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Coincidence Point Results for Different Types of H_b^+-contractions on m_b-Metric Spaces

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Abstract. In this paper, we give some properties of m_b-metric topology and prove Cantor's intersection theorem in m_b-metric spaces. Moreover, we introduce some new classes of H_b^+-contractions for a pair of multi-valued and single-valued mappings and discuss their coincidence points. Some examples are provided to justify the validity of our main results.

1. Introduction

It is well known that the convergence of sequences and the continuity of functions are two important concepts in real or complex analysis. Our main task in metric spaces is to introduce an abstract formulation of the notion of distance between two points of an arbitrary nonempty set. It is interesting to note that most of the central concepts of real or complex analysis can be generalized in metric spaces. Several authors successfully extended the notion of metric spaces in different directions such as G-metric space [22, 23, 25], cone metric space [15, 24], b-metric space [6, 9], C^*-algebra valued metric space [19, 20]. In 1994, Matthews [21] introduced the concept of a partial metric while studying denotational semantics of data flow networks and proved the well-known Banach contraction theorem in this setting. This framework is useful to model several complex problems in the theory of computation and have opened new avenues for application in different fields of mathematics and applied sciences. Thereafter, a lot of articles have been dedicated to the

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improvement of fixed point theory in various spaces (see [4, 11, 13, 16–18, 27, 28] and references therein). In 2012, Haghi et al.[12] proved that some fixed point results on partial metric spaces are fake generalizations. Moreover, they showed in [14] that some fixed point generalizations are not real generalizations. In this study, we are interested to work with m_b-metric as it is a more general setting than partial metric and utilize the same to extend and improve some relevant results of the existing literature.

Recently, Asadi et al. [5] extended the notion of partial metric spaces to m-metric spaces and proved that every partial metric is an m-metric but not every m-metric is a partial metric. Taking into account both of m-metric and b-metric on a nonempty set, Şahin et al. [29] introduced the notion of m_b-metric to extend both m-metric and b-metric. They showed that every m-metric and every b-metric on a nonempty set X are m_b-metrics, but the converse may not hold, in general. As a result, it follows that the class of m_b-metric spaces is strictly larger than that of m-metric spaces (or b-metric spaces or partial metric spaces).

Let (X, d) be a metric space and $CB(X)$ be the family of all nonempty closed and bounded subsets of X. For $E, F \in CB(X)$, define

$$\mathcal{H}(E,F) = \max\left\{\sup_{a\in E} d(a,F), \sup_{b\in F} d(b,E)\right\},$$

where $d(x, E) = \inf\{d(x, a) : a \in E\}$. It is known that $\mathcal{H}$ is a metric on $CB(X)$, called the Hausdorff metric induced by the metric d.

An element x in a nonempty set X is a fixed point of a multi-valued mapping $T : X \longrightarrow 2^X$ if $x \in Tx$, where 2^X is the set of all nonempty subsets of X.

Definition 1.1. A multi-valued mapping $T : X \longrightarrow CB(X)$ is called a contraction if

$$\mathcal{H}(Tx, Ty) \leq kd(x, y),$$

for some $k \in [0, 1)$ and for all $x, y \in X$.

In 1969, Nadler [26] initiated the study of fixed points for multi-valued mappings using the Hausdorff metric and proved that every multi-valued contraction on a complete metric space has a fixed point. Afterwards, several authors successfully established some interesting fixed point results for multi-valued mappings with application in control theory, differential equations and convex optimization (see [1, 3, 7, 8, 10]).

In this work, we shall establish some properties of m_b-metric and prove Cantor's intersection theorem in m_b-metric spaces. We also introduce different types of H_b^+-contractions and discuss their coincidence points. Our results extend several well known comparable results in the existing literature.

2. Basic Definitions and Results

We begin with some basic notations, definitions, and necessary results in m_b-metric spaces.

Definition 2.1 ([21])**.** A partial metric on a nonempty set X is a function $p : X \times X \to \mathbb{R}^+$ such that for all $x, y, z \in X$:

(p1) $p(x,x) = p(y,y) = p(x,y) \Leftrightarrow x = y$,
(p2) $p(x,x) \leq p(x,y)$,
(p3) $p(x,y) = p(y,x)$,
(p4) $p(x,y) \leq p(x,z) + p(z,y) - p(z,z)$.

A partial metric space is a pair (X,p) such that X is a nonempty set and p is a partial metric on X.

Definition 2.2 ([9])**.** Let X be a nonempty set and $s \geq 1$ be a given real number. A function $d : X \times X \to \mathbb{R}^+$ is said to be a b-metric on X if the following conditions hold:

(i) $d(x,y) = 0$ if and only if $x = y$;
(ii) $d(x,y) = d(y,x)$ for all $x, y \in X$;
(iii) $d(x,y) \leq s\,(d(x,z) + d(z,y))$ for all $x, y, z \in X$.

The pair (X,d) is called a b-metric space.

