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Using Frames in Steepest Descent-Based Iteration Method for Solving Operator Equations

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Abstract. In this paper, by using the concept of frames, two iterative methods are constructed to solve the operator equation $Lu = f$ where $L : H \to H$ is a bounded, invertible and self-adjoint linear operator on a separable Hilbert space H. These schemes are analogous with steepest descent method which is applied on a preconditioned equation obtained by frames instead. We then investigate their convergence via corresponding convergence rates, which are formed by the frame bounds. We also investigate the optimal case, which leads to the exact solution of the equation. The first scheme refers to the case where H is a real separable Hilbert space, but in the second scheme, we drop this assumption.

1. Introduction

Projection methods are the most recently practical iterative techniques for solving large linear system of equations

$$Lu = f, \tag{1.1}$$

where $L : H \to H$ is a bounded, invertible and self-adjoint linear operator on a separable Hilbert space H. By using this approach, we can extract canonically an approximation $\tilde{u}$ to the exact solution u of the linear system from a subspace $\mathcal{K} \subseteq H$, called search subspace, provided that

$$f - L\tilde{u} \perp \mathcal{L},$$

where $\mathcal{L} \subseteq H$ is another (maybe the same) subspace, called the subspace of constraints, of equal dimension. We refer the interested reader to the

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book by Saad [12]. In the meantime, steepest descent method is of the great importance between one dimensional projection methods which utilizes subspaces $\mathcal{L} = \mathcal{K} = \langle r_0 \rangle$, where $r_0 = f - Lu_0$ is the residual vector of equation (1.1) for any initial guess u_0.

The goal of this paper is to study the application of frames in steepest descent method for solving operator equation (1.1). In [1, 5–8, 11], some numerical algorithms for solving this system have been developed by using wavelets and frames. The great advantage of approach is that we can see the convergence rate in this approach formed by upper and lower bounds of the frame, so we can control the convergence rate by choosing an appropriate frame with desired values of bounds. Furthermore in this setting, as it will be discussed at the end of each section, by employing a tight frame for designing iteration, even the exact solution of (1.1) is obtained in the first step of the iteration. These properties turn this modified method into an applicable tool for approximating solutions of operator equations with prescribed accuracy or even finding the exact solution. The method is obtained on the basis of preconditioning the operator equation $Lu = f$ by using frames and then by applying steepest descent method on it.

2. Preliminaries

We give a brief review about the definitions and basic properties of frames. For more information, we refer the reader to the book by Christensen [4].
Throughout this paper, H will be a separable Hilbert space and Λ will denote a countable index set. In the next subsection, we introduce the notion of preconditioning of an operator equation and we describe how a frame is used to precondition an operator equation.

2.1. **Frames.** We begin with the definition of a frame.

Definition 2.1. Let $(\psi_\lambda)_{\lambda\in\Lambda} \subset H$ be a sequence of elements. Then $(\psi_\lambda)_{\lambda\in\Lambda}$ is a frame for H, if there exist constants $0 < A \leq B < \infty$ such that

$$A \parallel f \parallel_H^2 \leq \sum_{\lambda\in\Lambda} \mid \langle f, \psi_\lambda \rangle \mid^2 \leq B \parallel f \parallel_H^2, \quad \forall f \in H.$$

The constants A and B are called the lower and upper frame bounds, respectively. If $A = B$, the frame $(\psi_\lambda)_{\lambda\in\Lambda}$ is said to be an A-*tight frame*, and if $A = B = 1$, it is a *Parseval frame*.

For a frame $(\psi_\lambda)_{\lambda\in\Lambda}$, the following operators are defined. The *synthesis operator*:

$$T : \ell_2(\Lambda) \to H, \qquad T((c_\lambda)_{\lambda\in\Lambda}) = \sum_{\lambda\in\Lambda} c_\lambda \psi_\lambda,$$

and the *analysis operator*:

$$T^* : H \to \ell_2(\Lambda), \qquad T^*(f) = (\langle f, \psi_\lambda \rangle)_{\lambda \in \Lambda},$$

and the *frame operator*:

$$S = TT^* : H \to H, \qquad S(f) = \sum_{\lambda \in \Lambda} \langle f, \psi_\lambda \rangle \psi_\lambda.$$

The frame operator is positive, self-adjoint and invertible, and the following inequalities hold [3, 4]

$$AI_H \leq S \leq BI_H, \qquad B^{-1}I_H \leq S^{-1} \leq A^{-1}I_H. \tag{2.1}$$

In [9, 10], it has been shown that if $\Psi = (\psi_\lambda)_{\lambda \in \Lambda}$ is a frame for H, and if L is a bounded invertible operator on H, then the sequence $(L(\psi_\lambda))_{\lambda \in \Lambda}$ would be also a frame for H with frame operator $S' = LSL$ where S is frame operator of $\Psi = (\psi_\lambda)_{\lambda \in \Lambda}$. For more detail, we refer the reader to [4].

