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Ali Reza Janfada and Javad Farokhi-Ostad

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Two Equal Range Operators on Hilbert C^*-modules

Ali Reza Janfada[1] and Javad Farokhi-Ostad[2,*]

Abstract. In this paper, number of properties, involving invertibility, existence of Moore-Penrose inverse and etc for modular operators with the same ranges on Hilbert C^*-modules are presented. Natural decompositions of operators with closed range enable us to find some relations of the product of operators with Moore-Penrose inverses under the condition that they have the same ranges in Hilbert C^*-modules. The triple reverse order law and the mixed reverse order law in the special cases are also given. Moreover, the range property and Moore-Penrose inverse are illustrated.

1. Introduction and Preliminaries

Let $\mathcal{X}, \mathcal{Y}$ be two Hilbert $\mathcal{A}$-modules and $T : \mathcal{X} \to \mathcal{Y}$ be an operator. The general inverse of T is an operator $S : \mathcal{Y} \to \mathcal{X}$ such that $TST = T$ and $STS = S$. A category of generalized inversions is called Moore-Penrose inverse, precisely described in Definition 1.2, and denoted by †. Xu and Sheng [22] showed that a bounded adjointable operator between two Hilbert $\mathcal{A}$-modules admits a bounded Moore-Penrose inverse if and only if it has closed range. Investigation of the closedness of range of an operator and Moore-Penrose inverse are important in operator theory. Although, in general there is no relation between $(TS)^\dagger$ with $T^\dagger$ and $S^\dagger$ except in some especial cases. If the equivalence $(TS)^\dagger = S^\dagger T^\dagger$ is satisfied, we say the reverse order law holds. This problem, which is expressed as "when the reverse order law hold?" was first studied by Bouldin and Izumino for bounded operators between Hilbert spaces, see [1, 2, 7]. If the relationship $(TSU)^\dagger = U^\dagger S^\dagger T^\dagger$, is established we say that the triple reverse order law holds. If $T(SU)^\dagger V = TU^\dagger S^\dagger V$ we call it

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* Corresponding author.

mixed reverse order law. Recently, many authors such as Sharifi [18] and Mohammadzadeh Karizaki [10, 12, 13] studied Moore -Penrose inverse of product of the operators with closed range in Hilbert C^*-modules and discussed about various cases of these laws.

Furthermore, there is no relation between $(STT^*)^\dagger$ and $T^\dagger$, $(T^*)^\dagger$ and $S^\dagger$ except in some especial cases. In this paper, by using some block operator matrix techniques, we obtain some relation between $(STT^*)^\dagger$ with $T^\dagger$, $(T^*)^\dagger$ and $S^\dagger$ when $\mathrm{ran(T^*)} = \mathrm{ran(STT^*)}$ and $\mathrm{ran(T)} = \mathrm{ran(TT^*S^*)}$.

This paper is organized as follows. In the remainder of this section, some preliminaries are given, which are used in the following sections. Section 2 describes the triple reverse order law and the mixed reverse order law in the special cases and so discusses about the range property and Moore-Penrose inverse and invertibility of some modular operators.

Throughout the paper $\mathcal{A}$ is a C*-algebra (not necessarily unital). A (right) pre-Hilbert module over a C^*-algebra $\mathcal{A}$ is a complex linear space $\mathcal{X}$, which is an algebraic right $\mathcal{A}$-module equipped with an $\mathcal{A}$-valued inner product $\langle .,. \rangle : \mathcal{X} \times \mathcal{X} \to \mathcal{A}$ satisfying,

(i) $\langle x, x\rangle \geq 0$, and $\langle x, x\rangle = 0$ iff $x = 0$,
(ii) $\langle x, y + \lambda z\rangle = \langle x, y\rangle + \lambda\langle x, z\rangle$,
(iii) $\langle x, ya\rangle = \langle x, y\rangle a$,
(iv) $\langle y, x\rangle = \langle x, y\rangle^*$,

for all $x, y, z \in \mathcal{X}$, $\lambda \in \mathbb{C}$, $a \in \mathcal{A}$. A pre-Hilbert $\mathcal{A}$-module $\mathcal{X}$ is called a Hilbert $\mathcal{A}$-module if it is complete with respect to the norm $\|x\| = \|\langle x, x\rangle\|^{\frac{1}{2}}$. Left Hilbert $\mathcal{A}$-modules are defined in a similar way. For example, every C^*-algebra $\mathcal{A}$ is a left Hilbert $\mathcal{A}$-module with respect to the inner product $\langle x, y\rangle = x^*y$, and every inner product space is a left Hilbert $\mathbb{C}$-module.

