) = {} & a_1 \cdot T(a_2, \ldots, a_{n+1}) \\ & + \sum_{i=1}^{n} (-1)^n T(a_1, \ldots, a_i a_{i+1}, \ldots, a_{n+1}) \\ & + (-1)^{n+1} T(a_1, \ldots, a_n) \cdot a_{n+1},\end{aligned} \tag{2.5}$$

where $T \in \mathcal{C}^n_{\mathfrak{A}}(A,X)$ and $a_1, \ldots, a_{n+1} \in A$. It is easy to show that $\delta^{n+1} \circ \delta^n = 0$ for every $n \in \mathbb{Z}^+$. The space $\ker \delta^n$ of all bounded n-$\mathfrak{A}$-module cocycles is denoted by $\mathcal{Z}^n_{\mathfrak{A}}(A,X)$ and the space $\operatorname{Im} \delta^{n-1}$ of all bounded n-$\mathfrak{A}$-module coboundaries is denoted by $B^n_{\mathfrak{A}}(A,X)$. We also recall that $\mathcal{B}^n_{\mathfrak{A}}(A,X)$ is included in $\mathcal{Z}^n_{\mathfrak{A}}(A,X)$ and that the n-th $\mathfrak{A}$-module cohomology group $\mathcal{H}^n_{\mathfrak{A}}(A,X)$ is defined by the quotient

$$\mathcal{H}^n_{\mathfrak{A}}(A,X) = \mathcal{Z}^n_{\mathfrak{A}}(A,X)/\mathcal{B}^n_{\mathfrak{A}}(A,X).$$

The space $\mathcal{Z}^n_{\mathfrak{A}}(A,X)$ is a Banach space, but in general $\mathcal{B}^n_{\mathfrak{A}}(A,X)$ is not closed, we regard $\mathcal{H}^n_{\mathfrak{A}}(A,X)$ as a complete seminormed space with respect to the quotient seminorm. This seminorm is norm if and only if $\mathcal{B}^n_{\mathfrak{A}}(A,X)$ is a closed subspace of $\mathcal{Z}^n_{\mathfrak{A}}(A,X)$, which means that $\mathcal{H}^n_{\mathfrak{A}}(A,X)$ is a Banach space.

3. Induced Semigroup S_T with the Left Multiplier Map T

Let S be a semigroup, the set of all idempotent elements of S is denoted by $E(S) = E = \{e \in S : ee = e\}$. A map $T : S \longrightarrow S$ is called a left (right) multiplier operators on S if $T(st) = T(s)t$ $(T(st) = sT(t))$, for all $s, t \in S$. The class of left (right) multiplier operators on S is denoted by $\mathrm{Mul}_l(S)$ $(\mathrm{Mul}_r(S))$. The map T is called multiplier operator on S if $T \in \mathrm{Mul}_l(S) \cap \mathrm{Mul}_r(S)$. The space of all multiplier operator on S is denoted by $\mathrm{Mul}(S)$. Let $T \in \mathrm{Mul}_l(S)$, we define a new operation "$\circ$" on S by $s \circ t := sT(t)$ for every s and t in S. The semigroup S equipped with the new oparation $\circ$, is denoted by S_T. It's easy to check that S_T is a semigroup which is called induced semigroup dependent on left multiplier T. Let E and E_T are sets of idempotent elements in S and S_T, respectively.

It is worth mentioning that, this idea has started by Birtel in [3] and continued by Larsen in [5]. Also the relation between weak amenability (not weak module amenability) Banach algebra A and induced Banach algebra A_T studied by Laali indicated in [4], where T is a left multiplier on Banach algebra A. This notion developed by some authors, for more details see, [3–5, 9].