2.2. **Preconditioning.** Preconditioning is any form of implicit or explicit modification of an original linear system which yields easier solving or faster convergence by a given iterative method. This is an effective technique for solving differential equations, integral equations, and related problems [2, 4]. The abstract approach is to multiply both sides of (1.1) by an operator M, and then apply a suitable iterative method. We choose M formed by using a given frame. To begin with, we write $Lu = f$ as

$$u = (I - L)u + f,$$

then for given $u_0 \in H$, define for $k \geq 0$,

$$u_{k+1} = (I - L)u_k + f. \tag{2.2}$$

Since $Lu - f = 0$, we can write

$$\begin{aligned} u_{k+1} - u &= (I - L)u_k + f - u - (f - Lu) \\ &= (I - L)u_k - u + Lu \\ &= (I - L)(u_k - u). \end{aligned}$$

Hence

$$\| u_{k+1} - u \|_H \leq \| I - L \|_{H \to H} \| u_k - u \|_H,$$

where $\| \, . \, \|_{H \to H}$ denotes the operator norm. So the sequence (2.2) converges if

$$\| I - L \|_{H \to H} < 1. \tag{2.3}$$

Let M be an operator which approximates L^{-1} i.e. $M \approx L^{-1}$ or $ML \approx I$. Then the last term implies $\| I - ML \|_{H \to H} \ll 1$, so in view of (2.3), the convergence of iterative sequence (2.2) associated to preconditioned operator equation

$$MLu = Mf, \tag{2.4}$$

is much faster than the original one.

One way to obtain M is using frames. For this concern, consider the following lemma.

Lemma 2.2 ([9])**.** *Let $\Psi = (\psi_\lambda)_{\lambda\in\Lambda}$ be a frame for H with frame operatpr S, and let L be as in (1.1). Suppose that A and B are the frame bounds of the frame $L\Psi = (L(\psi_\lambda))_{\lambda\in\Lambda}$. Then*

$$\left\| I - \frac{2}{A+B} LSL \right\|_{H\to H} \leq \frac{B-A}{A+B}. \tag{2.5}$$

Since $\frac{B-A}{A+B} < 1$, we can take $M := \frac{2}{A+B}LS$ for preconditioning (1.1). In the reminder of the discussion, we consider alternatively the following operator equation

$$\frac{2}{A+B}LSLu = \frac{2}{A+B}LSf,$$

or for short

$$\mathcal{B}u = \mathcal{F}, \tag{2.6}$$

where $\mathcal{B} := \frac{2}{A+B}LSL$ and $\mathcal{F} := \frac{2}{A+B}LSf$.

3. Steepest Descent Method by Using Frames Part I

We start the discussion with *Kantorovich inequality.* Some results are given by this inequality. In this section, H is a real separable Hilbert space. We first recall the definition of the symmetric positive definite operator.

Definition 3.1. Let H be a real Hilbert space and let $\mathcal{B} : H \to H$ be a linear operator. The operator $\mathcal{B}$ is said to be symmetric if for all $x, y \in H$ we have

$$\langle \mathcal{B}x, y\rangle = \langle x, \mathcal{B}y\rangle .$$

Also, The operator $\mathcal{B}$ is said to be positive definite if for all nonzero $x \in H$ we have

$$\langle \mathcal{B}x, x\rangle > 0.$$

The following lemma illustrates Kantorovich inequality.