Definition 2.3 ([5])**.** Let X be a nonempty set. A function $\mu : X \times X \to \mathbb{R}^+$ is called an m-metric if the following conditions are satisfied:

(m1) $\mu(x,x) = \mu(y,y) = \mu(x,y) \Leftrightarrow x = y$,
(m2) $m_{xy} \leq \mu(x,y)$,
(m3) $\mu(x,y) = \mu(y,x)$,
(m4) $(\mu(x,y) - m_{xy}) \leq (\mu(x,z) - m_{xz}) + (\mu(z,y) - m_{zy})$,

where $m_{xy} := \min\{\mu(x,x), \mu(y,y)\}$. Then the pair (X,μ) is called an m-metric space.

Definition 2.4 ([29])**.** Let X be a nonempty set. A function $\mu_b : X \times X \to \mathbb{R}^+$ is called an m_b-metric if the following conditions are satisfied: for all $x, y, z \in X$,

(m_b1) $\mu_b(x,x) = \mu_b(y,y) = \mu_b(x,y) \Leftrightarrow x = y$,
(m_b2) $m_{bxy} \leq \mu_b(x,y)$,
(m_b3) $\mu_b(x,y) = \mu_b(y,x)$,
(m_b4) $(\mu_b(x,y) - m_{bxy}) \leq s\,[(\mu_b(x,z) - m_{bxz}) + (\mu_b(z,y) - m_{bzy})]$,

where $m_{bxy} := \min\{\mu_b(x,x), \mu_b(y,y)\}$. Then the pair (X,μ_b) is called an m_b-metric space. The following notation is useful in the sequel:

$$M_{bxy} := \max\{\mu_b(x,x), \mu_b(y,y)\}.$$

It is to be noted that every m-metric (or b-metric) on a nonempty set X is also an m_b-metric on X.

Example 2.5 ([29])**.** Let $X = [0, \infty)$ and define a mapping by

$$\mu_b(x, y) = \min\{x^p, y^p\} + |x - y|^p,$$

where $p > 1$. Then μ_b is an m_b-metric on X with the coefficient $s = 2^{p-1}$. But it is neither an m-metric nor a b-metric on X.

Proposition 2.6. *Let (X, μ_b) be an m_b-metric space and $x, y, z \in X$. Then we have*

1. $0 \leq M_{bxy} + m_{bxy} = \mu_b(x, x) + \mu_b(y, y)$;
2. $0 \leq M_{bxy} - m_{bxy} = |\mu_b(x, x) - \mu_b(y, y)|$;
3. $M_{bxy} - m_{bxy} \leq (M_{bxz} - m_{bxz}) + (M_{bzy} - m_{bzy})$.

Remark 2.7 ([29])**.** Let (X, μ_b) be an m_b-metric space with the coefficient $s \geq 1$. Then the function defined by

$$b_m(x, y) = \mu_b(x, y) - 2m_{bxy} + M_{bxy},$$

is a b-metric on X with the same coefficient s.

3. Topology and Some Results

Let (X, μ_b) be an m_b-metric space, $x \in X$ and $r > 0$. The open ball centered at $x \in X$ with radius $r > 0$ is denoted by

$$B_{\mu_b}(x, r) = \{y \in X : \mu_b(x, y) < m_{bxy} + r\}.$$

We now visualise the open balls in the following example.

Example 3.1. Let $X := [0, 1]$ and $\mu_b(x, y) = \min\{x^2, y^2\} + |x - y|^2$ on X. Then μ_b is an m_b-metric on X with the coefficient $s = 2$. In this case for $r > 0$, we have

$$\begin{aligned} B_{\mu_b}(x_0, r) &= \{x \in X : \mu_b(x_0, x) < m_{bx_0x} + r\} \\ &= \{x \in X : | x - x_0 |^2 < r\} \\ &= \{x \in X : | x - x_0 | < \sqrt{r}\} \\ &= (x_0 - \sqrt{r}, x_0 + \sqrt{r}) \cap X. \end{aligned}$$

Definition 3.2 ([29])**.** A subset U of an m_b-metric space (X, μ_b) is called open if and only if for all $x \in U$, there exists $r > 0$ such that $B_{\mu_b}(x, r) \subseteq U$.

It can be shown that the family of all open subsets of X is a topology on X, say τ_{m_b}. The complements of the elements of τ_{m_b} in X are called closed sets. Even though every partial metric p is an m_b-metric on a nonempty set X and every partial metric p generates a T_0 topology on X, the topology τ_{m_b} may not be a T_0 topology. The following example supports this fact.

Example 3.3 ([29]). Let $X = [0, 1]$ and $\mu_b(x, y) = \min\{x, y\}$, then μ_b is an m_b-metric on X with coefficient $s = 1$. In this case for every $r > 0$, we get

$$\begin{aligned} B_{\mu_b}(x, r) &= \{y \in X : \mu_b(x, y) < m_{bxy} + r\} \\ &= \{y \in X : 0 < r\} \\ &= X, \end{aligned}$$

for all $x \in X$. Therefore, $\tau_{m_b} = \{\emptyset, X\}$ is not a T_0 topology.