Throughout this paper, unless otherwise indicated, we will assume that S is a discrete semigroup, $T \in \mathrm{Mul}(S)$ and T is bijective. We know that the set of point masses $\{\delta_s; s \in S\}$ is dense in $\ell^1(S)$. So, since module actions and module derivations are continuous, we consider points masses as representing elements of semigroup algebras $(\ell^1(S), *)$ and $(\ell^1(S_T), \circledast)$, where $*$ is convolution on $\ell^1(S)$, as follow

$$\delta_s * \delta_t = \delta_{st}, \quad (s, t \in S), \tag{3.1}$$

and $\circledast$ is different convolution on $\ell^1(S_T)$, as follow

$$\begin{aligned} \delta_s \circledast \delta_t &= \delta_{s \circ t} \\ &= \delta_s * \delta_{T(t)} \\ &= \delta_{sT(t)}, \quad (s, t \in S). \end{aligned} \tag{3.2}$$

Lemma 3.1. *Let S be a semigroup and $T : S \to S$ be a bijective map, then*

(i) $T \in \mathrm{Mul}_l(S)$ *if and only if* $T^{-1} \in \mathrm{Mul}_l(S)$.
(ii) *If* $T \in \mathrm{Mul}_l(S)$, *then* $T(E_T) = E$ *and* $T^{-1}(E) = E_T$.
(iii) *If* $T \in \mathrm{Mul}(S)$, *then* $s \circ T(t) = T(s) \circ t$ *and* $s \circ T^{-1}(t) = T^{-1}(s) \circ t$ *for every* $s, t \in S$.

Proof. It is easy to prove and is left to the reader. □

The next examples show that, when T is not bijective or not multiplier, the previous lemma is not necessarily true, therefore, bijective and multiplier conditions are neccessary for T.

Example 3.2. Let $S = \left\{ \begin{bmatrix} x & 0 \\ y & 0 \end{bmatrix}, x, y \in \mathbb{R} \right\}$. S with matrix product is a semigroup and one can verify that, its idempotent set is

$$E = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ y & 0 \end{bmatrix}, y \in \mathbb{R} \right\}.$$

Now let $T : S \longrightarrow S$ be a left multiplier L_a, where $a = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. Indeed,

$$T\left(\begin{bmatrix} x & 0 \\ y & 0 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x & 0 \\ y & 0 \end{bmatrix}$$

$$= \begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix}.$$

Clearly T is not right multiplier and bijective. It easy to show that

$$T(E_T) = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right\} \neq E.$$

Now for every $s = \begin{bmatrix} x & 0 \\ y & 0 \end{bmatrix}, t = \begin{bmatrix} z & 0 \\ p & 0 \end{bmatrix} \in S$, where $y, z \neq 0$, a simple computation shows that $s \circ T(t) \neq T(s) \circ t$.

Example 3.3. Let $S = \left\{ \begin{bmatrix} x & y \\ 0 & z \end{bmatrix}, x, y, z \in \mathbb{R} \right\}$. S with matrix product is a semigroup which its idempotent set is

$$E = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & y \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & y \\ 0 & 1 \end{bmatrix} y \in \mathbb{R} \right\}.$$

Now let $T : S \longrightarrow S$ be a left multiplier L_a, where $a = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. Indeed,

$$T\left(\begin{bmatrix} x & y \\ 0 & z \end{bmatrix} \right) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x & y \\ 0 & z \end{bmatrix} = \begin{bmatrix} x & y+z \\ 0 & z \end{bmatrix}.$$

We know that T is bijective but not right multiplier. It is easy to show that

$$E_T = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & y \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & y \\ 0 & 1 \end{bmatrix} y \in \mathbb{R} \right\},$$

and $T(E_T) = E$, also $T^{-1} : S \longrightarrow S$ is a left multiplier $T^{-1} = L_b$, where $b = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$ and $T^{-1}(E) = E_T$. Now for every $s = \begin{bmatrix} x & y \\ 0 & z \end{bmatrix}, t = \begin{bmatrix} m & n \\ 0 & q \end{bmatrix} \in S$, where $x \neq z$, a simple computation shows that $s \circ T(t) \neq T(s) \circ t$. Similarly can be shown $s \circ T^{-1}(t) \neq T^{-1}(s) \circ t$.

In the following, we show the relationship between S and S_T by giving a simple example.