Lemma 3.2 ([12, Kantorovich inequality])**.** *Let $\mathcal{B}$ be a symmetric positive definite operator defined on a real Hilbert space, and $\lambda_{\max} := \sup \sigma(\mathcal{B})$,*

$\lambda_{\min} := \inf \sigma(\mathcal{B})$ *where* $\sigma(\mathcal{B})$ *is the spectrum of the positive definite self-adjoint operator* $\mathcal{B}$. *Then,*

$$\frac{\langle \mathcal{B}x, x\rangle \langle \mathcal{B}^{-1}x, x\rangle}{\langle x, x\rangle^2} \leq \frac{(\lambda_{\max} + \lambda_{\min})^2}{4\lambda_{\max}\lambda_{\min}}, \quad \forall x \neq 0. \tag{3.1}$$

Corollary 3.3. *Let* $(\psi_\lambda)_{\lambda\in\Lambda}$ *be a frame for* H *with frame operatpr* S, *and let* L *and* $\mathcal{B}$ *be as in (1.1) and (2.6), respectively. Suppose that* A *and* B *are the frame bounds of the frame* $(L(\psi_\lambda))_{\lambda\in\Lambda}$. *Then for each* $x \in H$, *the following statements hold.*

i. $\langle \mathcal{B}x, x\rangle \langle \mathcal{B}^{-1}x, x\rangle \leq \dfrac{(A+B)^2}{4AB} \|x\|_H^4$;

ii. $\inf_{\|x\|_H=1} \dfrac{\langle x, \mathcal{B}x\rangle^2}{\|\mathcal{B}x\|_H^2} \geq \dfrac{4AB}{(A+B)^2}$;

iii. $\dfrac{\|\mathcal{B}x\|_H^2}{\|x\|^2 \langle x, \mathcal{B}x\rangle^2} \leq \dfrac{(A+B)^2}{4AB}, \quad \forall x \neq 0.$

Proof. i. The expression is clear from Kantrorovich inequality and eigenvalues λ of the operator $\mathcal{B}$ satisfies the following

$$\frac{2A}{A+B} \leq \lambda \leq \frac{2B}{A+B}.$$

ii. Let $y = \frac{x}{\|x\|}$ for $x \neq 0$ (in this case $\|y\| = 1$), and let $z = B^{\frac{1}{2}}x$. By putting z in the inequality (i), we obtain

$$\begin{aligned}
\frac{4AB}{(A+B)^2} &\leq \frac{\|z\|^4}{\langle \mathcal{B}z, z\rangle \langle \mathcal{B}^{-1}z, z\rangle} \\
&= \frac{\left\langle \mathcal{B}^{\frac{1}{2}}x, \mathcal{B}^{\frac{1}{2}}x\right\rangle^2}{\left\langle \mathcal{B}^{\frac{3}{2}}x, \mathcal{B}^{\frac{1}{2}}x\right\rangle \left\langle \mathcal{B}^{-\frac{1}{2}}x, \mathcal{B}^{\frac{1}{2}}x\right\rangle} \\
&= \frac{\langle \mathcal{B}x, x\rangle^2}{\langle \mathcal{B}x, \mathcal{B}x\rangle \|x\|^2} \\
&= \frac{\langle \mathcal{B}y, y\rangle^2}{\|\mathcal{B}y\|^2}.
\end{aligned}$$

This completes the proof of (ii).

iii. At the end, we see that the expression (iii) follows immediately from (ii).

□

Since L is an injective operator, for $f = 0$ in (i), one obtains $u = 0$. Therefor without losing generality assume that $f \neq 0$ and thus $\mathcal{F} \neq 0$.

Let us consider the following equation

$$\mathcal{B}u = \langle \mathcal{B}u, \mathcal{F} \rangle \mathcal{F}, \quad (u \neq 0, \|\mathcal{F}\| = 1) . \tag{3.2}$$

In this case, if u' is the solution of (3.2), then $\frac{u'}{\langle \mathcal{B}u', \mathcal{F} \rangle}$ would be the solution of the equation (2.6).

Remark 3.4. Note that if $\langle \mathcal{B}u', \mathcal{F} \rangle = 0$, then by equation (3.2) $\mathcal{B}u' = 0$ and so $u' = 0$. Therefore, we assume without loss of generality that $\langle \mathcal{B}u', \mathcal{F} \rangle \neq 0$.

Now, for some initial guess u'_0 to the solution of (3.2) with the property that $\langle u'_0, \mathcal{F} \rangle \neq 0$, we define the sequence $\{u'_n\}$ by

$$u'_{n+1} = u'_n + \alpha_n R(u'_n),$$

where $\alpha_n = \frac{\langle R(u'_n), \mathcal{B}R(u'_n) \rangle}{\langle R^2(u'_n), R^2(u'_n) \rangle}$, and $R(u'_n) = \langle \mathcal{B}u'_n, \mathcal{F} \rangle \mathcal{F} - \mathcal{B}u'_n$.