Suppose that $\mathcal{X}$ and $\mathcal{Y}$ are Hilbert $\mathcal{A}$-modules. Then $\mathcal{L}(\mathcal{X}, \mathcal{Y})$ is the set of all maps $T : \mathcal{X} \to \mathcal{Y}$ for which there is a map $T^* : \mathcal{Y} \to \mathcal{X}$ such that $\langle Tx, y\rangle = \langle x, T^*y\rangle$ for all $x \in \mathcal{X}$, $y \in \mathcal{Y}$. It is known that any element T of $\mathcal{L}(\mathcal{X}, \mathcal{Y})$ must be a bounded linear operator, which is also $\mathcal{A}$-linear in the sense that $T(xa) = (Tx)a$ for all $x \in \mathcal{X}$ and $a \in \mathcal{A}$ [9, Page 8]. We use the notations $\mathcal{L}(\mathcal{X})$ in place of $\mathcal{L}(\mathcal{X}, \mathcal{X})$, and $\ker(\cdot)$ and $\mathrm{ran}(\cdot)$ for the kernel and the range of operators, respectively.

Suppose that $\mathcal{X}$ is a Hilbert $\mathcal{A}$-module and $\mathcal{Y}$ is a closed submodule of $\mathcal{X}$. We say that $\mathcal{Y}$ is orthogonally complemented if $\mathcal{X} = \mathcal{Y} \oplus \mathcal{Y}^\perp$, where $\mathcal{Y}^\perp := \{y \in \mathcal{X} : \ \langle x, y\rangle = 0 \ \text{ for all } x \in \mathcal{Y}\}$ denotes the orthogonal complement of $\mathcal{Y}$ in $\mathcal{X}$. The reader is referred to [9] for more details.

Recall that a closed submodule in a Hilbert module is not necessarily orthogonally complemented; however, Lance [9] proved that certain submodules are orthogonally complemented as follows.

Theorem 1.1 ([9, Theorem 3.2])**.** *Let $\mathcal{X}$, $\mathcal{Y}$ be Hilbert $\mathcal{A}$-modules and $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ have closed range. Then*

(i) $\mathrm{ker(T)}$ *is orthogonally complemented in $\mathcal{X}$, with complement* $\mathrm{ran(T^*)}$.
(ii) $\mathrm{ran(T)}$ *is orthogonally complemented in $\mathcal{Y}$, with complement* $\mathrm{ker(T^*)}$.
(iii) *The map $T^* \in \mathcal{L}(\mathcal{Y}, \mathcal{X})$ has closed range.*

Definition 1.2. Let $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$. The Moore-Penrose inverse $T^\dagger$ of T is an element in $\mathcal{L}(\mathcal{Y}, \mathcal{X})$ which satisfies:

(a) $T\,T^\dagger T = T$,
(b) $T^\dagger\, T\, T^\dagger = T^\dagger$,
(c) $(T\,T^\dagger)^* = T\,T^\dagger$,
(d) $(T^\dagger\, T)^* = T^\dagger T$.

Motivated by these conditions $T^\dagger$ is unique and $T^\dagger T$ and $T\,T^\dagger$ are orthogonal projections, in the sense that they are selfadjoint idempotent operators. Clearly, T is Moore-Penrose invertible if and only if T^* is Moore-Penrose invertible, and in this case $(T^*)^\dagger = (T^\dagger)^*$. The following theorem is known.

Theorem 1.3 ([22, Theorem 2.2])**.** *Suppose that $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$. Then the Moore-Penrose inverse $T^\dagger$ of T exists if and only if T has closed range.*

By Definition 1.2, we have

$$\begin{array}{ll} \mathrm{ran(T) = ran(T\,T^\dagger)}, & \mathrm{ran(T^\dagger) = ran(T^\dagger T) = ran(T^*)}, \\ \mathrm{ker(T) = ker(T^\dagger T)}, & \mathrm{ker(T^\dagger) = ker(T\,T^\dagger) = ker(T^*)}, \end{array}$$

and by Theorem 1.1, we have

$$\mathcal{X} = \mathrm{ker(T)} \oplus \mathrm{ran(T^\dagger)} = \mathrm{ker(T^\dagger T)} \oplus \mathrm{ran(T^\dagger T)},$$

$$\mathcal{Y} = \mathrm{ker(T^\dagger)} \oplus \mathrm{ran(T)} = \mathrm{ker(T\,T^\dagger)} \oplus \mathrm{ran(T\,T^\dagger)}.$$

A matrix form of a bounded adjointable operator $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ can be induced by some natural decompositions of Hilbert C^*-modules. Indeed, if $\mathcal{M}$ and $\mathcal{N}$ are closed orthogonally complemented submodules of $\mathcal{X}$ and $\mathcal{Y}$, respectively, and $\mathcal{X} = \mathcal{M} \oplus \mathcal{M}^\perp$, $\mathcal{Y} = \mathcal{N} \oplus \mathcal{N}^\perp$, then T can be written as the following 2×2 matrix

$$T = \begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix} \tag{1.1}$$

where, $T_1 \in \mathcal{L}(\mathcal{M}, \mathcal{N})$, $T_2 \in \mathcal{L}(\mathcal{M}^\perp, \mathcal{N})$, $T_3 \in \mathcal{L}(\mathcal{M}, \mathcal{N}^\perp)$ and $T_4 \in \mathcal{L}(\mathcal{M}^\perp, \mathcal{N}^\perp)$. Note that $P_{\mathcal{M}}$ denotes the projection corresponding to $\mathcal{M}$.