Example 3.4. The unit disk in $\mathbb{C}$, that is, $S = \{z \in \mathbb{C} : |z| \leq 1\}$ is a compact topological semigroup, under complex multiplication with idempotent elements $E = \{0, 1\}$, Put $T = L_i$ where $L_i(x) = ix, (i = \sqrt{-1})$. It is clear that $T \in \mathrm{Mul}_l(S)$ and $S_T = (S, \circ)$ is a compact topological semigroup with idempotent elements $E_T = \{0, -i\}$ so $\ell^1(E) \neq \ell^1(E_T)$.

4. Second Module Cohomology Group of Induced Semigroup Algebras

In this section, we will show that if T is a multiplier and bijective, then the second $\ell^1(E)$-module cohomology group $\ell^1(S)$ with coefficents in $\ell^\infty(S)$ is equivalent to the second $\ell^1(E_T)$-module cohomology group $\ell^1(S_T)$ with coefficents in $\ell^\infty(S_T)$.

According to (2.5), for $\psi \in \mathcal{Z}^2_{\ell^1(E)}(\ell^1(S), \ell^\infty(S)) = \ker \delta^2$, we have the following relationship

$$\delta_x * \psi(\delta_y, \delta_z) - \psi(\delta_x, \delta_y) * \delta_z = \psi(\delta_x * \delta_y, \delta_z) - \psi(\delta_x, \delta_y * \delta_z). \tag{4.1}$$

Similarly, since $\mathcal{B}^2_{\ell^1(E)}(\ell^1(S), \ell^\infty(S)) = \operatorname{Im} \delta^1$, if $\psi \in \mathcal{B}^2_{\ell^1(E)}(\ell^1(S), \ell^\infty(S))$, then there exists $\phi \in \mathcal{C}^1_{\ell^1(E)}(\ell^1(S), \ell^\infty(S))$, such that

$$\delta_x * \phi(\delta_y) - \phi(\delta_x * \delta_y) + \phi(\delta_x) * \delta_y = \psi(\delta_x * \delta_y). \tag{4.2}$$

Theorem 4.1. *Let S be a semigroup and $T : S \to S$ be a multiplier and bijective map. Then*

$$\mathcal{H}^2_{\ell^1(E)}(\ell^1(S), \ell^\infty(S)) \simeq \mathcal{H}^2_{\ell^1(E_T)}(\ell^1(S_T), \ell^\infty(S_T)).$$

Proof. Consider the map

$$\begin{aligned} \Gamma : \mathcal{Z}^2_{\ell^1(E)}(\ell^1(S), \ell^\infty(S)) &\longrightarrow \mathcal{H}^2_{\ell^1(E_T)}(\ell^1(S_T), \ell^\infty(S_T)) \\ \psi &\longrightarrow \widetilde{\psi} + \mathcal{B}^2_{\ell^1(E_T)}(\ell^1(S_T), \ell^\infty(S_T)), \end{aligned}$$

where

$$\begin{aligned} \widetilde{\psi} &: \ell^1(S_T) \times \ell^1(S_T) \longrightarrow \ell^\infty(S_T) \\ \widetilde{\psi}(\delta_x, \delta_y) &= \psi((\delta_{T(x)}, \delta_{T(y)}). \end{aligned}$$

In the first, we show that Γ is well define. For this purpose we prove in the first step, when ψ is a 2-$\ell^1(E)$-module cocycle then $\widetilde{\psi}$ is a 2-$\ell^1(E_T)$-module cocycle. It is easy to show that (2.1) is confirmed.

Let $e \in E_T$ and $x, y \in S_T$, since $T \in \mathrm{Mul}_l(S)$ and $T(e) \in E$, by (3.2) and Lemma 3.1, we have

$$\begin{aligned} \left[\delta_e \circledast \widetilde{\psi}(\delta_x, \delta_y)\right](\delta_t) &= \widetilde{\psi}(\delta_x, \delta_y)(\delta_t \circledast \delta_e) \\ &= \widetilde{\psi}(\delta_x, \delta_y)(\delta_{t \circ e}) \\ &= \psi(\delta_{T(x)}, \delta_{T(y)})(\delta_{tT(e)}) \\ &= \psi(\delta_{T(x)}, \delta_{T(y)})(\delta_t * \delta_{T(e)}) \\ &= (\delta_{T(e)} * \psi(\delta_{T(x)}, \delta_{T(y)}))(\delta_t) \\ &= \psi(\delta_{T(e)} * \delta_{T(x)}, \delta_{T(y)})(\delta_t) \\ &= \psi(\delta_{T(e)T(x)}, \delta_{T(y)})(\delta_t) \end{aligned}$$