Theorem 3.5. *Let $(\psi_\lambda)_{\lambda \in \Lambda}$ be a frame for H with frame bounds A and B, then for any initial guess u'_0, the residual vector $R(u'_n)$ of equation (3.2) satisfies the following inequality,*

$$\left\| R(u'_{n+1}) \right\| \leq \left(\frac{B - A}{A + B} \right) \left\| R(u'_n) \right\| . \tag{3.3}$$

Furthermore, the sequence $\{u'_n\}$ converges to the exact solution u' of (3.2).

Proof. By the definition of $R(u'_n)$, we first see

$$\begin{aligned} \left\| R(u'_n) \right\|^2 &= \left\langle \left\langle \mathcal{B}u'_n, \mathcal{F} \right\rangle \mathcal{F} - \mathcal{B}u'_n, \left\langle \mathcal{B}u'_n, \mathcal{F} \right\rangle \mathcal{F} - \mathcal{B}u'_n \right\rangle \\ &= \left\| \mathcal{B}u'_n \right\|^2 - \left\langle \mathcal{B}u'_n, \mathcal{F} \right\rangle^2 - \left\langle \mathcal{B}u'_n, \mathcal{F} \right\rangle \left\langle \mathcal{F}, \mathcal{B}u'_n \right\rangle \\ &\quad + \left\langle \mathcal{B}u'_n, \mathcal{F} \right\rangle^2 \left\langle \mathcal{F}, \mathcal{F} \right\rangle . \end{aligned}$$

Hence, since $\|\mathcal{F}\| = 1$ and H is a real Hilbert space, then

$$\left\| R(u'_n) \right\|^2 = \left\| \mathcal{B}u'_n \right\|^2 - \left\langle \mathcal{B}u'_n, \mathcal{F} \right\rangle^2 . \tag{3.4}$$

On the other hand, we can also easily see that

$$\begin{aligned} R(u'_{n+1}) &= R\left(u'_n + \alpha_n R(u'_n) \right) \\ &= R(u'_n) + \alpha_n R^2(u'_n). \end{aligned}$$

Therefore, we have altogether

$$\begin{aligned} \left\| R(u'_{n+1}) \right\|^2 &= \left\| R(u'_n) + \alpha_n R^2(u'_n) \right\|^2 \\ &= \left\| R(u'_n) + \alpha_n \left(\left\langle \mathcal{B}R(u'_n), \mathcal{F} \right\rangle \mathcal{F} - \mathcal{B}R(u'_n) \right) \right\|^2 \\ &= \left\| R(u'_n) \right\|^2 - 2\alpha_n (\left\langle \mathcal{B}R(u'_n), R(u'_n) \right\rangle \\ &\quad - \left\langle \mathcal{B}R(u'_n), \mathcal{F} \right\rangle \left\langle R(u'_n), \mathcal{F} \right\rangle) \end{aligned}$$

$$
\begin{aligned}
&+ \alpha_n^2(\left\|\mathcal{B}R(u_n')\right\|^2 - \left\langle \mathcal{B}R(u_n'), \mathcal{F}\right\rangle^2) \\
&= \left\|R(u_n')\right\|^2 - 2\alpha_n(\left\langle \mathcal{B}R(u_n'), R(u_n')\right\rangle \\
&- \left\langle \mathcal{B}R(u_n'), \mathcal{F}\right\rangle \left\langle R(u_n'), \mathcal{F}\right\rangle) + \alpha_n^2 \left\|R^2(u_n')\right\|.
\end{aligned}
$$

Since

$$
\left\langle R(u_n'), \mathcal{F}\right\rangle = \left\langle \mathcal{B}(u_n'), \mathcal{F}\right\rangle - \left\langle \mathcal{B}R(u_n'), \mathcal{F}\right\rangle \left\langle R(u_n'), \mathcal{F}\right\rangle
$$

by (3.4) we obtain

$$
\left\|R(u_{n+1}')\right\|^2 = \left\|R(u_n')\right\|^2 - \alpha_n \left\langle R(u_n'), LSLR(u_n')\right\rangle. \tag{3.5}
$$

This shows that $\{\|R(u_n')\|\}$ is monotonically decreasing. To prove that it vanishes, by (3.5) we have

$$
\left\|R(u_{n+1}')\right\|^2 = \left\|R(u_n')\right\|^2 - \frac{\langle R(u_n'), \mathcal{B}R(u_n')\rangle^2}{\|R^2(u_n')\|}.
$$