Remark 3.4. Let (X, μ_b) be an m_b-metric space, (x_n) be a sequence in X and $x \in X$. Then (x_n) converges to x with respect to(w.r.t.) τ_{m_b} if

$$\lim_{n\to\infty} (\mu_b(x_n, x) - m_{bx_nx}) = 0.$$

Suppose that $\lim_{n\to\infty} (\mu_b(x_n, x) - m_{bx_nx}) = 0$. We shall show that $x_n \to x$ w.r.t. τ_{m_b}. Let $U \in \tau_{m_b}$ and $x \in U$. Then there exists $\epsilon > 0$ such that $x \in B_{\mu_b}(x, \epsilon) \subseteq U$. Since $\lim_{n\to\infty} (\mu_b(x_n, x) - m_{bx_nx}) = 0$, there exists $n_0 \in \mathbb{N}$ such that $\mu_b(x_n, x) - m_{bx_nx} < \epsilon$ for all $n \geq n_0$. This ensures that $x_n \in B_{\mu_b}(x, \epsilon)$ for all $n \geq n_0$ and hence $x_n \in U$ for all $n \geq n_0$. Therefore, (x_n) converges to x w.r.t. τ_{m_b} on X.

In view of the above remark, we propose the following definition of convergence of a sequence and m_b-Cauchy sequence in m_b-metric spaces instead of that introduced by Şahin et al. [29].

Definition 3.5. Let (X, μ_b) be an m_b-metric space. Then:

1. A sequence (x_n) in an m_b-metric space (X, μ_b) converges to a point $x \in X$ if $\lim_{n\to\infty} (\mu_b(x_n, x) - m_{bx_nx}) = 0$.
2. A sequence (x_n) in an m_b-metric space (X, μ_b) is called an m_b-Cauchy sequence if $\lim_{n,m\to\infty} (\mu_b(x_n, x_m) - m_{bx_nx_m}) = 0$ and $\lim_{n,m\to\infty} (M_{bx_nx_m} - m_{bx_nx_m}) = 0$.
3. An m_b-metric space (X, μ_b) is said to be complete if every m_b-Cauchy sequence (x_n) in X converges to a point $x \in X$ such that

$$\lim_{n\to\infty} (\mu_b(x_n, x) - m_{bx_nx}) = 0, \qquad \lim_{n\to\infty} (M_{bx_nx} - m_{bx_nx}) = 0.$$

Lemma 3.6. *Let (X, μ_b) be an m_b-metric space. Then:*

(a) *(x_n) is an m_b-Cauchy sequence in (X, μ_b) if and only if it is a Cauchy sequence in the b-metric space (X, b_m).*

(b) *An m_b-metric space (X, μ_b) is complete if and only if the b-metric space (X, b_m) is complete. Furthermore,*

$$\lim_{n\to\infty} b_m(x_n, x) = 0$$

$$\Leftrightarrow \left\{ \lim_{n\to\infty} (\mu_b(x_n, x) - m_{bx_nx}) = 0, \lim_{n\to\infty} (M_{bx_nx} - m_{bx_nx}) = 0 \right\}.$$

Proof. (a) Suppose that (x_n) is an m_b-Cauchy sequence in (X, μ_b). Then,

$$\lim_{n,k\to\infty} (\mu_b(x_n, x_k) - m_{bx_nx_k}) = 0, \qquad \lim_{n,k\to\infty} (M_{bx_nx_k} - m_{bx_nx_k}) = 0.$$

Therefore,

$$\lim_{n,k\to\infty} (\mu_b(x_n, x_k) - 2m_{bx_nx_k} + M_{bx_nx_k}) = 0.$$

This gives that,

$$\lim_{n,k\to\infty} b_m(x_n, x_k) = 0,$$

that is, (x_n) is a Cauchy sequence in the b-metric space (X, b_m).

Conversely, suppose that (x_n) is a Cauchy sequence in the b-metric space (X, b_m). So,

$$\lim_{n,k\to\infty} b_m(x_n, x_k) = 0.$$

i.e., $\lim_{n,k\to\infty} (\mu_b(x_n, x_k) - 2m_{bx_nx_k} + M_{bx_nx_k}) = 0$. Since $\mu_b(x_n, x_k) - m_{bx_nx_k} \geq 0$ and $M_{bx_nx_k} - m_{bx_nx_k} \geq 0$, it follows that

$$\lim_{n,k\to\infty} (\mu_b(x_n, x_k) - m_{bx_nx_k}) = 0, \qquad \lim_{n,k\to\infty} (M_{bx_nx_k} - m_{bx_nx_k}) = 0.$$

This proves that (x_n) is an m_b-Cauchy sequence in (X, μ_b).

(b) Let (X, μ_b) be complete and (x_n) be a Cauchy sequence in (X, b_m). Then (x_n) is an m_b-Cauchy sequence in (X, μ_b). So there exists $x \in X$ such that

$$\lim_{n\to\infty} (\mu_b(x_n, x) - m_{bx_nx}) = 0, \qquad \lim_{n\to\infty} (M_{bx_nx} - m_{bx_nx}) = 0.$$

Therefore, $\lim_{n\to\infty} (\mu_b(x_n, x) - 2m_{bx_nx} + M_{bx_nx}) = 0$. This implies that

$$\lim_{n\to\infty} b_m(x_n, x) = 0.$$

Thus the b-metric space (X, b_m) is complete.