$$
\begin{aligned}
&= \psi(\delta_{T(eT(x))}, \delta_{T(y)})(\delta_t) \\
&= \widetilde{\psi}(\delta_{eT(x)}, \delta_y)(\delta_t) \\
&= \widetilde{\psi}(\delta_e \circledast \delta_x, \delta_y)(\delta_t),
\end{aligned}
$$

which shows

$$\widetilde{\psi}(\delta_e \circledast \delta_x, \delta_y) = \delta_e \circledast \widetilde{\psi}(\delta_x, \delta_y). \tag{4.3}$$

For the other equation, since $T \in \mathrm{Mul}_r(S)$, we have

$$
\begin{aligned}
\widetilde{\psi}(\delta_x \circledast \delta_e, \delta_y) &= \widetilde{\psi}(\delta_{xT(e)}, \delta_y) \\
&= \psi(\delta_{T(xT(e))}, \delta_{T(y)}) \\
&= \psi(\delta_{T(x)T(e)}, \delta_{T(y)}) \\
&= \psi(\delta_{T(x)} * \delta_{T(e)}, \delta_{T(y)}) \\
&= \psi(\delta_{T(x)}, \delta_{T(e)} * \delta_{T(y)}) \\
&= \psi(\delta_{T(x)}, \delta_{T(e)T(y)}) \\
&= \psi(\delta_{T(x)}, \delta_{T(eT(y))}) \\
&= \widetilde{\psi}(\delta_x, \delta_{eT(y)}) \\
&= \widetilde{\psi}(\delta_x, \delta_e \circledast \delta_y),
\end{aligned}
$$

this shows

$$\widetilde{\psi}(\delta_x \circledast \delta_e, \delta_y) = \widetilde{\psi}(\delta_x, \delta_e \circledast \delta_y). \tag{4.4}$$

Similarly, we can show that

$$\widetilde{\psi}(\delta_x, \delta_y \circledast \delta_e) = \widetilde{\psi}(\delta_x, \delta_y) \circledast \delta_e. \tag{4.5}$$

Now let $x, y, z, t \in S$ and $\Delta = \left[\delta_x \circledast \widetilde{\psi}(\delta_y, \delta_z) - \widetilde{\psi}(\delta_x, \delta_y) \circledast \delta_z\right]$, by (3.2),(4.1) and Lemma 3.1, we have

$$
\begin{aligned}
\Delta(\delta_t) &= \left[\delta_x \circledast \widetilde{\psi}(\delta_y, \delta_z)\right](\delta_t) - \left[\widetilde{\psi}(\delta_x, \delta_y) \circledast \delta_z\right](\delta_t) \\
&= \widetilde{\psi}(\delta_y, \delta_z)(\delta_t \circledast \delta_x) - \widetilde{\psi}(\delta_x, \delta_y)(\delta_z \circledast \delta_t) \\
&= \psi(\delta_{T(y)}, \delta_{T(z)})(\delta_{tT(x)}) - \psi(\delta_{T(x)}, \delta_{T(y)})(\delta_{T(z)t}) \\
&= \psi(\delta_{T(y)}, \delta_{T(z)})(\delta_t * \delta_{T(x)}) - \psi(\delta_{T(x)}, \delta_{T(y)})(\delta_{T(z)} * \delta_t) \\
&= \left[\delta_{T(x)} * \psi(\delta_{T(y)}, \delta_{T(z)})\right](\delta_t) - \left[\psi(\delta_{T(x)}, \delta_{T(y)}) * \delta_{T(z)}\right](\delta_t) \\
&= \psi(\delta_{T(x)} * \delta_{T(y)}, \delta_{T(z)})(\delta_t) - \psi(\delta_{T(x)}, \delta_{T(y)} * \delta_{T(z)})(\delta_t) \\
&= \psi(\delta_{T(x)T(y)}, \delta_{T(z)})(\delta_t) - \psi(\delta_{T(x)}, \delta_{T(y)T(z)})(\delta_t) \\
&= \psi(\delta_{T(xT(y))}, \delta_{T(z)})(\delta_t) - \psi(\delta_{T(x)}, \delta_{T(yT(z))})(\delta_t) \\
&= \widetilde{\psi}(\delta_{xT(y)}, \delta_z)(\delta_t) - \widetilde{\psi}(\delta_x, \delta_{yT(z)})(\delta_t)
\end{aligned}
$$