In fact T_1 is the restriction of $P_{\mathcal{N}}TP_{\mathcal{M}}$ on M, or briefly $T_1 = P_{\mathcal{N}}TP_{\mathcal{M}}$, and semilarly,

$$T_2 = P_{\mathcal{N}}T(1-P_{\mathcal{M}}), \quad T_3 = (1-P_{\mathcal{N}})TP_{\mathcal{M}}, \quad T_4 = (1-P_{\mathcal{N}})T(1-P_{\mathcal{M}}).$$

The interested reader, can be referred for more details to [13].

Recall that if $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ has closed range, then $TT^\dagger = P_{\mathrm{ran(T)}}$ and $T^\dagger T = P_{\mathrm{ran(T^*)}}$.

Lemma 1.4 ([13, Corollary 1.2]). *Suppose that $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ has closed range. Then T has the following matrix decomposition with respect to the orthogonal decompositions of closed submodules $\mathcal{X} = \mathrm{ran(T^*)} \oplus \mathrm{ker(T)}$ and $\mathcal{Y} = \mathrm{ran(T)} \oplus \mathrm{ker(T^*)}$:*

$$T = \begin{bmatrix} T_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathrm{ran(T^*)} \\ \mathrm{ker\,ker(T)} \end{bmatrix} \mapsto \begin{bmatrix} \mathrm{ran(T)} \\ \mathrm{ker(T^*)} \end{bmatrix},$$

where T_1 is invertible. Moreover

$$T^\dagger = \begin{bmatrix} T_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathrm{ran(T)} \\ \mathrm{ker(T^*)} \end{bmatrix} \mapsto \begin{bmatrix} \mathrm{ran(T^*)} \\ \mathrm{ker(T)} \end{bmatrix}.$$

Definition 1.5 ([17, Definition 2.1]). Let $\mathcal{X}$ be a Hilbert $\mathcal{A}$-modules. An operator $T \in \mathcal{L}(\mathcal{X})$ is called EP if ran(T) and ran(T*) have the same closure.

Obviously, by this definition we have $TT^\dagger = T^\dagger T$, whenever $T^\dagger$ exists. For further information, the interested reader is referred to [5], [6] and the references therein. Some characterizations and related results of EP operator can be found in [8, 11]

2. Main Results

In this section, we obtain valuable results about invertibility of operators, for which, we use the matrix form of operators and special techniques. In the next discussion we shall use the following result.

Lemma 2.1 ([14, Corollary 2.4]). *Suppose that $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ has closed range. Then $(TT^*)^\dagger = (T^*)^\dagger T^\dagger$.*

In the following theorem, we state the triple reverse order law for especial operators.

Theorem 2.2. *Suppose $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{Z}$ are Hilbert $\mathcal{A}$-modules, the operators $S \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$, $T \in \mathcal{L}(\mathcal{Y}, \mathcal{Z})$ have closed ranges and $(\mathrm{ker(T^*T)})^\perp = \mathrm{ran(S)}$. Then*

$$(T^*TS)^\dagger = S^\dagger T^\dagger (T^*)^\dagger.$$

Proof. By Theorem 3.2 in [12] and previous Lemma, the desired result follows. $\square$

Theorem 2.3. *Suppose that $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ and $S \in \mathcal{L}(\mathcal{Y}, \mathcal{X})$ have closed ranges and* $\mathrm{ran}(\mathrm{T}^*) = \mathrm{ran}(\mathrm{STT}^*)$ *and* $\mathrm{ran}(\mathrm{T}) = \mathrm{ran}(\mathrm{TT}^*\mathrm{S}^*)$. *Then*

(i) *The restriction of $TT^*S^*STT^*$ on* $\mathrm{ran}(\mathrm{T})$ *is invertible,*
(ii) *$STT^\dagger = T^\dagger TSTT^\dagger$,*
Moreover, if $T^\dagger TS = T^\dagger TSTT^\dagger$ then
(iii) *$(STT^*)(STT^*)^\dagger = STT^\dagger S^\dagger$,*
(iv) *ST has a closed range and $(ST)^\dagger ST = T^\dagger S^\dagger ST$.*