Conversely, let (X, b_m) be a complete b-metric space and (x_n) be an m_b-Cauchy sequence in (X, μ_b). Then (x_n) is a Cauchy sequence in (X, b_m). Since (X, b_m) is complete, there exists $x \in X$ such that

$$\begin{aligned}\lim_{n\to\infty} b_m(x_n, x) &= \lim_{n\to\infty} (\mu_b(x_n, x) - 2m_{bx_nx} + M_{bx_nx}) \\ &= 0.\end{aligned}$$

Since $\mu_b(x_n, x) - m_{bx_nx} \geq 0$ and $M_{bx_nx} - m_{bx_nx} \geq 0$, it follows that

$$\lim_{n\to\infty}(\mu_b(x_n, x) - m_{bx_nx}) = 0, \qquad \lim_{n\to\infty}(M_{bx_nx} - m_{bx_nx}) = 0.$$

This ensures that the m_b-Cauchy sequence (x_n) in (X, μ_b) converges to x. So (X, μ_b) is complete.

□

Definition 3.7. Let (X, μ_b) be an m_b-metric space and $A \subseteq X$. The closure of A, denoted by $\overline{A}$ or $cl(A)$ is the intersection of all closed subsets of X which contains A. Clearly, $cl(A)$ is always a closed set. Moreover, A is closed if and only if $A = \overline{A}$.

Theorem 3.8. *Let (X, μ_b) be an m_b-metric space, τ_{m_b} be the topology defined above and A be any nonempty subset of X. Then,*

(i) *A is closed if and only if for any sequence (x_n) in A which converges to x, we have $x \in A$;*
(ii) *for any $x \in \overline{A}$ and for any $\epsilon > 0$, we have $B_{\mu_b}(x, \epsilon) \cap A \neq \emptyset$.*

Proof. (i) Suppose that A is a closed subset of X. Let (x_n) be a sequence in A such that $x_n \to x$ as $n \to \infty$. We shall show that $x \in A$. If possible, suppose that $x \notin A$. So $x \in X \setminus A$ and $X \setminus A$ is open. Then there exists $\epsilon > 0$ such that $B_{\mu_b}(x, \epsilon) \subseteq X \setminus A$. Therefore, $B_{\mu_b}(x, \epsilon) \cap A = \emptyset$. Since $x_n \to x$ as $n \to \infty$, we have $\lim_{n\to\infty}(\mu_b(x_n, x) - m_{bx_nx}) = 0$. So for $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $\mu_b(x_n, x) - m_{bx_nx} < \epsilon$, for all $n \geq n_0$. Therefore, $x_n \in B_{\mu_b}(x, \epsilon)$, for all $n \geq n_0$. Hence $x_n \in B_{\mu_b}(x, \epsilon) \cap A$, for all $n \geq n_0$, which leads to a contradiction that $B_{\mu_b}(x, \epsilon) \cap A = \emptyset$. So, $x \in A$.

Conversely, assume that the condition holds i.e., for any sequence (x_n) in A which converges to x, we have $x \in A$. Let us prove that A is closed. In fact, we have to show that $X \setminus A$ is open. So for any $x \in X \setminus A$, we need to prove that there exists $\epsilon > 0$ such that $B_{\mu_b}(x, \epsilon) \subseteq X \setminus A$ i.e., $B_{\mu_b}(x, \epsilon) \cap A = \emptyset$. If possible, suppose that for any $\epsilon > 0$, we have $B_{\mu_b}(x, \epsilon) \cap A \neq \emptyset$. So for any $n \geq 1$, choose $x_n \in B_{\mu_b}(x, \frac{1}{n}) \cap A$. Then $x_n \in A$ for all $n \geq 1$ and $\mu_b(x_n, x) - m_{bx_nx} < \frac{1}{n}$ for all $n \geq 1$. Therefore, $\lim_{n\to\infty}(\mu_b(x_n, x) - m_{bx_nx}) = 0$ i.e., $x_n \to x$ as $n \to \infty$ in (X, μ_b). Hence, by assumption $x \in A$, which is a contradiction. So for any $x \in X \setminus A$, there exists $\epsilon > 0$ such that $B_{\mu_b}(x, \epsilon) \subseteq X \setminus A$ i.e., $X \setminus A$ is open and hence A is closed in X.

(ii) It follows from definition that $\overline{A}$ is the smallest closed subset which contains A. Set

$$A^* = \{x \in X : \text{ for any } \epsilon > 0, \exists a \in A \text{ such that } \mu_b(x,a) < m_{bxa} + \epsilon\}.$$

Obviously, $A \subseteq A^*$. Next we prove that A^* is closed. Let (x_n) be a sequence in A^* such that $x_n \to x$ as $n \to \infty$. We have to prove that $x \in A^*$. Since $x_n \to x$ as $n \to \infty$, we have $\lim_{n\to\infty} (\mu_b(x_n,x) - m_{bx_nx}) = 0$.