$$
\begin{aligned}
&= \widetilde{\psi}(\delta_{x\circ y}, \delta_z)(\delta_t) - \widetilde{\psi}(\delta_x, \delta_{y\circ z})(\delta_t) \\
&= \widetilde{\psi}(\delta_x \circledast \delta_y, \delta_z)(\delta_t) - \widetilde{\psi}(\delta_x, \delta_y \circledast \delta_z)(\delta_t) \\
&= \left[\widetilde{\psi}(\delta_x \circledast \delta_y, \delta_z) - \widetilde{\psi}(\delta_x, \delta_y \circledast \delta_z)\right](\delta_t).
\end{aligned}
$$

Therefore, $\delta_x \circledast \widetilde{\psi}(\delta_y, \delta_z) - \widetilde{\psi}(\delta_x, \delta_y) \circledast \delta_z = \widetilde{\psi}(\delta_x \circledast \delta_y, \delta_z) - \widetilde{\psi}(\delta_x, \delta_y \circledast \delta_z)$, and so $\widetilde{\psi} \in \mathcal{Z}^2_{\ell^1(E_T)}(\ell^1(S_T), \ell^\infty(S_T))$ and Γ is well define. Clearly Γ is linear.

For the surjectivity of Γ, let $P \in \mathcal{Z}^2_{\ell^1(E_T)}(\ell^1(S_T), \ell^\infty(S_T))$. Define

$$
\begin{aligned}
&\psi : \ell^1(S) \times \ell^1(S) \longrightarrow \ell^\infty(S), \\
&\psi(\delta_x, \delta_y) := P(\delta_{T^{-1}(x)}, \delta_{T^{-1}(y)}).
\end{aligned}
$$

It is easy to show that ψ is two admissible. Let $e \in E$ and $x, y, z, t \in S$, since $T \in \mathrm{Mul}_l(S)$, by (3.2), (4.3) and lemma 3.1, we have

$$
\begin{aligned}
[\delta_e * \psi(\delta_x, \delta_y)](\delta_t) &= \psi(\delta_x, \delta_y)(\delta_t * \delta_e) \\
&= \psi(\delta_x, \delta_y)(\delta_{te}) \\
&= P(\delta_{T^{-1}(x)}, \delta_{T^{-1}(y)})(\delta_{t\circ T^{-1}(e)}) \\
&= P(\delta_{T^{-1}(x)}, \delta_{T^{-1}(y)})(\delta_t \circledast \delta_{T^{-1}(e)}) \\
&= (\delta_{T^{-1}(e)} \circledast P(\delta_{T^{-1}(x)}, \delta_{T^{-1}(y)})(\delta_t) \\
&= P(\delta_{T^{-1}(e)} \circledast \delta_{T^{-1}(x)}, \delta_{T^{-1}(y)})(\delta_t) \\
&= P(\delta_{T^{-1}(e)x}, \delta_{T^{-1}(y)})(\delta_t) \\
&= \psi(\delta_{T(T^{-1}(e)x), \delta_y})(\delta_t) \\
&= \psi(\delta_{ex}, \delta_y)(\delta_t) \\
&= \psi(\delta_e * \delta_x, \delta_y)(\delta_t),
\end{aligned}
$$

which shows

$$
\psi(\delta_e * \delta_x, \delta_y) = \delta_e * \psi(\delta_x, \delta_y).
$$

For the other equation, since $T^{-1} \in \mathrm{Mul}_l(S)$, by (4.4)

