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Interior Schauder-Type Estimates for Higher-Order Elliptic Operators in Grand-Sobolev Spaces

Bilal T. Bilalov[1*] and Sabina R. Sadigova[2]

ABSTRACT. In this paper an elliptic operator of the m-th order L with continuous coefficients in the n-dimensional domain $\Omega \subset R^n$ in the non-standard Grand-Sobolev space $W^m_{q)}(\Omega)$ generated by the norm $\|\cdot\|_{q)}$ of the Grand-Lebesgue space $L_{q)}(\Omega)$ is considered. Interior Schauder-type estimates play a very important role in solving the Dirichlet problem for the equation $Lu = f$. The considered non-standard spaces are not separable, and therefore, to use classical methods for treating solvability problems in these spaces, one needs to modify these methods. To this aim, based on the shift operator, separable subspaces of these spaces are determined, in which finite infinitely differentiable functions are dense. Interior Schauder-type estimates are established with respect to these subspaces. It should be noted that Lebesgue spaces $L_q(G)$ are strict parts of these subspaces. This work is a continuation of the authors of the work [28], which established the solvability in the small of higher order elliptic equations in grand-Sobolev spaces.

1. Introduction

Solvability problems of partial differential equations in one sense or another has a long and rich history. Remarkable monographs of various well-known mathematicians such as I.G. Petrovsky [1], L.A. Ladyzhenskaya, N.N. Uraltseva [2], L. Bers, F. John, M. Schechter [3], L. Hörmander [4], S.L. Sobolev [5], K. Moren [6], J.L. Lions, E. Magenes [10], V.P. Mikhaylov [7], C. Miranda [29] and others are devoted to this

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direction. This direction was started with the study of the solvability of the considered differential equation in the classical sense. Since, problems of an applied nature required the definition of solutions not only in the classical, but also in some generalized sense (for example, a generalized solution, a weak solution, a strong solution, etc.). It should be noted that questions of the solvability of elliptic equations play an exceptional role in this theory. In this case, interior Schauder-type estimates play an important role for elliptic operators (see, for example, [2, 3]). Naturally, with the appearance of new functional spaces dictated by applied problems, an appropriate theory should be developed. Naturally, with the appearance of new functional spaces dictated by applied problems, an appropriate theory should be developed.

In connection with concrete problems of mechanics and mathematical physics, interest in the study of various questions of mathematics in non-standard spaces of functions has greatly increased. Such spaces include the Lebesgue space with variable summability exponent, the Morrey space, the Grand Lebesgue space, etc. In this regard, various problems of mathematics in such spaces began to be intensively studied. More details on concerning results of recent years can be obtained from books [8, 11–14, 27]. Questions of analysis and approximation theory have been comparatively well studied in Lebesgue and Morrey spaces with variable summability exponents (see, for example, [8, 11, 18–27]). Along with this, questions of harmonic analysis and approximation began to be studied in grand-Lebesgue spaces, and significant results were obtained in this direction (see, for example, [13]). It should be noted that the Morrey and Grand Lebesgue spaces are not separable and, therefore, infinitely differentiable, finite functions are not dense in them. This circumstance makes it impossible to directly apply existing methods to the study of solvability in these spaces. Therefore, we should choose the appropriate subspaces of these spaces and develop an appropriate theory for this case.

In this paper an elliptic operator of the m-th order L with continuous coefficients in the n-dimensional domain $\Omega \subset R^n$ in the non-standard Grand-Sobolev space $W^m_{q)}(\Omega)$ generated by the norm $\|\cdot\|_{q)}$ of the Grand-Lebesgue space $L_{q)}(\Omega)$ is considered. Interior Schauder-type estimates play a very important role in solving the Dirichlet problem for the equation $Lu = f$. The considered non-standard spaces are not separable, and therefore, to use classical methods for treating solvability problems in these spaces, one needs to modify these methods. To this aim, based on the shift operator, separable subspaces of these spaces are determined, in which finite infinitely differentiable functions are dense. Interior Schauder-type estimates are established with respect to these subspaces. It should be noted that Lebesgue spaces $L_q(G)$ are strict parts of these subspaces. This work is a continuation of the authors of the work [28], which established the solvability in the small of higher order elliptic equations in grand-Sobolev spaces.

2. Needful Information

We will use the following standard notations. Z_+ will be the set of non-negative integers. $B_r(x_0) = \{x \in R^n : |x - x_0| < r\}$ will denote the open ball in R^n centered at x_0, where $|x| = \sqrt{x_1^2 + \cdots + x_n^2}$, $x = (x_1, \ldots, x_n)$. $\Omega_r(x_0) = \Omega \bigcap B_r(x_0)$, $B_r = B_r(0)$, $\Omega_r = \Omega_r(0)$. $mes\,(M)$ will stand for the Lebesgue measure of the set M; $\partial\Omega$ will be the boundary of the domain Ω; $\bar{\Omega} = \Omega \bigcup \partial\Omega$; $M_1 \Delta M_2$ will denote the symmetric difference between the sets M_1 and M_2; $diam\,\Omega$ will stand for the diameter of the set Ω; $\rho(x; M)$ will be the distance between x and the set M; and $\|T\|_{X \to Y}$ will denote the norm of the operator T, acting boundedly from X to Y.