Let $\epsilon > 0$ be given. Then there exists $n_0 \in \mathbb{N}$ such that $\mu_b(x_n,x) - m_{bx_nx} < \frac{\epsilon}{2s}$, for all $n \geq n_0$. As $x_n \in A^*$, there exists $a_n \in A$ such that $\mu_b(x_n,a_n) < m_{bx_na_n} + \frac{\epsilon}{2s}$. Hence,

$$\begin{aligned}\mu_b(x,a_n) - m_{bxa_n} &\leq s(\mu_b(x,x_n) - m_{bxx_n}) + s(\mu_b(x_n,a_n) - m_{bx_na_n}) \\ &< s[\frac{\epsilon}{2s} + \frac{\epsilon}{2s}] \\ &= \epsilon, \quad \text{for all } n \geq n_0.\end{aligned}$$

In particular, $\mu_b(x,a_{n_0}) - m_{bxa_{n_0}} < \epsilon$, which implies that $x \in A^*$. Therefore, by part (i), it follows that A^* is closed and contains A. The definition of $\overline{A}$ assures that $\overline{A} \subseteq A^*$, which implies the conclusion of (ii).

□

Theorem 3.9. *Let (X,μ_b) be an m_b-metric space and $A \subseteq X$. Then $x \in \overline{A}$ iff every open set U containing x intersects A.*

Proof. We shall show that $x \notin \overline{A}$ if and only if there exists an open set U containing x which does not intersect A.

If $x \notin \overline{A}$, then the set $U = X \setminus \overline{A}$ is an open set containing x that does not intersect A, as desired.

Conversely, if there exists an open set U containing x which does not intersect A, then $X \setminus U$ is a closed set containing A. By definition of $\overline{A}$, it must be the case that $\overline{A} \subseteq X \setminus U$. Therefore, x cannot be in $\overline{A}$. □

Definition 3.10. Let (X,μ_b) be an m_b-metric space and $A \subseteq X$. The diameter of A, denoted by $diam(A)$, is defined by

$$diam(A) = \sup\{\mu_b(x,y) : x,y \in A\}.$$

Clearly, $0 \leq diam(A) \leq \infty$.

It follows from the above definition that if $A \subseteq B$, then $diam(A) \leq diam(B)$. Hence, it is worth mentioning that $diam(A) \leq diam(\overline{A})$.

We now prove Cantor's intersection theorem in m_b-metric spaces.

Theorem 3.11. *Let (X,μ_b) be a complete m_b-metric space and let (A_n) be a descending sequence of nonempty closed subsets of X with* *diam*$(A_n) \to 0$ *as* $n \to \infty$. *Then the intersection* $A = \bigcap_{n=1}^{\infty} A_n$ *consists of exactly one point.*

Proof. Let (X, μ_b) be a complete m_b-metric space and (A_n) be a descending sequence of nonempty closed sets with $diam(A_n) \to 0$ as $n \to \infty$. As each A_n is nonempty, we choose a point $x_n \in A_n$, for each $n \in \mathbb{N}$. We shall show that (x_n) is m_b-Cauchy in (X, μ_b). For $m, n \in \mathbb{N}$ with $m > n$, we have $A_m \subseteq A_n$ which gives that $x_m, x_n \in A_n$. Therefore,

$$\begin{aligned}\mu_b(x_n, x_m) - m_{bx_nx_m} &\leq \mu_b(x_n, x_m)\\ &\leq diam(A_n) \to 0 \text{ as } n \to \infty,\end{aligned}$$

i.e., $\lim_{n,m\to\infty}(\mu_b(x_n, x_m) - m_{bx_nx_m}) = 0$.

Moreover,

$$\begin{aligned}0 &\leq \mu_b(x_n, x_n)\\ &\leq diam(A_n) \to 0 \text{ as } n \to \infty,\end{aligned}$$

i.e., $\lim_{n\to\infty} \mu_b(x_n, x_n) = 0$. This gives that

$$\begin{aligned}\lim_{n,m\to\infty}(M_{bx_nx_m} - m_{bx_nx_m}) &= \lim_{n,m\to\infty} |\mu_b(x_n, x_n) - \mu_b(x_m, x_m)|\\ &= 0.\end{aligned}$$

Therefore, (x_n) is an m_b-Cauchy sequence in (X, μ_b). Then by hypothesis, (x_n) converges to a point $x \in X$ such that

$$\lim_{n\to\infty}(\mu_b(x_n, x) - m_{bx_nx}) = 0, \qquad \lim_{n\to\infty}(M_{bx_nx} - m_{bx_nx}) = 0.$$

We prove that $x \in \bigcap_{n=1}^{\infty} A_n$. Let $U \in \tau_\mu$ and $x \in U$. Then there exists $\epsilon > 0$ such that $B_{\mu_b}(x, \epsilon) \subseteq U$. As $\lim_{n\to\infty}(\mu_b(x_n, x) - m_{bx_nx}) = 0$, there exists $n_0 \in \mathbb{N}$ such that $\mu_b(x_n, x) - m_{bx_nx} < \epsilon$, for all $n \geq n_0$. Therefore, $x_n \in B_{\mu_b}(x, \epsilon) \subseteq U$, for all $n \geq n_0$. Again, $x_m \in A_n$, for all $m \geq n$ as $x_m \in A_m \subseteq A_n$, for all $m \geq n$. So, $U \cap A_n \neq \emptyset$, for all $n \in \mathbb{N}$. This proves that $x \in \overline{A_n} = A_n, \forall n$, A_n being closed. Hence $x \in \bigcap_{n=1}^{\infty} A_n$.