$$
\begin{aligned}
\psi(\delta_x * \delta_e, \delta_y) &= \psi(\delta_{xe}, \delta_y) \\
&= P(\delta_{T^{-1}(xe)}, \delta_{T^{-1}(y)}) \\
&= P(\delta_{T^{-1}(x)e}, \delta_{T^{-1}(y)}) \\
&= P(\delta_{T^{-1}(x)} \circledast \delta_{T^{-1}(e)}, \delta_{T^{-1}(y)}) \\
&= P(\delta_{T^{-1}(x)}, \delta_{T^{-1}(e)} \circledast \delta_{T^{-1}(y)}) \\
&= P(\delta_{T^{-1}(x)}, \delta_{T^{-1}(e)\circ T^{-1}(y)}) \\
&= P(\delta_{T^{-1}(x)}, \delta_{T^{-1}(e)y}) \\
&= \psi(\delta_x, \delta_{ey})
\end{aligned}
$$

$$= \psi(\delta_x, \delta_e * \delta_y),$$

so obtained

$$\psi(\delta_x * \delta_e, \delta_y) = \psi(\delta_x, \delta_e * \delta_y).$$

Similarly, by (4.5) we can show that

$$\psi(\delta_x, \delta_y \circledast \delta_e) = \psi(\delta_x, \delta_y) \circledast \delta_e.$$

Let $e \in E$ and $x, y, z, t \in S$ and $\Theta = [\delta_x * \psi(\delta_y, \delta_z) - \psi(\delta_x, \delta_y) * \delta_z]$, since $T^{-1} \in \mathrm{Mul}_l(S)$, by (3.2), (4.1) and Lemma 3.1 we have

$$\begin{aligned}
\Theta(\delta_t) &= [\delta_x * \psi(\delta_y, \delta_z)](\delta_t) - [\psi(\delta_x, \delta_y) * \delta_z](\delta_t) \\
&= \psi(\delta_y, \delta_z)(\delta_t * \delta_x) - \psi(\delta_x, \delta_y)(\delta_z * \delta_t) \\
&= \psi(\delta_y, \delta_z)(\delta_{tx}) - \psi(\delta_x, \delta_y)(\delta_{zt}) \\
&= P(\delta_{T^{-1}(y)}, \delta_{T^{-1}(z)})(\delta_t \circledast \delta_{T^{-1}(x)}) \\
&\quad - P(\delta_{T^{-1}(x)}, \delta_{T^{-1}(y)})(\delta_{T^{-1}(z)} \circledast \delta_t) \\
&= (\delta_{T^{-1}(x)} \circledast P(\delta_{T^{-1}(y)}, \delta_{T^{-1}(z)}))(\delta_t) \\
&\quad - (P(\delta_{T^{-1}(x)}, \delta_{T^{-1}(y)}) \circledast \delta_{T^{-1}(z)})(\delta_t) \\
&= P(\delta_{T^{-1}(x)} \circledast \delta_{T^{-1}(y)}, \delta_{T^{-1}(z)})(\delta_t) \\
&\quad - P(\delta_{T^{-1}(x)}, \delta_{T^{-1}(y)} \circledast \delta_{T^{-1}(z)})(\delta_t) \\
&= P(\delta_{T^{-1}(x)y}, \delta_{T^{-1}(z)})(\delta_t) - P(\delta_{T^{-1}(x)}, \delta_{T^{-1}(y)z})(\delta_t) \\
&= \psi(\delta_{xy}, \delta_z)(\delta_t) - \psi(\delta_x, \delta_{yz})(\delta_t) \\
&= [\psi(\delta_x * \delta_y, \delta_z) - \psi(\delta_x, \delta_y * \delta_z)](\delta_t).
\end{aligned}$$

This shows that

$$\delta_x * \psi(\delta_y, \delta_z) - \psi(\delta_x, \delta_y) * \delta_z = \psi(\delta_x * \delta_y, \delta_z) - \psi(\delta_x, \delta_y * \delta_z),$$

so $\psi \in \mathcal{Z}^2_{\ell^1(E)}(\ell^1(S), \ell^\infty(S))$ and $\Gamma(\psi) = P + \mathcal{B}^2_{\ell^1(E_T)}(\ell^1(S_T), \ell^\infty(S_T))$.