2.1. **Elliptic Operator of m-th Order.** Let $\Omega \subset R^n$ be some bounded domain with the rectifiable boundary $\partial\Omega$. We will use the notations of [3]. $\alpha = (\alpha_1, \ldots, \alpha_n)$ will be the multiindex with the coordinates $\alpha_k \in Z_+$, $\forall k = \overline{1, n}$; $\partial_i = \frac{\partial}{\partial x_i}$ will denote the differentiation operator, $\partial^\alpha = \partial_1^{\alpha_1} \partial_2^{\alpha_2} \ldots \partial_n^{\alpha_n}$. For every $\xi = (\xi_1, \ldots, \xi_n)$ we assume $\xi^\alpha = \xi_1^{\alpha_1} \xi_2^{\alpha_2} \ldots \xi_n^{\alpha_n}$. Let L be an elliptic differential operator of m-th order:

$$L = \sum_{|p| \le m} a_p(x)\, \partial^p, \tag{2.1}$$

where $p = (p_1, \ldots, p_n)$, $p_k \in Z_+$, $\forall k = \overline{1, n}$, $a_p(\cdot) \in L_\infty(\Omega)$ are real functions, i.e. the characteristic form

$$Q(x, \xi) = \sum_{|p| = m} a_p(x)\, \xi^p,$$

is defined at every point $x \in \Omega$. It is known that in this case m is even. Let $m = 2m'$, and assume without loss of generality that

$$(-1)^{m'} Q(x, \xi) > 0, \quad \forall \xi \ne 0, \forall x \in \Omega.$$

Consider the elliptic operator L_0:

$$L_0 = \sum_{|p| = m} a_p^0\, \partial^p, \tag{2.2}$$

with the constant coefficients a_p^0.

In what follows, by solution of the equation $Lu = f$ we mean a strong solution (see [3]), i.e. u belongs to the corresponding space and satisfies a.e. equality $Lu = f$.

We will need the following classical result of [3].

Theorem 2.1 ([3])**.** *For an arbitrary elliptic operator of m-th order L_0 of the form (2.2) with the constant coefficients, the function $J(x)$ can be constructed which has the following properties:*

i) If n is odd or if n is even and $n > m$, then

$$J(x) = \frac{\omega(x)}{|x|^{n-m}},$$

where $\omega(x)$ is a positive homogeneous function of degree zero (i.e. $\omega(tx) = \omega(x)$, $\forall t > 0$). If n is even and $n \le m$, then

$$J(x) = q(x) \log|x| + \frac{\omega(x)}{|x|^{n-m}},$$

where q is a homogeneous polynomial of degree $m - n$.

ii) The function $J(x)$ satisfies (in a generalized sense) the equation

$$L_0 J(x) = \delta(x),$$

(δ is a Dirac function) so the following equality is true for every infinitely differentiable function $\varphi(\cdot)$ with compact support

$$\begin{aligned} \varphi(x) &= \int [L_0 \varphi(y)]\; J(x-y)\, dy \\ &= L_0 \int \varphi(y)\, J(x-y)\, dy. \end{aligned}$$

Let's consider the elliptic operator (2.1) and assign a "tangential operator"

$$L_{x_0} = \sum_{|p|=m} a_p\,(x_0)\, \partial^p, \tag{2.3}$$

to it at every point $x_0 \in \Omega$. Denote by $J_{x_0}(\cdot)$ the fundamental solution of the equation $L_{x_0}\varphi = 0$ in accordance with Theorem 2.1. The function $J_{x_0}(\cdot)$ is called a parametrics for the equation $L\varphi = 0$ with a singularity at the point x_0. Let

$$\begin{aligned} S_{x_0}\varphi &= \psi(x) \\ &= \int J_{x_0}(x-y)\; \varphi(y)\, dy, \end{aligned}$$

and

$$T_{x_0} = S_{x_0}(L_{x_0} - L). \tag{2.4}$$

In establishing the existence of the solution to the equation $Lu = f$, the significant role is played by the following lemma.

Lemma 2.2 ([3])**.** *If φ has compact support, then*

$$\varphi = T_{x_0}\varphi + S_{x_0}L\varphi,$$

and if

$$\varphi = T_{x_0}\varphi + S_{x_0}f,$$

then $L\varphi = f$.

2.2. **Grand-Sobolev spaces $W_{q)}^{m}(\Omega)$ and $WG_{q)}^{m}(\Omega)$.** Let's first define the grand-Lebesgue space $L_{q)}(\Omega)$, $1 < q < +\infty$ (throughout the paper we will assume that this assumption holds true). Grand-Lebesgue space $L_{q)}(\Omega)$ is a Banach space of (Lebesgue) measurable functions f on Ω with the norm

$$\|f\|_{q)} = \sup_{0<\varepsilon<q-1}\left(\varepsilon\int_{\Omega}|f|^{q-\varepsilon}\,dx\right)^{\frac{1}{q-\varepsilon}}, \quad 1 < q < +\infty.$$

It should be noted that, in generally, the norm of the space $L_{q)}(\Omega)$ has a form

$$\|f\|_{q),\,\Omega} = \sup_{0<\varepsilon<q-1}\left(\frac{\varepsilon}{|\Omega|}\int_{\Omega}|f|^{q-\varepsilon}\,dx\right)^{\frac{1}{q-\varepsilon}}.$$

It is easy to see that the norms $\|f\|_{q)}$ and $\|f\|_{q),\,\Omega}$ are equivalent and that is why for simplicity we will consider the norm $\|f\|_{q)}$.

The following continuous embeddings hold

$$L_q(\Omega) \subset L_{q)}(\Omega) \subset L_{q-\varepsilon}(\Omega),$$

where $\varepsilon \in (0, q-1)$ is an arbitrary number. The space $L_{q)}(\Omega)$ is not separable and the following theorem is true.

Theorem 2.3 ([15, Theorem 7.34, p. 304])**.** *The set $C_0^{\infty}(\Omega)$ of finite and infinitely differentiable functions in Ω is not dense in $L_{q)}(\Omega)$. The closure $\overline{C_0^{\infty}(\Omega)}$ in $L_{q)}(G)$ consists of the functions f which satisfy*

$$\lim_{\varepsilon\to 0}\varepsilon\int_{\Omega}|f|^{q-\varepsilon}\,dx = 0.$$

Below in this section we will assume that every function defined on Ω is extended by zero to $R^n\backslash\bar{\Omega}$. Let T_δ be a shift operator, i.e. $(T_\delta f)(x) = f(\delta + x)$, $\forall x \in \Omega$, where $\delta \in R^n$ is an arbitrary vector. Let

$$G_{q)}(\Omega) = \left\{f \in L_{q)}(\Omega) : \|T_\delta f - f\|_{q)} \to 0,\ \delta \to 0\right\}.$$

With the norm $\|\cdot\|_{q)}$, $G_{q)}(\Omega)$ becomes a Banach space (i.e. the subspace of $L_{q)}(\Omega)$).