Therefore

$$
\begin{aligned}
\frac{\left\|R(u_{n+1}')\right\|^2}{\|R(u_n')\|^2} &= 1 - \frac{\langle R(u_n'), \mathcal{B}R(u_n')\rangle^2}{\|R(u_n')\|^2\,\|R(u_n')\|^2} \\
&= 1 - \left(\frac{\langle R(u_n'), \mathcal{B}R(u_n')\rangle^2}{\|R(u_n')\|^2\,\|\mathcal{B}R(u_n')\|^2} \times \frac{\|\mathcal{B}R(u_n')\|^2}{\|R(u_n')\|^2}\right). \qquad (3.6)
\end{aligned}
$$

But we can see

$$
\begin{aligned}
\frac{\|\mathcal{B}R(u_n')\|^2}{\|R(u_n')\|^2} &= \frac{\|\mathcal{B}R(u_n')\|^2 - \langle \mathcal{B}R(u_n'), \mathcal{F}\rangle^2 + \langle \mathcal{B}R(u_n'), \mathcal{F}\rangle^2}{\|\mathcal{B}R(u_n')\|^2 - \langle \mathcal{B}R(u_n'), \mathcal{F}\rangle^2} \\
&= 1 + \frac{\langle \mathcal{B}R(u_n'), \mathcal{F}\rangle^2}{\|\mathcal{B}R(u_n')\|^2 - \langle \mathcal{B}R(u_n'), \mathcal{F}\rangle^2}.
\end{aligned}
$$

Thus, by importing preceding relation into (3.6) we arrive at

$$
\begin{aligned}
\left\|R(u_{n+1}')\right\|^2 = \Bigg[1 &- \frac{\langle R(u_n'), \mathcal{B}R(u_n')\rangle^2}{\|R(u_n')\|^2\,\|\mathcal{B}R(u_n')\|^2} \\
&\times \left(1 + \frac{\langle \mathcal{B}R(u_n'), \mathcal{F}\rangle^2}{\|\mathcal{B}R(u_n')\|^2 - \langle \mathcal{B}R(u_n'), \mathcal{F}\rangle^2}\right)\Bigg] \left\|R(u_n')\right\|^2.
\end{aligned}
$$

Now, together with the last equality, if we employ third part of Corollary (3.3) we observe that

$$
\begin{aligned}
\left\|R(u_{n+1}')\right\|^2 &\leq \left[1 - \frac{4AB}{(A+B)^2}\left(1 + \frac{\langle \mathcal{B}R(u_n'), \mathcal{F}\rangle^2}{\|\mathcal{B}R(u_n')\|^2 - \langle \mathcal{B}R(u_n'), \mathcal{F}\rangle^2}\right)\right] \left\|R(u_n')\right\|^2 \\
&\leq \left(1 - \frac{4AB}{(A+B)^2}\right) \left\|R(u_n')\right\|^2
\end{aligned}
$$

$$= \left(\frac{B-A}{A+B}\right)^2 \left\|R(u_n')\right\|^2,$$

which proves the first part of the theorem as well as $R(u_n') \to 0$ and thus $u_n' \to u'$. This completes the proof. □

For the reminder of the discussion we only need to show that the sequence $\left\{u_n := \frac{u_n'}{\langle \mathcal{B}u_n', \mathcal{F}\rangle}\right\}$ converges to the exact solution u of (2.6). It turns out that this sequence converges at least as fast as $\left(\frac{B-A}{A+B}\right)$. To this end, we state the following theorem.

Theorem 3.6. *Let $(\psi_\lambda)_{\lambda\in\Lambda}$ be a frame for H with frame bounds A and B, then for any initial guess u_0, the residual vector $\mathcal{R}(u_n)$ of equation (2.6) satisfies the following inequality*

$$\|\mathcal{R}(u_{n+1})\| \leq \left(\frac{B-A}{A+B}\right) \|\mathcal{R}(u_n)\|. \tag{3.7}$$

Furthermore, the sequence $\{u_n\}$ converges to the exact solution u of (2.6).