Proof. (i) The operator T has the following matrix form with respect to the orthogonal sum of submodules:

$$T = \begin{bmatrix} T_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathrm{ran}(\mathrm{T}^*) \\ \ker(\mathrm{T}) \end{bmatrix} \to \begin{bmatrix} \mathrm{ran}(\mathrm{T}) \\ \ker(\mathrm{T}^*) \end{bmatrix},$$

where T_1 is invertible. Also S has the form

$$\begin{bmatrix} S_1 & S_2 \\ S_3 & S_4 \end{bmatrix} : \begin{bmatrix} \mathrm{ran}(\mathrm{T}) \\ \ker(\mathrm{T}^*) \end{bmatrix} \to \begin{bmatrix} \mathrm{ran}(\mathrm{T}^*) \\ \ker(\mathrm{T}) \end{bmatrix}.$$

Since $\mathrm{ran}(\mathrm{T}) = \mathrm{ran}(\mathrm{TT}^*\mathrm{S}^*)$, then $TT^\dagger = (TT^*S^*)(TT^*S^*)^\dagger$. According to calculation the Moore-Penrose inverse of display of the matrix forms of operators as in [19] and [3], we have

$$\begin{aligned}
\begin{bmatrix} T_1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} T_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} &= \begin{bmatrix} T_1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} T_1^* & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} S_1^* & S_3^* \\ S_2^* & S_4^* \end{bmatrix} \\
&\quad \times \left(\begin{bmatrix} T_1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} T_1^* & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} S_1^* & S_3^* \\ S_2^* & S_4^* \end{bmatrix} \right)^\dagger \\
&= \begin{bmatrix} T_1T_1^*S_1^* & T_1T_1^*S_3^* \\ 0 & 0 \end{bmatrix} \left(\begin{bmatrix} T_1T_1^*S_1^* & T_1T_1^*S_3^* \\ 0 & 0 \end{bmatrix} \right)^\dagger \\
&= \begin{bmatrix} T_1T_1^*S_1^* & T_1T_1^*S_3^* \\ 0 & 0 \end{bmatrix} \times \begin{bmatrix} S_1T_1T_1^*D^\dagger & 0 \\ S_3T_1T_1^*D^\dagger & 0 \end{bmatrix} \\
&= \begin{bmatrix} DD^\dagger & 0 \\ 0 & 0 \end{bmatrix},
\end{aligned}$$

where, $D = T_1T_1^*S_1^*S_1T_1T_1^* + T_1T_1^*S_3^*S_3T_1T_1^*$. Therefore, $DD^\dagger = 1_{\mathrm{ran(T)}}$.

Since D is self-adjoint, then $D^\dagger D = 1_{\mathrm{ran(T)}}$. Therefore, D is invertible and $D^\dagger = D^{-1}$ on ran(T).

On the other hand, since $\mathrm{ran}(\mathrm{T}^*) = \mathrm{ran}(\mathrm{STT}^*)$, we have $(T^*)(T^*)^\dagger = (STT^*)(STT^*)^\dagger$. Using the same matrix form as above for operators T and S in the equation, we have

$$\begin{bmatrix} T_1^* & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} (T_1^*)^{-1} & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} S_1 & S_2 \\ S_3 & S_4 \end{bmatrix}\begin{bmatrix} T_1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} T_1^* & 0 \\ 0 & 0 \end{bmatrix}$$

$$\times \left(\begin{bmatrix} S_1 & S_2 \\ S_3 & S_4 \end{bmatrix} \begin{bmatrix} T_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} T_1^* & 0 \\ 0 & 0 \end{bmatrix} \right)^\dagger$$
$$= \begin{bmatrix} S_1T_1T_1^* & 0 \\ S_3T_1T_1^* & 0 \end{bmatrix} \left(\begin{bmatrix} S_1T_1T_1^* & 0 \\ S_3T_1T_1^* & 0 \end{bmatrix} \right)^\dagger$$
$$= \begin{bmatrix} S_1T_1T_1^* & 0 \\ S_3T_1T_1^* & 0 \end{bmatrix} \begin{bmatrix} D^\dagger T_1T_1^*S_1^* & D^\dagger T_1T_1^*S_3^* \\ 0 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} S_1T_1T_1^*D^\dagger T_1T_1^*S_1^* & S_1T_1T_1^*D^\dagger T_1T_1^*S_3^* \\ S_3T_1T_1^*D^\dagger T_1T_1^*S_1^* & S_3T_1T_1^*D^\dagger T_1T_1^*S_3^* \end{bmatrix}.$$

Therefore,

$$S_1T_1T_1^*D^\dagger T_1T_1^*S_1^* = 1_{\mathrm{ran(T)}}, \tag{2.1}$$
$$S_3T_1T_1^*D^\dagger T_1T_1^*S_3^* = 0. \tag{2.2}$$

Hence, the matrix representation of the operator $TT^*S^*STT^*$ is:

$$TT^*S^*STT^* = \begin{bmatrix} T_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} T_1^* & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} S_1^* & S_3^* \\ S_2^* & S_4^* \end{bmatrix} \begin{bmatrix} S_1 & S_2 \\ S_3 & S_4 \end{bmatrix}$$
$$\times \begin{bmatrix} T_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} T_1^* & 0 \\ 0 & 0 \end{bmatrix}.$$

Since the operator $D = T_1T_1^*(S_1^*S_1 + S_3^*S_3)T_1T_1^*$ is invertible on ran(T) and $P_{\mathrm{ran(T)}} = TT^\dagger$, according to what we discussed above, (i) followes.