In [29] the following lemma is established.

Lemma 2.4. $\overline{C_0^\infty(\Omega)} = G_{q)}(\Omega)$ *(the closure is taken with regard to the norm $\|\cdot\|_{q)}$).*

Moreover, the following lemma is true.

Lemma 2.5. *The following embeddings hold*

$$L_q(\Omega) \subset G_{q)}(\Omega) \subset L_{q)}(\Omega) \subset L_1(\Omega),$$

and every inclusion is strict.

Proof. It is sufficient to carry out the proof for the one-dimensional case, since the presented examples can be easily carried over to the n-dimensional case. So, consider the case $\Omega = (0,1)$. Since the space $L_{q)}(0,1)$ is not separable, then from Lemma 2.4 it immediately follows that the embedding

$$G_{q)}(0,1) \subset L_{q)}(0,1),$$

is strict. We have

$$\begin{aligned} \|f\|_{L_{q)}(0,1)} &\leq \sup_{0<\varepsilon<q-1} \varepsilon^{\frac{1}{q-\varepsilon}} \|f\|_{L_{q-\varepsilon}(0,1)} \\ &\leq \|f\|_{L_q(0,1)} \sup_{0<\varepsilon<q-1} \varepsilon^{\frac{1}{q-\varepsilon}} 2^{\frac{\varepsilon}{q(q-\varepsilon)}} \\ &\leq (q-1)\, 2^{-\frac{1}{q}} \|f\|_{L_q(0,1)}. \end{aligned} \tag{2.5}$$

Considering that

$$T_\delta f - f, \delta \to 0 \text{ in } L_q(0,1), \quad \forall f \in L_q(0,1),$$

from inequality (2.5) it directly follows that $L_q(0,1) \subset G_{q)}(0,1)$. Let us show that this inclusion is strict. Let's construct a corresponding example. Consider the following sequence of functions

$$f_n(t) = \begin{cases} t^{-\frac{1}{q}}, & t \in \left[e^{-n^{2q}}, 1\right], \\ 0, & t \in \left[0, e^{-n^{2q}}\right). \end{cases}$$

For norms $\|f_n\|_{L_{q)}(0,1)}$ and $\|f_n\|_{L_q(0,1)}$ we have

$$\begin{aligned} \|f_n\|_{L_{q)}(0,1)} &\leq \sup_{0<\varepsilon<q-1} \left(\varepsilon \int_0^1 t^{-1+\frac{\varepsilon}{q}} dt\right)^{\frac{1}{q-\varepsilon}} \\ &= q, \\ \|f_n\|_{L_q(0,1)} &= \left(\int_{e^{-n^{2q}}}^1 t^{-1} dt\right)^{\frac{1}{q}} \\ &= n^2. \end{aligned}$$

Therefore, the series

$$\sum_{n=1}^{\infty} \frac{\|f_n\|_{L_{q)}(0,1)}}{n^2},$$

converges. This immediately implies that the series

$$\sum_{n=1}^{\infty} \frac{f_n(t)}{n^2},$$

converges in $L_{q)}(0,1)$ and its sum is denoted by $f(t)$:

$$f(t) = \sum_{n=1}^{\infty} \frac{f_n(t)}{n^2}.$$

Since $f_n \in G_{q)}(0,1)$, $\forall n \in N$, then it is clear that $f \in G_{q)}(0,1)$. Let us show that $f \notin L_q(0,1)$. Let

$$S_m(t) = \sum_{n=1}^{m} \frac{f_n(t)}{n^2}, \quad \forall m \in N.$$

We have

$$0 \leq S_m(t) \leq S_{m+1}(t), \text{ a.e. } t \in (0,1), \quad \forall m \in N.$$

As $S_m \to f$, $m \to \infty$, in $L_q(0,1)$, $\forall \varepsilon \in (0, q-1)$, then it is clear that $|S_m(t)|^q \to |f(t)|^q$, $m \to \infty$, a.e. $t \in (0,1)$.

Then, by Levi's theorem, we obtain

$$\int_0^1 |S_m(t)|^q \, dt \to \int_0^1 |f(t)|^q \, dt, m \to \infty.$$

On the other hand, from the relations

$$\begin{aligned}
\int_0^1 |S_m(t)|^q \, dt &= \int_0^1 \left| \sum_{n=1}^{m} \frac{f_n(t)}{n^2} \right|^q dt \\
&\geq \sum_{n=1}^{m} \frac{\int_0^1 |f_n(t)|^q \, dt}{n^{2q}} \\
&= \sum_{n=1}^{m} \frac{n^{2q}}{n^{2q}} \\
&= m \to \infty,
\end{aligned}$$

it follows that

$$\int_0^1 |S_m(t)|^q \, dt \to \infty, \ m \to \infty.$$

Consequently, $f \notin L_q(0,1)$. Lemma is proved. □

Denote by $W^m_{q)}(\Omega)$ the grand-Sobolev space generated by the norm

$$\|f\|_{W^m_{q)}} = \sum_{k=0}^{m} \left\| f^{(k)} \right\|_{q)} .$$

Let

$$WG^m_{q)}(\Omega) = \left\{ f \in W^m_{q)}(\Omega) : \|T_\delta f - f\|_{W^m_{q)}} \to 0,\, \delta \to 0 \right\}.$$

Consider the following singular kernel

$$k(x) = \frac{\omega(x)}{|x|^n},$$

where $\omega(x)$ is a positive homogeneous function of degree zero, which is infinitely differentiable and satisfies

$$\int_{|x|=1} \omega(x)\, d\sigma = 0,$$

$d\sigma$ being a surface element on the unit sphere. Denote by K the corresponding singular integral

$$\begin{aligned}(Kf)(x) &= k * f(x) \\ &= \int_\Omega f(y)\, k(x-y)\, dy.\end{aligned}$$

The theorem below was proved in [13].

Theorem 2.6 ([13])**.** *The singular operator K is acting boundedly in $L_{q)}(\Omega)$, $1 < q < +\infty$, i.e. $\exists c > 0$:*

$$\|Kf\|_{q)} \le c\, \|f\|_{q)}, \quad \forall f \in L_{q)}(\Omega) .$$

Here c is a constant independent of f (in what follows, c or C will denote constants, may be different in different places) . The following lemma from [28] is true.