Finally, we prove that $\ker \Gamma = \mathcal{B}^2_{\ell^1(E)}(\ell^1(S), \ell^\infty(S))$, or equivalently $\psi \in \mathcal{B}^2_{\ell^1(E)}(\ell^1(S), \ell^\infty(S))$ if and only if $\widetilde{\psi} \in \mathcal{B}^2_{\ell^1(E_T)}(\ell^1(S_T), \ell^\infty(S_T))$. To prove this, we first assume that $\psi \in \mathcal{B}^2_{\ell^1(E)}(\ell^1(S), \ell^\infty(S))$ then there exists $\phi \in \mathcal{C}^1_{\ell^1(E)}(\ell^1(S), \ell^\infty(S))$ such that $\delta^1(\phi) = \psi$, and by (4.2),

$$\psi(\delta_x, \delta_y) = \delta_x * \phi(\delta_y) - \phi(\delta_x * \delta_y) - \phi(\delta_x) * \delta_y.$$

Now let $\Delta = \widetilde{\psi}(\delta_{T^{-1}(x)}, \delta_{T^{-1}(y)})$, by by using the relation between ψ and $\widetilde{\psi}$, we have

$$\begin{aligned}
\Delta(\delta_t) &= [\psi(\delta_x, \delta_y)](\delta_t) \\
&= [\delta_x * \phi(\delta_y) - \phi(\delta_x * \delta_y) - \phi(\delta_x) * \delta_y](\delta_t) \\
&= (\delta_x * \phi(\delta_y))(\delta_t) - (\phi(\delta_x * \delta_y))(\delta_t) + (\phi(\delta_x) * \delta_y)(\delta_t)
\end{aligned}$$

$$
\begin{aligned}
&= \phi(\delta_y)(\delta_t * \delta_x) - (\phi(\delta_x * \delta_y))(\delta_t) + \phi(\delta_x)(\delta_y * \delta_t) \\
&= \phi(\delta_y)(\delta_{tx}) - (\phi(\delta_x * \delta_y))(\delta_t) + \phi(\delta_x)(\delta_{yt}), \\
&= \widetilde{\phi}(\delta_{T^{-1}(y)})(\delta_t \circledast \delta_{T^{-1}(x)}) - \widetilde{\phi}(\delta_{T^{-1}(x)} \circledast \delta_{T^{-1}(y)})(\delta_t) \\
&\quad + \widetilde{\phi}(\delta_{T^{-1}(x)})(\delta_{T^{-1}(y)} \circledast \delta_t) \\
&= (\delta_{T^{-1}(x)} \circledast \widetilde{\phi}(\delta_{T^{-1}(y)})(\delta_t) - \widetilde{\phi}(\delta_{T^{-1}(x)} \circledast \delta_{T^{-1}(y)})(\delta_t) \\
&\quad + (\widetilde{\phi}(\delta_{T^{-1}(x)}) \circledast \delta_{T^{-1}(y)})(\delta_t) \\
&= [(\delta_{T^{-1}(x)} \circledast \widetilde{\phi}(\delta_{T^{-1}(y)}) - \widetilde{\phi}(\delta_{T^{-1}(x)} \circledast \delta_{T^{-1}(y)}) \\
&\quad + (\widetilde{\phi}(\delta_{T^{-1}(x)}) \circledast \delta_{T^{-1}(y)})](\delta_t),
\end{aligned}
$$

that shows $\widetilde{\psi} = \delta^1(\widetilde{\phi})$ and so $\widetilde{\psi} \in \mathcal{B}^2_{\ell^1(E_T)}(\ell^1(S_T), \ell^\infty(S_T))$. Similarly, we can show that ψ is a 2-$\ell^1(E)$-module coboundary if $\widetilde{\psi}$ is a 2-$\ell^1(E_T)$-module coboundary. □

5. Second Module Cohomology Group of Induced Inverse Semigroup Algebras

A discrete semigroup S is call an inverse semigroup if for each $a \in S$ there is a unique element $a^* \in S$ such that $a = a \cdot a^* \cdot a$ and $a^* = a^* \cdot a \cdot a^*$. In this section, we show that if S is a commutative inverse semigroup, then $\mathcal{H}^2_{\ell^1(E_T)}(\ell^1(S_T), \ell^1(S_T)^{(n)})$ is a Banach space for every odd $n \in \mathbb{N}$.