(ii) D is an invertible and positive operator, because

$$D = T_1T_1^*S_1^*(T_1T_1^*S_1^*)^* + T_1T_1^*S_3^*(T_1T_1^*S_3^*)^*.$$

Now by equation (2.2) we have

$$\begin{aligned} S_3T_1T_1^*D^{-1}T_1T_1^*S_3^* &= S_3\left(T_1T_1^*D^{-1}T_1T_1^*\right)^{\frac{1}{2}}\left(T_1T_1^*D^{-1}T_1T_1^*\right)^{\frac{1}{2}}S_3^* \\ &= \left(S_3(T_1T_1^*D^{-1}T_1T_1^*)^{\frac{1}{2}}\right)\left(S_3\left(T_1T_1^*D^{-1}T_1T_1^*\right)^{\frac{1}{2}}\right)^* \\ &= 0. \end{aligned}$$

Invertibility of $T_1T_1^*D^{-1}T_1T_1^*$, implies that $S_3 = 0$. We know that, $S_3 = \left(1 - P_{\mathrm{ran(T^*)}}\right)SP_{\mathrm{ran(T)}} = 0$ or equivalently $S_3 = (1 - T^\dagger T)STT^\dagger = 0$ which implies that $STT^\dagger = T^\dagger TSTT^\dagger$.

(iii) Using $T^\dagger TS = T^\dagger TSTT^\dagger$, we have $P_{\mathrm{ran(T^*)}}S = P_{\mathrm{ran(T^*)}}SP_{\mathrm{ran(T)}}$ or equivalently $S_2 = P_{\mathrm{ran(T^*)}}S(1 - P_{\mathrm{ran(T)}}) = 0$. From the proof of (ii) we have $S_3 = 0$. Since, ran(S) is closed, we get that both ran(S$_1$) and ran(S$_4$) are closed [4]. Therefore, obviously

$S^\dagger = \begin{bmatrix} S_1^\dagger & 0 \\ 0 & S_4^\dagger \end{bmatrix}$ is the Moore-Penrose of $S = \begin{bmatrix} S_1 & 0 \\ 0 & S_4 \end{bmatrix}$.
On the other hand,

$$\begin{aligned} STT^* &= \begin{bmatrix} S_1 & 0 \\ 0 & S_4 \end{bmatrix} \begin{bmatrix} T_1T_1^* & 0 \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} S_1T_1T_1^* & 0 \\ 0 & 0 \end{bmatrix}, \end{aligned}$$

and we have

(a) $$\begin{bmatrix} S_1T_1T_1^* & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} (T_1^*)^{-1}T_1^{-1}S_1^\dagger & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} S_1T_1T_1^* & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} S_1T_1T_1^* & 0 \\ 0 & 0 \end{bmatrix}$$

(b) $$\begin{bmatrix} (T_1^*)^{-1}T_1^{-1}S_1^\dagger & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} S_1T_1T_1^* & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} (T_1^*)^{-1}T_1^{-1}S_1^\dagger & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} (T_1^*)^{-1}T_1^{-1}S_1^\dagger & 0 \\ 0 & 0 \end{bmatrix}$$

(c) $$\begin{bmatrix} S_1T_1T_1^* & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} (T_1^*)^{-1}T_1^{-1}S_1^\dagger & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} S_1S_1^\dagger & 0 \\ 0 & 0 \end{bmatrix}.$$

Therefore, condition (iii) in [19, Theorem 2.1] holds. By (iii)⇒(i) of [19, Theorem 2.1], $STT^\dagger = T^\dagger TSTT^\dagger$.

(iv) The operators ST and $T^\dagger S^\dagger$ have the following matrix representations

$$\begin{aligned} ST &= \begin{bmatrix} S_1 & 0 \\ 0 & S_4 \end{bmatrix} \begin{bmatrix} T_1 & 0 \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} S_1T_1 & 0 \\ 0 & 0 \end{bmatrix}, \end{aligned} \tag{2.3}$$

$$\begin{aligned} T^\dagger S^\dagger &= \begin{bmatrix} T_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} S_1^\dagger & 0 \\ 0 & S_4^\dagger \end{bmatrix} \\ &= \begin{bmatrix} T_1^{-1}S_1^\dagger & 0 \\ 0 & 0 \end{bmatrix}. \end{aligned} \tag{2.4}$$