Lemma 2.7 ([28])**.** *$G_{q)}(\Omega)$, $1 < q < +\infty$, is an invariant subspace of the singular operator K in $L_{q)}(\Omega)$.*

To obtain our main results, we also need the following Minkowski inequality for convolution in R^n.

Theorem 2.8 ([17])**.** *Let $f \in L_1(R^n)$ and $g \in L_q(R^n)$, $1 \le q \le +\infty$. Then $f * g \in L_q(R^n)$ and*

$$\|f * g\|_{L_q(R^n)} \le \|f\|_{L_1(R^n)} \|g\|_{L_q(R^n)} .$$

Concerning this result see, for example, [17, pp. 19].

In the sequel, when $\Omega = B_r$ the spaces $L_{q)}(\Omega)$, $G_{q)}(\Omega)$, $W_{q)}(\Omega)$ and $WG_{q)}(\Omega)$ will be redenoted by $L_{q)}(r)$, $G_{q)}(r)$, $W_{q)}^m(r)$ and $WG_{q)}^m(r)$, respectively. Along with $WG_{q)}^m(\Omega)$, consider the following space of functions $N_{q)}^m(\Omega)$ equipped with the norm

$$\|f\|_{N_{q)}^m(\Omega)} = \sum_{|p|\le m} d_\Omega^{|p|-\frac{n}{q}} \|\partial^p f\|_{L_{q)(\Omega)}},$$

where $d_\Omega = diam\,\Omega$. Also accept

$$\|f\|_{N_q^m(\Omega)} = \sum_{|p|\le m} d_\Omega^{|p|-\frac{n}{q}} \|\partial^p f\|_{L_q(\Omega)},$$

$\|\cdot\|_{L_q(\Omega)}$where is an ordinary norm in $L_q(\Omega)$.

It is not difficult to see that the norms of the spaces $WG_{q)}^m(\Omega)$ and $N_{q)}^m(\Omega)$ are equivalent to each other, and therefore their stocks of functions coincide. Consequently, it suffices to prove the existence in the small of the solution to the equation $Lu = f$ in $WG_{q)}^m(\Omega)$ for the space $N_{q)}^m(\Omega)$. But first let's introduce the following definition.

Definition 2.9. We will say that the operator L has the property $P_{x_0})$ if its coefficients satisfy the conditions: i) $a_p \in L_\infty(B_r(x_0))$, $\forall\,|p| \le m$, for some $r > 0$; ii) $\exists r > 0$: for $|p| = m$ the coefficient $a_p(\cdot)$ coincides a.e. in $B_r(x_0)$ with some function bounded and continuous at the point x_0.

It is absolutely clear that if $a_p \in C(\Omega)$, $\forall\,|p| \le m$, then L has the property $P_{x_0})$ for $\forall x_0 \in \Omega$.

Let's consider the m-th order elliptic operator L with the coefficients $a_p(x)$ defined by (2.1), and the corresponding operator T_{x_0} defined by (2.4). Denote the operators S_{x_0}, L_{x_0} and T_{x_0}, corresponding to the point $x_0 = 0$, by S_0, L_0 and T_0, respectively.

The following lemma from the monograph [3, p. 225] has a crucial role in subsequent studies.

Lemma A. Let L be m-th order elliptic differential operator, L_{x_0} be the corresponding tangential operator at the point $x_0 \in \Omega$, $J_{x_0}(\cdot)$ be the fundamental solution of the equation $L_{x_0}\varphi = 0$; $S_{x_0}\varphi = J_{x_0} * \varphi$ and $T_{x_0} = S_{x_0}(L_{x_0} - L)$. Then, if φ has a compact support, then

$$\varphi = T_{x_0}\varphi + S_{x_0}L\varphi,$$

and if $\varphi = T_{x_0}\varphi + S_{x_0}f$, then φ is a solution of the equation $L\varphi = f$.

In obtaining the main results, we will essentially use the following Main Lemma, proved in [28].

Main Lemma. *Let the m-th order elliptic operator L have the property $P_{x_0})$ at the point x_0. Let $\varphi \in N^m_{q)}(B_r(x_0))$ and φ vanish in a neighborhood of $|x - x_0| = r$. Then for $q > 1$ it holds*

$$\|T_{x_0}\varphi\|_{N^m_{q)}(B_r(x_0))} \leq \sigma(r) \|\varphi\|_{N^m_{q)}(B_r(x_0))},$$

where the function $\sigma(r) \to 0$, $r \to 0$, depends only on the ellipticity constant L_{x_0}, on the coefficients of L and their moduli of continuity.

We will also need a space $C^m(\Omega)$ consisting of m-times continuously differentiable functions in Ω with the norm

$$\|f\|_{C^m(\Omega)} = \sum_{|p| \leq m} d_\Omega^{|p|} \|\partial^p f\|_{L_\infty(\Omega)}.$$

And also assume $C^m(R) = C^m(B_R)$.

3. Main Results

Before formulating the main results in accordance with the monograph [3], we prove the following lemmas, which we will make essential use of.