Lemma 5.1. *Let S be a semigroup and $T \in \mathrm{Mul}_l(S)$ be bijective, then S is a commutative semigroup if and only if S_T is a commutative semigroup.*

Proof. It is easy to prove and is left to the reader. □

Lemma 5.2. *Let S be a semigroup and $T \in \mathrm{Mul}_l(S)$ be bijective, then S is an inverse semigroup if and only if S_T is an inverse semigroup.*

Proof. Let $(S, \cdot)$ be an inverse semigroup and $a \in S$. Suppose that $a^* \in S$ is the unique element of S such that $a = a \cdot a^* \cdot a$ and $a^* = a^* \cdot a \cdot a^*$. We define $a^\star := T^{-1}(b^*)$, where $b = T(a)$, we have

$$
\begin{aligned}
a \circ a^\star \circ a &= a \cdot T(a^\star) \cdot T(a) \\
&= T^{-1}(b) \cdot b^* \cdot b \\
&= T^{-1}(b \cdot b^* \cdot b) \\
&= T^{-1}(b) \\
&= a.
\end{aligned}
$$

Similarly we can show that $a^\star \circ a \circ a^\star = a^\star$. Therefore, $(S_T, \star)$ is an inverse semigroup. Similarly, we can show that S is an inverse semigroup when S_T is an inverse semigroup. □

Theorem 5.3 ([8, Thorem 2.3]). *Let S be a semigroup and $T \in \mathrm{Mul}_l(S)$ be bijective. Then for every $n \in \mathbb{N}$, $\mathcal{H}^2_{\ell^1(E)}(\ell^1(S), X^*)$ is a Banach space, where $X = (\ell^1(S))^{(2n)}$.*

Theorem 5.4. *Let S be a commutative inverse semigroup. Then for every odd $n \in \mathbb{N}$, $\mathcal{H}^2_{\ell^1(E_T)}\left(\ell^1(S_T), \ell^1(S_T)^{(n)}\right)$ is a Banach space.*

Proof. Let S be a commutative inverse semigroup, by Lemmas 5.1 and 5.2, S_T is a commutative inverse semigroup, now by Theorem 5.3

$$\mathcal{H}^2_{\ell^1(E_T)}\left(\ell^1(S_T), \ell^1(S_T)^{(n)}\right),$$

is a Banach space for every odd $n \in \mathbb{N}$. □

References

1. M. Amini, *Module amenability fore semigroup algebra*, Semigroup Forum., 69 (2004), pp. 243-254.
2. M. Amini and D.E. Bagha, *Weak module amenability fore semigroup algebra*, Semigroup Forum., 71 (2005), pp. 18-26.
3. F.T. Birtel, *Banach algebra of multiplier*, Duke Math. J, 28 (1961), pp. 203-211.
4. J. Laali, *The multipliers related products in Banach algebras*, Quaestion Mathematicae., 37 (2014), pp. 1-17.
5. R. Larsen, *An introduction to the theory of multipliers*, Springer-verlag, New York., (1971).
6. E. Nasrabadi, *First and second module cohomology groups for non commutative semigroup algebras*, Sahand Commun. Math. Anal., 17 (2020), pp. 39-47.
7. E. Nasrabadi and A. Pourabbas, *Module cohomology group of inverse semigroup algebra*, Bulletin of Iranian Mathematical Society., 37 no. 4 (2011), pp. 157-169.
8. E. Nasrabadi and A. Pourabbas, *Second module cohomology group of inverse semigroup algebra*, Semigroup Fourm., 81 no. 1 (2010), pp. 269-278.
9. A.L. Paterson, *Amenability*, American Mathematical Society, (1988).
10. M.H. Sattari and H. Shafieasl, *Symmetric module and Connes amenability*, Sahand Commun. Math. Anal., 5 (2017), pp. 49-59.

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