Proof. First, note that since

$$u_n' = u_0' - \sum_{i=0}^{n-1} \alpha_i R(u_i'),$$

and for each i, $\langle R(u_i'), \mathcal{F}\rangle = 0$, then for any $n \in \mathbb{N}$ we have

$$\left\langle u_n', \mathcal{F}\right\rangle = \left\langle u_0', \mathcal{F}\right\rangle.$$

Thus

$$\inf\left\{\left\|u_n'\right\| \,:\, n = 1, 2, \ldots\right\} > 0. \tag{3.8}$$

Supposing on the contrary and taking any subsequence $\{u_{n_k}'\}$ such that $u_{n_k}' \to 0$, implies

$$\left\langle u_0', \mathcal{F}\right\rangle = \left\langle u_{n_k}', \mathcal{F}\right\rangle \to 0,$$

which contradicts the condition $\langle u_0', \mathcal{F}\rangle \neq 0$. In the one hand, since L is a bounded operator, then by (2.1), (3.8), and the fact that LSL is the frame operator of $L\Psi = (L(\psi_\lambda))_{\lambda\in\Lambda}$ we see

$$\begin{aligned}\left\|\mathcal{B}u_n'\right\|^2 &\geq \frac{4A^2}{(A+B)^2}\left\|u_n'\right\|^2 \\ &> 0.\end{aligned} \tag{3.9}$$

On the other hand, since $\|R(u_n')\|^2 \to 0$, then by (3.4) we have

$$\left\|\mathcal{B}u_n'\right\|^2 - \left\langle \mathcal{B}u_n', \mathcal{F}\right\rangle^2 \to 0.$$

Hence, in view of above relations altogether with the relation (3.9), we conclude

$$\inf\left\{\left\langle \mathcal{B}u'_n, \mathcal{F}\right\rangle \,:\, n = 1, 2, \ldots\right\} > 0.$$

This proves the last part of the theorem. For the rest of the theorem, since $\frac{\langle \mathcal{B}u'_n, \mathcal{F}\rangle}{\left\langle \mathcal{B}u'_{n+1}, \mathcal{F}\right\rangle} < 1$, then by virtue of preceding theorem we consider the following relations

$$\begin{aligned}
\|\mathcal{R}(u_{n+1})\| &= \left\| \mathcal{R}\left(\frac{u'_{n+1}}{\left\langle \mathcal{B}u'_{n+1}, \mathcal{F}\right\rangle}\right)\right\| \\
&= \left\| \mathcal{F} - \mathcal{B}\left(\frac{u'_{n+1}}{\left\langle \mathcal{B}u'_{n+1}, \mathcal{F}\right\rangle}\right)\right\| \\
&= \left\| \mathcal{F} - \frac{\mathcal{B}\left(u'_{n+1}\right)}{\left\langle \mathcal{B}u'_{n+1}, \mathcal{F}\right\rangle}\right\| \\
&= \frac{1}{\left\langle \mathcal{B}u'_{n+1}, \mathcal{F}\right\rangle} \left\| \left\langle \mathcal{B}u'_{n+1}, \mathcal{F}\right\rangle \mathcal{F} - \mathcal{B}\left(u'_{n+1}\right)\right\| \\
&= \frac{1}{\left\langle \mathcal{B}u'_{n+1}, \mathcal{F}\right\rangle} \left\| R(u'_{n+1})\right\| \\
&\leq \frac{1}{\left\langle \mathcal{B}u'_{n+1}, \mathcal{F}\right\rangle} \left(\frac{B-A}{A+B}\right) \left\| R(u'_n)\right\| \\
&= \frac{\langle \mathcal{B}u'_n, \mathcal{F}\rangle}{\left\langle \mathcal{B}u'_{n+1}, \mathcal{F}\right\rangle} \left(\frac{B-A}{A+B}\right) \|\mathcal{R}(u_n)\| \\
&\leq \left(\frac{B-A}{A+B}\right) \|\mathcal{R}(u_n)\|.
\end{aligned}$$

This completes the proof. □

This convergence rate suggests also that if we turn to use a tight frame, then we can enjoy obtaining the exact solution of (3.2).

4. Steepest Descent Method by Using Frames Part II

In this section we introduce a general version of *Steepest Descent method* in the framework of *Frame Theory.* It is worth saying that no restriction on Hilbert space H is no longer assumed, but the convergence rate obtained here is not as efficient as the previous one.