Since we have

$$\begin{aligned} STT^\dagger S^\dagger ST &= \begin{bmatrix} (S_1T_1)T_1^{-1}S_1^\dagger(S_1T_1) & 0 \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} S_1T_1 & 0 \\ 0 & 0 \end{bmatrix}, \end{aligned}$$

$$
\begin{aligned}
T^\dagger S^\dagger STT^\dagger S^\dagger &= \begin{bmatrix} T^{-1}S_1^\dagger & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} S_1T_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} T^{-1}S_1^\dagger & 0 \\ 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} T_1^{-1}S_1^\dagger & 0 \\ 0 & 0 \end{bmatrix}
\end{aligned}
$$

and also

$$
\begin{aligned}
(STT^\dagger S^\dagger)^* &= \begin{bmatrix} S_1T_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} T^{-1}S_1^\dagger & 0 \\ 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} S_1S_1^\dagger & 0 \\ 0 & 0 \end{bmatrix},
\end{aligned}
$$

then we conclude $T^\dagger S^\dagger$ satisfies conditions (a), (b), (d) of Definition 1.2. Hence ST has closed range and Theorem 2.2 in [19] implies that $(ST)^\dagger ST = T^\dagger S^\dagger ST$.

□

Throughout this work, we use the matrix techniques which is simpler than functional analysis methods to machine language. Although the direct proof could be presented.

Remark 2.4. Direct proof of (i): Let $A = TT^*S^*STT^*TT^\dagger|_{\mathrm{ran(T)}}$. For any $x \in \mathrm{ran(T)}$ with $Ax = 0$, we have $TT^\dagger x = x$. Hence, $TT^*S^*STT^*x = 0$, which gives $STT^*x = 0$. It followes from $\mathrm{ran(T)} = \mathrm{ran(TT^*S^*)}$ that $\mathrm{ker(T^*)} = \mathrm{ker(STT^*)}$. So, $T^*x = 0$, and thus $T^\dagger x = 0$. Therefore, $x = TT^\dagger x = 0$, which implies that A is injective. Secondly, by assumptions, $\mathrm{ran(TT^*S^*)} = \mathrm{ran(T)}$, which is closed. Hence,

$$
\begin{aligned}
\mathrm{ran(A)} &= \mathrm{ran(TT^*S^*(TT^*S^*)^*)} \\
&= \mathrm{ran(TT^*S^*)} \\
&= \mathrm{ran(T)}.
\end{aligned}
$$

This shows that A is surjective.

For a direct proof of (ii) note that,

$$
\begin{aligned}
\mathrm{ran(STT^\dagger)} &= \mathrm{ran(STT^*)} \\
&= \mathrm{ran(T^*)} \\
&= \mathrm{ran(T^\dagger T)}.
\end{aligned}
$$

Item (iii) can, also, be derived directly. By assumption, we have: $T^\dagger TS = T^\dagger TSTT^\dagger = STT^\dagger$. This together with $(TT^*)(TT^*)^\dagger = TT^\dagger$ and $\mathrm{ran(T^*)} = \mathrm{ran(STT^*)} \subseteq \mathrm{ran(S)}$ yields

$$
STT^*(T^*)^\dagger T^\dagger S^\dagger = STT^*(TT^*)^\dagger S^\dagger
$$

$$\begin{aligned}&= STT^\dagger S^\dagger\\&= T^\dagger TSS^\dagger\\&= T^\dagger T\\&= STT^*(STT^*)^\dagger.\end{aligned}$$

Now, by matrix form of EP operator in [17, Lemma 3.6] we have the following corollary.

Corollary 2.5. *Let $T, S \in \mathcal{L}(\mathcal{X})$ be such that T and S are Moore-Penrose invertible, and T is an EP operator. Suppose that* $\mathrm{ran(T^*)} = \mathrm{ran(STT^*)}$ *and* $\mathrm{ran(T)} = \mathrm{ran(TT^*S)}$. *Then*

(i) $TT^*S^*(TT^*S^*)^*$ *is invertible,*
(ii) $ST^\dagger T = TT^\dagger ST^\dagger T$,
Moreover, if $T^\dagger TS = T^\dagger TSTT^\dagger$ then
(iii) $(STT^*)(STT^*)^\dagger = STT^*(T^*)^\dagger T^\dagger S^\dagger$,
(iv) ST *has a closed range and* $(ST)^\dagger ST = T^\dagger S^\dagger ST$.