Let $\omega(\cdot)$ be an infinitely differentiable function on $[0,1]$ such that for $0 \leq t < \frac{1}{3}$, $\omega(t) \equiv 1$ and for $\frac{2}{3} < t \leq 1$, $\omega(t) \equiv 0$. For $0 < R_1 < R_2$ we put

$$\xi(x) = \begin{cases} 1, & |x| \leq R_1, \\ \omega\left(\frac{|x| - R_1}{R_2 - R_1}\right), & R_1 < |x| \leq R_2. \end{cases} \tag{3.1}$$

Lemma 3.1. *There is a constant $C > 0$ depending only on R_2 and $\omega(\cdot)$, such that for $\forall R_1 : 0 < R_1 < R_2$, there is*

$$\|\xi\|_{C^m(R_2)} \leq C\left(1 - \frac{R_1}{R_2}\right)^{-m}. \tag{3.2}$$

Proof. It should be noted that for $|x| < \frac{1}{3}R_2 + \frac{2}{3}R_1$ it holds $\xi(x) \equiv 1$ and consequently, for $|x| < \frac{1}{3}R_2 + \frac{2}{3}R_1$ the relation $\partial^p\xi(x) \equiv 0$ is true. We have

$$\begin{aligned} \partial_{x_k}\xi(x) &= \frac{1}{R_2 - R_1}\omega'\left(\frac{|x| - R_1}{R_2 - R_1}\right)\partial_{x_k}|x| \\ &\leq \frac{C}{R_2 - R_1}, \end{aligned}$$

as $|\partial_{x_k}|x|| = \left|\frac{x_k}{|x|}\right| \leq 1$. Similarly

$$\partial^2_{x_k x_j}\xi(x) = \frac{1}{(R_2 - R_1)^2}\omega''\left(\frac{|x| - R_1}{R_2 - R_1}\right)\ \partial_{x_k}|x|\,\partial_{x_j}|x| \\ + \frac{1}{R_2 - R_1}\omega'\left(\frac{|x| - R_1}{R_2 - R_1}\right)\ \partial^2_{x_k x_j}|x|$$

then

$$\left|\partial^2_{x_k x_j}\xi\right| \leq \frac{C}{(R_2 - R_1)^2} + \frac{C}{R_2 - R_1}\frac{1}{|x|} \\ \leq \frac{1}{(R_2 - R_1)^2}\frac{C}{|x|},$$

for sufficiently small $R_2 > 0$ (e.g., $R_2 \leq 1$). Continuing this process, we establish

$$|\partial^p \xi(x)| \leq \frac{1}{(R_2 - R_1)^{|p|}}\frac{C}{|x|^{|p|-1}}, \quad \forall p : |p| \leq m.$$

Putting $M_{R_2} = B_{R_2} \backslash B_{\frac{1}{3}R_2}$, we have

$$R_2^{|p|}\|\partial^p\xi\|_{C(R_2)} \leq \frac{CR_2^{|p|}}{(R_2 - R_1)^{|p|}}\left\|\frac{1}{|x|^{|p|-1}}\right\|_{C\left(M_{R_2}\right)} \\ \leq C\left(1 - \frac{R_1}{R_2}\right)^{-|p|}\left\|\frac{1}{|x|^{|p|-1}}\right\|_{C\left(M_{R_2}\right)}.$$

As a result

$$\|\xi\|_{C^m(R_2)} \leq C\left(1 - \frac{R_1}{R_2}\right)^{-m}, \quad \forall R_1 : 0 < R_1 < R_2,$$

where $C > 0-$dependent only on R and $\omega(\cdot) -$ constant.

Lemma is proved. □

We also need the following

Lemma 3.2. *Let L be an m-th order elliptic differential operator whose coefficients satisfy the conditions: $a_p(\cdot) \in C\left(\overline{B_{R_2}}\right), \forall p : |p| = m$; $a_p(\cdot) \in L_\infty(B_{R_2}), \forall p : |p| < m$. Then there exists $C > 0$ depending only on R_2 and on the coefficients L, such that for $\forall u \in N^m_{q)}(R_2)$ the following inequality holds $(1 < q < +\infty)$*

$$\|u\|_{N^m_{q)}(R_1)} \leq C\left(1 - \frac{R_1}{R_2}\right)^{-m}\left(\|Lu\|_{L_{q)}(R_2)} + \|u\|_{N^{m-1}_{q)}(R_2)}\right), \tag{3.3}$$

for all $R_1 : 0 < R_1 < R_2$.

Proof. Let $u \in N^m_{q)}(R_2)$ and consider the function $\xi(\cdot)$, defined by expression (3.1). Assume $\varphi = \xi u$. It is obvious that $\operatorname{supp}\varphi \subset B_{R_2}$. Therefore, by Lemma A we have

$$\varphi = T_0\varphi + S_0 L\varphi. \tag{3.4}$$

By the Main Lemma, $\exists R > 0-$ is so small that the relation

$$\|T_0\varphi\|_{N^m_{q)}(R_2)} \leq \frac{1}{2}\|\varphi\|_{N^m_{q)}(R_2)},$$

holds true for $\forall R_2 < R$. R_2 is selected based on this relation. Then from (3.4) we obtain

$$\|\varphi\|_{N^m_{q)}(R_2)} \leq 2\|S_0 L\varphi\|_{N^m_{q)}(R_2)}.$$

So

$$S_0 L\varphi = \int_{B_{R_2}} J_0(x-y)\, L\varphi(y)\, dy.$$

As shown in the monograph [3, p. 235], for $|p| = m$ we have the formula

$$\partial^p S_0 L\varphi = \int_{B_{R_2}} \partial^p_x J_0(x-y)\, L\varphi(y)\, dy + C L\varphi(x), \tag{3.5}$$

where $C \neq 0-$ is a constant. The kernel $\partial^p J_0(x)$ is singular for $|p| = m$ and applying Theorem 2.6 to (3.5) we obtain

$$\|\partial^p S_0 L\varphi\|_{L_{q)}(R_2)} \leq C\|L\varphi\|_{L_{q)}(R_2)}. \tag{3.6}$$

Consider the case $|p| < m$. We have

$$\partial^p S_0 L\varphi = \int_{B_{R_2}} \partial^p_x J_0(x-y)\, L\varphi(y)\, dy.$$

In this case a kernel has a weak singularity and it was shown in [28] that the inequality

$$R_2^{|p|}\|\partial^p S_0 L\varphi\|_{L_{q)}(R_2)} \leq C R_2^m \|L\varphi\|_{L_{q)}(R_2)}, \tag{3.7}$$

holds. Taking into account inequalities (3.6) and (3.7) for the norm $\|S_0 L\varphi\|_{N^m_{q)}(R_2)}$, we obtain

$$\|S_0 L\varphi\|_{N^m_{q)}(R_2)} \leq C R_2^m \|L\varphi\|_{L_{q)}(R_2)}. \tag{3.8}$$