First of all, we note that since L and S are self-adjoint and positive definite operators, then $\mathcal{B} = \frac{2}{A+B}LSL$ is a positive definite operator. Hence, we can define the following $\mathcal{B}$-norm by

$$\|f\|_{\mathcal{B}} = \langle f, \mathcal{B}f\rangle^{\frac{1}{2}}$$

$$= \left\| (\mathcal{B})^{\frac{1}{2}} f \right\|, \quad \forall f \in H,$$

with corresponding inner product

$$\langle f, g \rangle_{\mathcal{B}} = \langle f, \mathcal{B} g \rangle, \quad \forall f, g \in H.$$

To begin with, let us define the sequence $\{u_n\}$ induced by steepest descent method on equation (2.6) as

$$u_{n+1} = u_n + \alpha_n r_n, \quad n = 0, 1, 2, \ldots,$$

where u_0 is an arbitrary vector, $r_n = \mathcal{F} - \mathcal{B} u_n$ is the residual vector and

$$\alpha_n = \frac{\langle r_n, r_n \rangle}{\langle r_n, \mathcal{B} r_n \rangle}.$$

First of all, we investigate the convergence of the steepest descent method.

Theorem 4.1. *Let $\{u_n\}$ be the sequence induced by steepest descent method, then for any initial guess u_0, the residual vector r_n satisfies the following inequality*

$$\|r_{n+1}\|_{\mathcal{B}} \leq \left(\frac{B - A}{B} \right)^{\frac{1}{2}} \|r_n\|_{\mathcal{B}}. \tag{4.1}$$

Furthermore, $\{u_n\}$ converges to the exact solution u_* of the equation (2.6).

Proof. We first consider the sequence $\{\|e_n\|_{\mathcal{B}}\}_n$, where $e_n = u_* - u_n$ is the estimation error. Since $\mathcal{B} e_n = r_n$, then

(4.2)

$$\begin{aligned}
\|e_n\|_{\mathcal{B}} - \|e_{n+1}\|_{\mathcal{B}} &= \langle e_n, \mathcal{B} e_n \rangle - \langle e_{n+1}, \mathcal{B} e_{n+1} \rangle \\
&= \langle e_n, r_n \rangle - \langle u_* - u_{n+1}, \mathcal{F} - \mathcal{B} u_{n+1} \rangle \\
&= \langle e_n, r_n \rangle - \langle u_* - u_n - \alpha_n r_n, \mathcal{F} - \mathcal{B} (u_n + \alpha_n r_n) \rangle \\
&= \langle e_n, r_n \rangle - \langle u_* - u_n - \alpha_n r_n, r_n - \alpha_n \mathcal{B} r_n \rangle \\
&= \langle e_n, r_n \rangle - \langle e_n, r_n \rangle + \alpha_n \langle r_n, r_n \rangle \\
&\quad + \alpha_n \langle e_n, \mathcal{B} r_n \rangle + \alpha_n^2 \langle r_n, \mathcal{B} r_n \rangle \\
&= \frac{\langle r_n, r_n \rangle^2}{\langle r_n, \mathcal{B} r_n \rangle} \\
&= \frac{\|r_n\|^4}{\langle r_n, \mathcal{B} r_n \rangle} \\
&= \frac{\|r_n\|^4}{\|r_n\|_{\mathcal{B}}} \\
&> 0.
\end{aligned}$$

Therefore the sequence $\{\|e_n\|_{\mathcal{B}}\}_n$ is monotonically decreasing, bounded below by zero and thus convergent. To continue, by relation (4.2) and by

$$\begin{aligned}\langle e_n, r_n\rangle &= \langle e_n, \mathcal{B}e_n\rangle \\ &= \left\langle \mathcal{B}^{-1}r_n, r_n\right\rangle \\ &= \left\langle r_n, \mathcal{B}^{-1}r_n\right\rangle,\end{aligned}$$

we see

$$\begin{aligned}\langle e_n, r_n\rangle - \langle e_{n+1}, r_{n+1}\rangle &= \langle e_n, \mathcal{B}e_n\rangle - \langle e_{n+1}, \mathcal{B}e_{n+1}\rangle \\ &= \frac{\langle r_n, r_n\rangle \langle r_n, r_n\rangle}{\langle r_n, \mathcal{B}r_n\rangle} \\ &= \frac{\langle r_n, r_n\rangle}{\langle r_n, \mathcal{B}r_n\rangle} \times \frac{\langle r_n, r_n\rangle \langle e_n, r_n\rangle}{\langle r_n, \mathcal{B}^{-1}r_n\rangle} \\ &= \frac{\langle e_n, r_n\rangle}{\beta_n\gamma_n},\end{aligned}$$

where $\beta_n = \frac{\langle r_n, \mathcal{B}r_n\rangle}{\langle r_n, r_n\rangle}$ and $\gamma_n = \frac{\left\langle r_n, \mathcal{B}^{-1}r_n\right\rangle}{\langle r_n, r_n\rangle}$, or equivalently

$$\langle e_{n+1}, r_{n+1}\rangle = \left(1 - \frac{1}{\beta_n\gamma_n}\right)\langle e_n, r_n\rangle. \tag{4.3}$$