Theorem 2.6. *Let $\mathcal{X}$ be a Hilbert $\mathcal{A}$-module, $T \in L(\mathcal{X})$ has closed range. If* $\mathrm{ran(T)} \oplus \mathrm{ran(T^*)} = \mathcal{X}$, *then $T^\dagger T + TT^*$ is an invertible operator.*

Proof. Since $\mathrm{ran(T)} \oplus \mathrm{ran(T^*)} = \mathcal{X}$, we have $TT^\dagger + T^\dagger T = 1$, hence $T^\dagger T = 1 - TT^\dagger$. Let $C := 1 - TT^\dagger + TT^*$ is $K := 1 - TT^\dagger + (TT^*)^\dagger$, sine

$$\begin{aligned}KC &= (1 - TT^\dagger + (TT^*)^\dagger)(1 - TT^\dagger + TT^*)\\&= 1 - TT^\dagger + TT^* - TT^\dagger + TT^\dagger TT^\dagger - TT^\dagger TT^*\\&\quad + (TT^*)^\dagger - (TT^*)^\dagger TT^\dagger + (TT^*)^\dagger TT^*\\&= 1 - TT^\dagger + (TT^*)^\dagger - (TT^*)^\dagger TT^\dagger + (TT^*)^\dagger TT^*\\&= 1 - TT^\dagger + (T^\dagger)^*T^\dagger - (T^\dagger)^*T^\dagger TT^\dagger + (T^\dagger)^*T^\dagger TT^*\\&= 1 - TT^\dagger + (T^\dagger)^*T^\dagger - (T^\dagger)^*T^\dagger + (T^\dagger)^*T^\dagger TT^*\\&= 1 - TT^\dagger + (T^\dagger)^*T^*\\&= 1 - TT^\dagger + (TT^\dagger)^*\\&= 1,\end{aligned}$$

and

$$\begin{aligned}CK &= (1 - TT^\dagger + TT^*)(1 - TT^\dagger + (TT^*)^\dagger)\\&= 1 - TT^\dagger + (TT^*)^\dagger - TT^\dagger(1 - TT^\dagger + (TT^*)^\dagger)\\&\quad + TT^*(1 - TT^\dagger + (TT^*)^\dagger)\end{aligned}$$

$$\begin{aligned}
&= 1 - TT^\dagger + (TT^*)^\dagger - TT^\dagger + TT^\dagger - TT^\dagger(TT^*)^\dagger \\
&\quad + TT^* - TT^*TT^\dagger + TT^*(TT^*)^\dagger \\
&= 1 - TT^\dagger + (T^*)^\dagger T^\dagger - TT^\dagger(T^*)^\dagger T^\dagger \\
&\quad + TT^* - TT^* + TT^*(T^*)^\dagger T^\dagger \\
&= 1 - TT^\dagger + (T^*)^\dagger T^\dagger - TT^\dagger(T^*)^\dagger T^\dagger + TT^*(T^*)^\dagger T^\dagger \\
&= 1 - TT^\dagger + (T^*)^\dagger T^\dagger - TT^\dagger(T^*)^\dagger T^\dagger + TT^\dagger \\
&= 1 + (T^*)^\dagger T^\dagger - TT^\dagger(T^*)^\dagger T^\dagger \\
&= 1 + (T^*)^\dagger T^\dagger - (T^*)^\dagger T^\dagger \\
&= 1.
\end{aligned}$$

Then $1 - TT^\dagger + TT^* = T^\dagger T + TT^*$ is an invertible operator. □

Consider the C^*-algebra $\mathcal{A}$ as a Hilbert $\mathcal{A}$-module. For each $a \in \mathcal{A}$, the operator $T_a : \mathcal{A} \to \mathcal{A}$ with $x \mapsto ax$ is an element of $\mathcal{L}(\mathcal{A})$. Note that $T_a^* = T_{a^*}$ and T_a is Moore-Penrose invertible if and only if a is Moore-Penrose invertible and in this case $(T_a)^\dagger = T_{a^\dagger}$ (adapted from[10–13]). In addition, since every C^*-algebra admits an approximate identity [15], one can easily see $T_a = T_b$ if and only if $a = b$. In the following, we explain how to apply the results obtained in this paper. Many researchers, including Feng, Xu, and Vosugh [21] and Niazi Moghani [16] have provided similar applications in their works as solutions of equations of operation.

3. Conclusion

In last decades, solving matrix equations was interesting for to mathematicians, and with the expansion of mathematical structures, mathematicians were keen on generalizing matrix equations in new structures regardless of application. This process is common in the development of mathematical science. In this regard, the results of this field of mathematics can be used to generalize the methods of solving matrix equations to the context of operators. This can be seen in the research of many mathematicians. See [16, 20, 21] for example. Since the matrix space is finite dimensional, so if we extend these equations to the space of operators, then also it will be able to solve equations in infinite dimensional cases. It may be assumed that the applied equations have a matrix representation, and the generalization of equations to higher spaces is studied merely for the abstract mathematical discussion. Therefore, in order to show that this suspicion is false, we present here an example of functional equations appeared in physics. The Kadomtsev-Petviashvili operator equation $v_{xt} = \frac{1}{4}\left(v_{xxx} + 6v_x^2\right)_x + \frac{3}{4}v_{yy} + \frac{3}{2}\left(v_y v_x - v_x v_y\right)$ as an example, which is different from the finite case (as mentioned in [20]). The interested reader, for more detail and informations can be referred to [20, 21].