On the other hand, it is easy to see that $L\varphi$ can be represented in the form

$$L\varphi = \xi\, Lu + M(u;\xi), \tag{3.9}$$

where $M(u;\xi)$ is a linear combination of derivatives $\partial^p u$, the order $|p|$ of which does not exceed $m-1$, multiplied by the derivatives of ξ of order at most m, and the derivatives $\partial^p u$ of the order $|p|$ are multiplied by the derivatives of ξ of order $(m-|p|)$, i.e.

$$M(u;\xi)=\sum_{|p|<m} C_p(x)\,\partial^{\tilde{p}}\xi\partial^p u,$$

holds, where $|\tilde{p}|=m-|p|$ and $C_p(\cdot)-$ is some linear combination of the coefficients of the operator L. Consequently

$$R_2^m M(u;\xi)=\sum_{|p|<m} C_p(x)\,R_2^{|\tilde{p}|}\partial^{\tilde{p}}\xi\,R_2^{|p|}\partial^p u$$

then

$$R_2^m M(u;\xi)=R_2^{\frac{n}{q}}\sum_{|p|<m} C_p(x)\,R_2^{|\tilde{p}|}\partial^{\tilde{p}}\xi\,R_2^{|p|-\frac{n}{q}}\partial^p u.$$

Hence we immediately get

$$R_2^m\left\|M(u;\xi)\right\|_{L_{q)}(R_2)}\leq C\left\|\xi\right\|_{C^m(R_2)}\left\|u\right\|_{N_{q)}^{m-1}(R_2)}.$$

As a result, it follows from (3.9) that

$$\begin{aligned}
&R_2^m\left\|L\varphi\right\|_{L_{q)}(R_2)}\\
&\quad\leq R_2^m\left(\left\|\xi\right\|_{C(R_2)}\left\|Lu\right\|_{L_{q)}(R_2)}+\left\|M(u;\xi)\right\|_{L_{q)}(R_2)}\right)\\
&\quad\leq C\left\|\xi\right\|_{C^m(R_2)}\left(\left\|Lu\right\|_{L_q(R_2)}+\left\|u\right\|_{N_{q)}^{m-1}(R_2)}\right)\\
&\quad\leq C\left(1-\frac{R_1}{R_2}\right)^{-m}\left(\left\|Lu\right\|_{L_{q)}(R_2)}+\left\|u\right\|_{N_{q)}^{m-1}(R_2)}\right),\ \text{(by Lemma 3.1)}
\end{aligned}$$

where C is the constant depending only on R_2 and coefficients $a_p(\cdot)$.

Lemma is proved. □

Before formulating the next lemma, we accept the following property with respect to the domain Ω.

Property 3.3. We say that the domain Ω admits the continuation of functions of the space $N_{q)}^k(\Omega)$ if there exists a domain $\Omega'\supset\bar{\Omega}$ and a linear mapping θ of the space $N_{q)}^k(\Omega)$ into $N_{q)}^k(\Omega')$ such that $\theta u=u\in\Omega$,

$$\left\|\theta u\right\|_{N_{q)}^k(\Omega')}\leq const,\left\|u\right\|_{N_{q)}^k(\Omega)},\tag{3.10}$$

holds.

So, let us prove the following

Lemma 3.4. *Let the domain Ω have the Property α) with respect to space $N^k_{q)}(\Omega)$. Then $\exists C>0$ depending only on n, q and on a constant from* (3.10)*, which holds*

$$\|\varphi\|_{N^k_{q)}(\Omega)} \le \varepsilon \|\varphi\|_{N^{k+1}_{q)}(\Omega)} + C\varepsilon^{-k} \|\varphi\|_{L_{q)}(\Omega)}, \tag{3.11}$$

for $\forall k=\overline{1,m-1}$ and $\forall \varepsilon>0$.

Proof. First, without loss of generality we can assume that $d_\Omega = diam\,\Omega = 1$. Indeed, making the change of variables $y = d_\Omega^{-1}x$ we have

$$\begin{aligned}\left|y'-y''\right| &= d_\Omega^{-1}\left|x'-x''\right| \\ &\le 1, \quad \forall x',x''\in\Omega.\end{aligned}$$

It is clear that

$$\begin{aligned}dx &= d_\Omega^n dy, \partial_x^p u \\ &= d_\Omega^{-|p|}\partial_y^p v,\end{aligned}$$

where $v(y) = u(d_\Omega y)$. Consequently

$$\begin{aligned}\|\partial^p u\|_{L_r(\Omega)} &= \left(\int_\Omega |\partial_x^p u(x)|^r dx\right)^{1/r} \\ &= d_\Omega^{-|p|+\frac{n}{r}}\left(\int_{\Omega'} \left|\partial_y^p v(y)\right|^r dy\right)^{1/r} \\ &= d_\Omega^{-|p|+\frac{n}{r}}\|v\|_{L_r(\Omega')},\end{aligned}$$

where $\Omega' : \Omega \overset{y=d_\Omega^{-1}x}{\longrightarrow} \Omega'$ is the image of Ω under the mapping $y = d_\Omega^{-1}x$. Putting here $r = q-\varepsilon$ by definition of the norm of $N^m_{q)}$ we have

$$\begin{aligned}\|u\|_{N^m_{q)}(\Omega)} &= \sup_{0<\varepsilon<q-1} \varepsilon^{\frac{1}{q-\varepsilon}}\|u\|_{N^m_{q-\varepsilon}(\Omega)} \\ &= \sup_{0<\varepsilon<q-1} d_\Omega^{\frac{n}{q-\varepsilon}}\varepsilon^{\frac{1}{q-\varepsilon}}\|v\|_{N^m_{q)}(\Omega')}.\end{aligned}$$

It directly follows that

$$\|u\|_{N^m_{q)}(\Omega)} \sim \|u(d_\Omega\,\cdot)\|_{N^m_{q)}(\Omega')},$$

i.e. the norms $\|\cdot\|_{N^m_{q)}(\Omega)}$ and $\|\cdot\|_{N^m_{q)}(\Omega')}$ are equivalent. As a result, it is clear that we can assume $d_\Omega = 1$.