But in view of (2.1) we have

$$\begin{aligned}\beta_n\gamma_n &= \frac{\langle r_n, \mathcal{B}r_n\rangle \left\langle r_n, \mathcal{B}^{-1}r_n\right\rangle}{\langle r_n, r_n\rangle \langle r_n, r_n\rangle} \\ &\leq \|\mathcal{B}\| \left\|\mathcal{B}^{-1}\right\| \\ &\leq \frac{B}{A},\end{aligned}$$

and therefore by (4.3) we have

$$\begin{aligned}\langle e_{n+1}, r_{n+1}\rangle &\leq \left(1 - \frac{A}{B}\right)\langle e_n, r_n\rangle \\ &= \left(\frac{B-A}{B}\right)\langle e_n, r_n\rangle,\end{aligned} \tag{4.4}$$

or $\langle \mathcal{B}r_{n+1}, r_{n+1}\rangle \leq \left(\frac{B-A}{B}\right)\langle \mathcal{B}r_n, r_n\rangle$ and thus

$$\|r_{n+1}\|_{\mathcal{B}} \leq \left(\frac{B-A}{B}\right)^{\frac{1}{2}} \|r_n\|_{\mathcal{B}}.$$

This proves the relation (4.1). For the reminder of the theorem, since $\left(\frac{B-A}{B}\right) < 1$, we infer that $r_n \to 0$ as n grows. From this we have

$\mathcal{B}u_n \to \mathcal{F}$. At the end of the proof, we see

$$\begin{aligned}\|u_n - u_*\|_{\mathcal{B}} &= \left\|\mathcal{B}^{-1}\mathcal{B}\,(u_n - u_*)\right\|_{\mathcal{B}}\\ &\le \left\|\mathcal{B}^{-1}\right\|_{\mathcal{B}} \|\mathcal{B}\,(u_n - u_*)\|_{\mathcal{B}}\\ &= \left\|\mathcal{B}^{-1}\right\|_{\mathcal{B}} \|\mathcal{B}u_n - \mathcal{F}\|_{\mathcal{B}}\,.\end{aligned}$$

Thus $u_n \to u_*$, this completes the proof. □

In the setting of frame-based version of steepest descent method, we can see the ratio $\left(\frac{B-A}{B}\right)^{\frac{1}{2}}$ is applicable for both residual convergence analysis, as stated in preceding theorem, and error convergence analysis, as follows.

Corollary 4.2. *The sequence $\{u_n\}$ induced by steepest descent method converges with the rate of a geometric progression with ratio*

$$\|u_{n+1} - u_*\|_{\mathcal{B}} \le \left(\frac{B-A}{B}\right)^{\frac{1}{2}} \|u_n - u_*\|_{\mathcal{B}}\,. \tag{4.5}$$

Proof. First of all, we see that

$$\begin{aligned}\langle e_n, r_n\rangle &= \langle u_* - u_n, \mathcal{B}\,(u_* - u_n)\rangle\\ &= \|u_n - u_*\|_{\mathcal{B}}\,.\end{aligned}$$

Therefore, in view of (4.4) we can write

$$\begin{aligned}\|u_* - u_{n+1}\|_{\mathcal{B}}^2 &= \langle e_{n+1}, r_{n+1}\rangle\\ &\le \left(\frac{B-A}{B}\right)\langle e_n, r_n\rangle\\ &\le \left(\frac{B-A}{B}\right)\|u_* - u_n\|_{\mathcal{B}}^2\,.\end{aligned}$$

This implies inequality (4.5) as we desired. □

Similar to the case of the previous section, the coefficient $\left(\frac{B-A}{B}\right)^{\frac{1}{2}}$ can also develop an exact solution where a tight frame is used for constructing the sequence $\{u_n\}$. Needless to say, loosely speaking, this coefficient is less efficient than the preceding one for the same values of A and B. But this approach takes the advantage of being not restricted for real Hilbert spaces.

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