Now, let $y \in \bigcap_{n=1}^{\infty} A_n$ with $y \neq x$. Then for each $n \in \mathbb{N}$, we have $x, y \in A_n$. Therefore,

$$\begin{aligned}0 &\leq \mu_b(x, y)\\ &\leq diam(A_n) \to 0 \text{ as } n \to \infty\end{aligned}$$

which gives that $\mu_b(x,y)=0$. Similarly,

$$0 \le \mu_b(x,x) \le diam(A_n), \qquad 0 \le \mu_b(y,y) \le diam(A_n)$$

imply that $\mu_b(x,x)=0$ and $\mu_b(y,y)=0$. Therefore, $\mu_b(x,x)=\mu_b(y,y)=\mu_b(x,y)$ and so, $x=y$, a contradiction. This proves that A contains exactly one point. □

Lemma 3.12 ([29])**.** *Let (X,μ_b) be an m_b-metric space, $A \subseteq X$, $x \in X$ and $\mu_b(x,A)=\inf\{\mu_b(x,a): a \in A\}$. If $\mu_b(x,A)=0$ then $x \in \overline{A}$.*

4. Coincidence Point Results

In this section, we prove some coincidence point results for a pair of multi-valued and single-valued mappings in m_b-metric spaces.

Definition 4.1. A subset A of an m_b-metric space (X,μ_b) is called bounded if there exist $x \in X$ and $r>0$ such that $a \in B_{\mu_b}(x,r)$, i.e., $\mu_b(a,x) < m_{bax}+r$, for all $a \in A$.

Let $CB_{\mu_b}(X)$ denote the family of all nonempty, bounded and closed subsets in an m_b-metric space (X,μ_b). For every $P,Q \in CB_{\mu_b}(X)$, define

$$H_b^+(P,Q)=\frac{1}{2}[\delta_b(P,Q)+\delta_b(Q,P)],$$

where

$$\delta_b(P,Q)=\sup\{\mu_b(a,Q): a \in P\}, \qquad \mu_b(a,Q)=\inf\{\mu_b(a,b): b \in Q\}.$$

Proposition 4.2. *For any $P,Q,R \in CB_{\mu_b}(X)$ and for any $x,y \in X$, the following are true*

(*a*) $H_b^+(P,P)=\delta_b(P,P)$;
(*b*) $H_b^+(P,Q)=H_b^+(Q,P)$;
(*c*) $H_b^+(P,Q)=0 \quad \Rightarrow \quad P=Q$;
(*d*) $\left(\mu_b(x,P)-\sup_{a\in P} m_{bxa}\right) \le s\left[(\mu_b(x,y)-m_{bxy})+\left(\mu_b(y,P)-\inf_{a\in P} m_{bya}\right)\right]$.

Proof. (*a*) and (*b*) are immediate consequences of the definition of H_b^+.

(*c*) We first prove that $\delta_b(P,Q)=0 \Rightarrow P \subseteq Q$. Suppose that $\delta_b(P,Q)=0$. Then

$$\begin{aligned}\delta_b(P,Q)&=\sup\{\mu_b(a,Q): a \in P\}=0\\ &\Rightarrow \mu_b(a,Q)=0, \quad \forall a \in P\\ &\Rightarrow a \in \overline{Q}=Q, \quad \forall a \in P\\ &\Rightarrow P \subseteq Q.\end{aligned}$$

Now, $H_b^+(P,Q)=0 \Rightarrow \delta_b(P,Q)=0$ and $\delta_b(Q,P)=0 \Rightarrow P \subseteq Q$ and $Q \subseteq P \Rightarrow P=Q$.

(d) We have

$$
\begin{aligned}
\mu_b(x,P) - \sup_{a\in P} m_{bxa} & \\
&= \inf_{a\in P}\mu_b(x,a) + \inf_{a\in P}(-m_{bxa}) \\
&\leq \inf_{a\in P}[\mu_b(x,a) - m_{bxa}] \\
&\leq \mu_b(x,a) - m_{bxa}, \quad \forall a\in P \\
&\leq s\left[\mu_b(x,y) - m_{bxy} + \mu_b(y,a) - m_{bya}\right], \quad \forall a\in P \\
&\leq s\left[\mu_b(x,y) - m_{bxy} + \mu_b(y,a) - \inf_{a\in P} m_{bya}\right], \quad \forall a\in P.
\end{aligned}
$$

This implies that

$$\mu_b(x,P) - \sup_{a\in P} m_{bxa} \leq s\left[(\mu_b(x,y) - m_{bxy}) + \left(\mu_b(y,P) - \inf_{a\in P} m_{bya}\right)\right].$$

□

Remark 4.3. In general, $H_b^+(P,P)\neq 0$ for $P\in CB_{\mu_b}(X)$. It can be verified through the following example.