Let $\sigma : 0<\sigma<q-1$–be an arbitrary number. As established in monograph [3] (see p. 261; inequality (3.2)), $\forall h>0$:

$$\|\partial^p f\|^{q_\sigma}_{L_{q_\sigma}(\infty)} \le 2^{q_\sigma-1}h^{q_\sigma}\|\partial^{p_2} f\|^{q_\sigma}_{L_{q_\sigma}(\infty)} + 2^{2(q_\sigma-1)}h^{-q_\sigma}\|\partial^{p_1} f\|^{q_\sigma}_{L_{q_\sigma}(\infty)},$$

is true, where $p_k : |p_k - p| = 1$, $k = 1;2$; $p_2 \geq p \geq p_1-$are arbitrary multindexes (we assume that the multindex p_1 is less than or equal to the multindex p_2 if each component of p_1 is less than or equal to the corresponding component of p_2). Paying attention to the inequality

$$(a+b)^r \leq a^r + b^r,$$

for $\forall a;b \geq 0$ and $0 \leq r \leq 1$, we have

$$\|\partial^p f\|_{L_{q_\sigma}(\infty)} \leq 2^{1-\frac{1}{q_\sigma}} h \|\partial^{p_2} f\|_{L_{q_\sigma}(\infty)} + 2^{2\left(1-\frac{1}{q_\sigma}\right)} h^{-1} \|\partial^{p_1} f\|_{L_{q_\sigma}(\infty)} .$$

Consequently

$$\|\partial^p f\|_{L_{q_\sigma}(\infty)} \leq 2^{1-\frac{1}{q}} h \|\partial^{p_2} f\|_{L_{q_\sigma}(\infty)} + 2^{2\left(1-\frac{1}{q}\right)} h^{-1} \|\partial^{p_1} f\|_{L_{q_\sigma}(\infty)} .$$

Multiplying both parts by $\varepsilon^{\frac{1}{q_\sigma}}$ and taking $\sup\limits_{0<\varepsilon<q-1}$ we get

$$\|\partial^p f\|_{L_{q)}(\infty)} \leq 2^{1-\frac{1}{q}} h \|\partial^{p_2} f\|_{L_{q)}(\infty)} + 2^{2\left(1-\frac{1}{q}\right)} h^{-1} \|\partial^{p_1} f\|_{L_{q)}(\infty)} .$$

Summing over all $p_2 : |p_2| = k+1$, for $\forall h > 0$ we have

$$\begin{aligned} n \sum_{|p|=k} \|\partial^p f\|_{L_{q)}(\infty)} \leq 2^{1-\frac{1}{q}} h &\sum_{|p|=k+1} \|\partial^p f\|_{L_{q)}(\infty)} \\ &+ n^2 2^{2\left(1-\frac{1}{q}\right)} h^{-1} \sum_{|p|=k-1} \|\partial^p f\|_{L_{q)}(\infty)} . \end{aligned}$$

Denoting $\frac{2^{1-\frac{1}{q}}}{n} h = \varepsilon$ and $C_{n;q} = n 2^{3\left(1-\frac{1}{q}\right)}$, we obtain

$$\sum_{|p|=k} \|\partial^p f\|_{L_{q)}(\infty)} \leq \varepsilon \sum_{|p|=k+1} \|\partial^p f\|_{L_{q)}(\infty)} + C_{n;q}\varepsilon^{-1} \sum_{|p|=k-1} \|\partial^p f\|_{L_{q)}(\infty)} .$$

As a result for $\forall k = \overline{1, m-1}$ and $\forall \varepsilon > 0$ we have

$$\|f\|_{N^k_{q)}(\infty)} \leq \varepsilon \|f\|_{N^{k+1}_{q)}(\infty)} + C_{n;q}\varepsilon^{-1} \|f\|_{N^{k-1}_{q)}(\infty)} , \tag{3.12}$$

where $C_{n;q}-$ is a constant depending only on n and q.

Assume $A_k = \|f\|_{N^k_{q)}(\infty)}$. Then from (3.12) we obtain

$$\begin{aligned} A_1 &\leq \varepsilon_1 A_2 + C_{n;q}\ \varepsilon_1^{-1} A_0, \\ A_2 &\leq \varepsilon_2 A_3 + C_{n;q}\ \varepsilon_2^{-1} A_1, \end{aligned}$$

where $\varepsilon_1; \varepsilon_2 > 0-$are arbitrary numbers. Consequently

$$A_2 \leq \varepsilon_2 A_3 + C_{n;q}\ \varepsilon_1 \varepsilon_2^{-1} A_2 + C^2_{n;q} \varepsilon_1^{-1} \varepsilon_2^{-1} A_0.$$

Taking $\varepsilon_1 = \frac{\varepsilon_2}{2C_{n;q}}$ and $\varepsilon_2 = \frac{\varepsilon}{2}$ we obtain

$$A_2 \leq \varepsilon A_3 + C_{2;3}\varepsilon^{-2} A_0,$$

where $C_{2;3}-$is a constant depending only on n and q, $\varepsilon > 0-$is an arbitrary number. Continuing this process, we obtain the validity of the estimate

$$A_k \leq \varepsilon A_{k+1} + C_{k;k+1}\varepsilon^{-k} A_0,$$

for $\forall k = \overline{1, m-1}$; $\forall \varepsilon > 0$, where $C_{k;k+1}$ is a constant depending only on n and q. Taking $\max\limits_k C_{k;k+1}$ as the final result we obtain

$$\|f\|_{N^k_{q)}(\infty)} \leq \varepsilon \|f\|_{N^{k+1}_{q)}(\infty)} + C\varepsilon^{-k} \|f\|_{L_{q)}(\infty)},$$

for $\varepsilon > 0$, where $C-$is a constant depending only on n and q. Further, paying attention to the Property (3.3) from the same considerations given in the monograph [3], we obtain the validity of the estimate (3.11).