Similar to the matrix form, here too many of the solutions are found using general inverses, and in particular the Moore-Penrose inverse. Specifically, with the help of the results obtained in Theorem 2.3 and Corollary 2.5 and by using some block matrix technique some of that operator equations can be solved.

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References

1. R. Bouldin, *Closed range and relative regularity for products*, J. Math. Anal. Appl., 61, (1977), pp. 397-403.
2. R. Bouldin, *The product of operators with closed range*, Tohoku Math. J., 25(2), (1973), pp. 359-363.
3. C.Y. Deng and H.K. Du, *Representations of the Moore-Penrose inverse for a class of 2 by 2 block operator valued partial matrices*, Linear Multilinear Algebra, 58, (2010), pp. 15-26.
4. D.S. Djordjević, *Further results on the reverse order law for generalized inverses*, SIAM J. Matrix Anal. Appl., 29(4), (2007), pp. 1242-1246.
5. J. Farokhi-Ostad and A.R. Janfada, *Products of EP operators on Hilbert C^*-Modules*, Sahand Commun. Math. Anal., 10, (2018), pp. 61-71.
6. J. Farokhi-Ostad and A.R. Janfada, *On closed range C^*-modular operators*, Aust. J. Math. Anal. Appl., 15(2), (2018), pp. 1-9.
7. S. Izumino, *The product of operators with closed range and an extension of the reverse order law*, Tohoku Math. J., 34(2), (1982), pp. 43-52.
8. M. Jalaeian, M. Mohammadzadeh Karizaki and M. Hassani, *Conditions that the product of operators is an EP operator in Hilbert C^*-module*, Linear Multilinear Algebra, 68(10), (2020), pp. 1990-2004.
9. E.C. Lance, *Hilbert C^*-Modules*, LMS Lecture Note Series 210, Cambridge Univ. Press, 1995.
10. M. Mohammadzadeh Karizaki, *Antisymmetric relations of operators which satisfy specified conditions*, Funct. Anal. Approx. Comput., 10(3), (2018), pp. 15-20
11. M. Mohammadzadeh Karizaki and D.S. Djordjevic, *Commuting C^*- modular operators*, Aequationes mathematicae, 90(6), pp. 1103-1114.

12. M. Mohammadzadeh Karizaki, M. Hassani and M. Amyari, *Moore-Penrose inverse of product operators in Hilbert C^*-modules*, Filomat, 30(13), (2016), pp. 3397-3402.
13. M. Mohammadzadeh Karizaki, M. Hassani, M. Amyari and M. Khosravi, *Operator matrix of Moore-Penrose inverse operators on Hilbert C^*-modules*, Colloq. Math., 140, (2015), pp. 171-182.
14. M.S. Moslehian, K. Sharifi, M. Forough and M. Chakoshi, *Moore-Penrose inverse of Gram operator on Hilbert C^*-modules*, Studia Math., 210(2), (2012), 189-196.
15. G.J. Murphy, *C^*-algebras and operator theory*, Academic Press Inc., Boston, MA, 1990.
16. Z. Niazi Moghani, M. Khanehgir and M. Mohammadzadeh Karizaki, *explicit solution to the operator equation $AXD+FX^*B = C$ over Hilbert C^*-modules*, J. Math. Anal. Appl., 10(1), (2019), pp. 52-64.
17. K. Sharifi, *EP modular operators and their products*, J. Math. Anal. Appl., 419, (2014), pp. 870-877.
18. K. Sharifi, *The product of operators with closed range in Hilbert C^*-modules*, Linear Algebra Appl., 435, (2011), pp. 1122-1130.
19. K. Sharifi and B. Ahmadi Bonakdar, *The reverse order law for Moore-Penrose inverses of operators on Hilbert C^*-modules*, Bull. Iran. Math. Soc., 42(1), (2016), pp. 53-60.
20. M. Vosough and M.S. Moslehian, *Operator and Matrix Equations*, Ph.d thesis, Ferdowsi University of Mashhad, (2017).
21. M. Vosough and M.S. Moslehian, *Solutions of the system of operator equations $BXA = B = AXB = AXB$ via $*$-order*,Electron. J. Linear Algebra, 32, (2017), pp. 172-183.
22. Q. Xu and L. Sheng, *Positive semi-definite matrices of adjointable operators on Hilbert C^*-modules*, Linear Algebra Appl., 428, (2008), pp. 992-1000.

[1] Department of Mathematics, Faculty of Mathematics Science and Statistics, University of Birjand, Birjand 9717851367, Iran.

E-mail address: ajanfada@birjand.ac.ir

[2] Department of Basic sciences, Birjand University of Technology, Birjand 9719866981, Iran

E-mail address: j.farrokhi@birjandut.ac.ir