Example 4.4. Let $X := [0,1]$ and $\mu_b(x,y) = |x-y|^2 + \min\left\{x^2,y^2\right\}$ on X. Then μ_b is an m_b-metric on X with the coefficient $s=2$. Let $P = \left[0,\frac{1}{3}\right]\in CB_{\mu_b}(X)$. Then,

$$
\begin{aligned}
H_b^+(P,P) &= \delta_b(P,P) \\
&= \sup_{\alpha\in\left[0,\frac{1}{3}\right]} \mu_b\left(\alpha,\left[0,\frac{1}{3}\right]\right).
\end{aligned}
$$

Now,

$$
\begin{aligned}
\mu_b\left(\alpha,\left[0,\frac{1}{3}\right]\right) &= \inf\left\{\mu_b(\alpha,x) : x\in\left[0,\frac{1}{3}\right]\right\} \\
&= \inf\left\{|\ \alpha - x\ |^2 + \min\left\{\alpha^2,x^2\right\} : x\in\left[0,\frac{1}{3}\right]\right\} \\
&= \inf\left\{g(x) : x\in\left[0,\frac{1}{3}\right]\right\},
\end{aligned}
$$

where $g(x) = |\ \alpha - x\ |^2 + \min\left\{\alpha^2,x^2\right\}$. We note that g is monotonic decreasing on $[0,\frac{\alpha}{2}]$ and monotonic increasing on $[\frac{\alpha}{2},\frac{1}{3}]$. Further note that $g(0) = \alpha^2, g(\frac{\alpha}{2}) = \frac{\alpha^2}{2}, g(\frac{1}{3}) = \left(\frac{1}{3}-\alpha\right)^2 + \alpha^2$. So, $\inf\left\{g(x) : x\in\left[0,\frac{1}{3}\right]\right\} = \frac{\alpha^2}{2}$. Therefore,

$$H_b^+(P,P) = \sup_{\alpha\in\left[0,\frac{1}{3}\right]} \mu_b\left(\alpha,\left[0,\frac{1}{3}\right]\right)$$

$$= \sup_{\alpha\in[0,\frac{1}{3}]} \frac{\alpha^2}{2}$$
$$= \frac{1}{18}$$
$$\neq 0.$$

Definition 4.5. Let (X,μ_b) be an m_b-metric space and $T : X \to CB_{\mu_b}(X)$ and $f : X \to X$ be two mappings. If $y = fx \in Tx$ for some x in X, then x is called a coincidence point of T and f and y is called a point of coincidence of T and f.

Definition 4.6. Let (X,μ_b) be an m_b-metric space with the coefficient $s \geq 1$. Then a multi-valued mapping $T : X \to CB_{\mu_b}(X)$ and a single-valued mapping $f : X \to X$ are called Banach type H_b^+-contraction if the following conditions hold:

($C1$) there exists $k \in [0, \frac{1}{s^2})$ such that

$$H_b^+(Tx,Ty) \leq k\mu_b(fx,fy), \quad \forall x,y \in X,$$

($C2$) for every $x \in X$ and $\epsilon > 0$, there exist $fy \in Tx, fz \in Ty$ such that

$$\mu_b(fz,fz) \leq \mu_b(fy,fz) \leq H_b^+(Tx,Ty) + \epsilon.$$

Definition 4.7. Let (X,μ_b) be an m_b-metric space with the coefficient $s \geq 1$. Then a multi-valued mapping $T : X \to CB_{\mu_b}(X)$ and a single-valued mapping $f : X \to X$ are called Kannan type H_b^+-contraction if the conditions ($C2$) and the following hold:

($K1$) there exists $k \in [0, \frac{1}{(s+1)^2})$ such that

$$H_b^+(Tx,Ty) \leq k[\mu_b(fx,Tx) + \mu_b(fy,Ty)], \quad \forall x,y \in X.$$

Definition 4.8. Let (X,μ_b) be an m_b-metric space with the coefficient $s \geq 1$. Then a multi-valued mapping $T : X \to CB_{\mu_b}(X)$ and a single-valued mapping $f : X \to X$ are called Fisher type H_b^+-contraction if the conditions ($C2$) and the following hold:

($F1$) there exists $k \geq 0$ satisfying $(s^2+2s)\sqrt{k} < 1$ such that

$$H_b^+(Tx,Ty) \leq k[\mu_b(fx,Ty) + \mu_b(fy,Tx)], \quad \forall x,y \in X.$$

Theorem 4.9. *Let (X,μ_b) be an m_b-metric space with the coefficient $s \geq 1$ and the mappings $T : X \to CB_{\mu_b}(X)$ and $f : X \to X$ be Banach type H_b^+-contraction with the constant $k \in [0, \frac{1}{s^2})$. If $f(X)$ is a complete m_b-metric subspace of X, then f and T have a point of coincidence u(say) in $f(X)$ with $\mu_b(u,u) = 0$.*