Lemma is proved. □

The main result of this work is the following

Theorem 3.5. *Let the coefficients of m-th order elliptic operator L satisfy the following conditions: i) $a_p(\cdot) \in C\left(\bar{\Omega}\right)$, $\forall p : |p| = m$; ii) $a_p(\cdot) \in L_\infty(\Omega)$, $\forall p : |p| < m$; where $\Omega \subset R^n-$ bounded domain with rectifiable boundary $\partial\Omega$. Let $\Omega_0 \subset \Omega$ be an arbitrary compact. Then for $\forall u \in WG^m_{q)}(\Omega)$, $1 < q < +\infty$, the following a priori estimate holds*

$$\|u\|_{N^m_{q)}(\Omega_0)} \leq C\left(\|Lu\|_{L_{q)}(\Omega)} + \|u\|_{L_{q)}(\Omega)}\right),$$

where the constant C depends only on the ellipticity constant m, Ω, Ω_0 of L, on the coefficients of the operator L.

Proof. Let us carry out the proof according to the scheme of the monograph [3, p. 243]. Let a domain Ω and a compact $\Omega_0 \subset \Omega$ be given. It is clear that Ω_0 can cover a finite number of open balls whose closures are contained in Ω. Therefore, it suffices to prove the theorem for the case when Ω and Ω_0 are concentric balls of small radius centered at a point $x_0 \in \Omega$ and without loss of generality we assume that $x_0 = 0$.

So, let $R > 0-$ be sufficiently small number. Let us prove that the following estimate holds for $\forall r : 0 < r < R$:

$$\|u\|_{N^m_{q)}(r)} \leq C\left(1 - \frac{r}{R}\right)^{-m^2}\left(\|Lu\|_{L_{q)}(R)} + \|u\|_{L_{q)}(R)}\right),$$

where $C > 0-$is a constant depending on R (independent of r and u) . Assume

$$A = \sup_{0\leq r\leq R}\left(1 - \frac{r}{R}\right)^{m^2} \|u\|_{N^m_{q)}(r)}$$
$$\leq \|u\|_{N^m_{q)}(R)}.$$

Suppose that $\|u\|_{N^m_{q)}(R)} \neq 0$. Otherwise, there is nothing to prove. Then it is easy to see $\exists R_1 : 0 < R_1 < R$, such that

$$A \leq 2\left(1-\frac{R_1}{R}\right)^{m^2} \|u\|_{N^m_{q)}(R_1)}.$$

Then, for $R_2 : R_1 < R_2 < R$, by Lemma 3.2, inequality (3.3) holds. Taking this inequality into account, we have

$$\begin{aligned} A &\leq 2\left(1-\frac{R_1}{R}\right)^{m^2} C_1\left(1-\frac{R_1}{R_2}\right)^{-m}\left(\|Lu\|_{L_{q)}(R_2)} + \|u\|_{N^{m-1}_{q)}(R_2)}\right) \\ &\leq 2C_1\left(1-\frac{R_1}{R}\right)^{m^2}\left(1-\frac{R_1}{R_2}\right)^{-m}\left(\|Lu\|_{L_{q)}(R)} + \|u\|_{N^{m-1}_{q)}(R_2)}\right). \end{aligned}$$

Taking into account inequality (3.11) for $k = m-1$, we obtain

$$\begin{aligned} A \leq{}& 2C_1\left(1-\frac{R_1}{R}\right)^{m^2}\left(1-\frac{R_1}{R_2}\right)^{-m} \\ &\times\left(\|Lu\|_{L_{q)}(R)} + \varepsilon\|u\|_{N^m_{q)}(R_2)} + C_2\varepsilon^{-m+1}\|u\|_{L_{q)}(R_2)}\right). \end{aligned}$$

Paying attention to the fact that

$$\left(1-\frac{R_2}{R}\right)^{m^2}\|u\|_{N^m_{q)}(R_2)} \leq A,$$

we have

$$\begin{aligned} A \leq{}& 2C_1\left(1-\frac{R_1}{R}\right)^{m^2}\left(1-\frac{R_1}{R_2}\right)^{-m}\|Lu\|_{L_{q)}(R)} \\ &+ 2\varepsilon\, C_1\left(1-\frac{R_1}{R}\right)^{m^2}\left(1-\frac{R_1}{R_2}\right)^{-m}\left(1-\frac{R_2}{R}\right)^{-m^2} A \\ &+ 2C_1C_2\varepsilon^{-m+1}\left(1-\frac{R_1}{R}\right)^{m^2}\left(1-\frac{R_1}{R_2}\right)^{-m}\|u\|_{L_{q)}(R_2)}, \end{aligned}$$

where $\varepsilon > 0-$is an arbitrary number. Let $\delta = 1-\frac{R_1}{R}$ and choose R_2 and ε from the relations

$$1-\frac{R_2}{R_1} = \frac{\delta}{2}, \qquad \varepsilon = 2^{-2-m-m^2}\frac{\delta^m}{C_1}.$$

We have

$$0 < \delta < 1, \qquad \frac{\delta}{2} < 1-\frac{R_1}{R_2} < \delta.$$

Consequently

$$2\varepsilon\, C_1\left(1-\frac{R_1}{R}\right)^{m}\left(1-\frac{R_1}{R_2}\right)^{-m}\left(1-\frac{R_2}{R}\right)^{-m^2} < \frac{1}{2},$$

and as a result

$$\frac{1}{2}A \le 2C_1\left(1-\frac{R_1}{R}\right)^{m^2}\left(1-\frac{R_1}{R_2}\right)^{-m}\|Lu\|_{L_{q)}(R)}$$
$$+\,2C_1C_2\varepsilon^{-m+1}\left(1-\frac{R_1}{R}\right)^{m^2}\left(1-\frac{R_1}{R_2}\right)^{-m}\|u\|_{L_{q)}(R_2)}$$
$$\le C\left(\|Lu\|_{L_{q)}(R)}+\|u\|_{L_{q)}(R)}\right).$$

Taking into account the expression for A we have

$$\|u\|_{N^m_{q)}(r)} \le C\left(1-\frac{r}{R}\right)^{-m^2}\left(\|Lu\|_{L_{q)}(R)}+\|u\|_{L_{q)}(R)}\right),$$

for $\forall r : 0 < r < R$, where $C > 0-$is a constant independent of r.

Theorem is proved.

□

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