Proof. Let $\epsilon > 0$ be given and take $x_0 \in X$ to be arbitrary. By condition (C2), there exist $fx_1 \in Tx_0, fx_2 \in Tx_1$ such that

$$\mu_b(fx_2, fx_2) \leq \mu_b(fx_1, fx_2) \leq H_b^+(Tx_0, Tx_1) + \epsilon.$$

At this step, we could choose ϵ depending on x_0 and x_1. In general, by condition (C2), there exist $fx_n \in Tx_{n-1}, fx_{n+1} \in Tx_n$ such that

$$\mu_b(fx_{n+1}, fx_{n+1}) \leq \mu_b(fx_n, fx_{n+1}) \leq H_b^+(Tx_{n-1}, Tx_n) + \epsilon, \tag{4.1}$$

for all $n \in \mathbb{N}$. At each step, we could choose ϵ depending on x_{n-1} and x_n. Observe that if $H_b^+(Tx_{n-1}, Tx_n) = 0$ for some n, then $Tx_{n-1} = Tx_n$ and the proof is complete. Therefore, we assume that $H_b^+(Tx_{n-1}, Tx_n) > 0$ for all n $\in \mathbb{N}$.

We set $\epsilon = \left(\frac{1}{\sqrt{k}} - 1\right) H_b^+(Tx_{n-1}, Tx_n) > 0$. Then by using conditions (4.1) and $(C1)$, we obtain that

$$\begin{aligned}
\mu_b(fx_n, fx_{n+1}) &\leq H_b^+(Tx_{n-1}, Tx_n) + \left(\frac{1}{\sqrt{k}} - 1\right) H_b^+(Tx_{n-1}, Tx_n) \\
&= \frac{1}{\sqrt{k}} H_b^+(Tx_{n-1}, Tx_n) \\
&\leq \frac{1}{\sqrt{k}} k \mu_b(fx_{n-1}, fx_n) \\
&= \sqrt{k} \mu_b(fx_{n-1}, fx_n),
\end{aligned}$$

for all $n \in \mathbb{N}$. Thus, we have

$$\mu_b(fx_n, fx_{n+1}) \leq r \mu_b(fx_{n-1}, fx_n), \tag{4.2}$$

where $r = \sqrt{k} < \frac{1}{s}$.
By repeated use of condition (4.2), we get

$$\begin{aligned}
0 &\leq \mu_b(fx_n, fx_{n+1}) \\
&\leq r^n \mu_b(fx_0, fx_1).
\end{aligned} \tag{4.3}$$

Taking limit as $n \to \infty$ in the above ineqality, we obtain

$$\lim_{n \to \infty} \mu_b(fx_n, fx_{n+1}) = 0.$$

Let $\mu_b^*(fx_n, fx_m) = \mu_b(fx_n, fx_m) - m_{bfx_nfx_m}$.

For $m, n \in \mathbb{N}$ with $m > n$, by using conditions (4.3) and (m_b4) we have

$$\begin{aligned}
\mu_b^*(fx_n, fx_m) &\leq s\left[\mu_b^*(fx_n, fx_{n+1}) + \mu_b^*(fx_{n+1}, fx_m)\right] \\
&\leq s\mu_b^*(fx_n, fx_{n+1}) \\
&\quad + s^2\left[\mu_b^*(fx_{n+1}, fx_{n+2}) + \mu_b^*(fx_{n+2}, fx_m)\right]
\end{aligned}$$

$$\begin{aligned}
&\vdots\\
&\le s\mu_b^*(fx_n, fx_{n+1}) + s^2\mu_b^*(fx_{n+1}, fx_{n+2}) + \cdots\\
&\quad + s^{m-n-1}\mu_b^*(fx_{m-2}, fx_{m-1})\\
&\quad + s^{m-n-1}\mu_b^*(fx_{m-1}, fx_m)\\
&\le s\mu_b(fx_n, fx_{n+1}) + s^2\mu_b(fx_{n+1}, fx_{n+2}) + \cdots\\
&\quad + s^{m-n-1}\mu_b(fx_{m-2}, fx_{m-1})\\
&\quad + s^{m-n}\mu_b(fx_{m-1}, fx_m)\\
&\le \left(sr^n + s^2r^{n+1} + \cdots + s^{m-n}r^{m-1}\right)\mu_b(fx_0, fx_1)\\
&= sr^n\left[1 + sr + (sr)^2 + \cdots + (sr)^{m-n-1}\right]\mu_b(fx_0, fx_1)\\
&\le \frac{sr^n}{1-sr}\mu_b(fx_0, fx_1).
\end{aligned}$$

Since $r < 1$, it follows that

$$\lim_{n,m\to\infty}(\mu_b(fx_n, fx_m) - m_{bfx_nfx_m}) = 0.$$

On the other hand, we have

$$\begin{aligned}
0 &\le \mu_b(fx_{n+1}, fx_{n+1})\\
&\le \mu_b(fx_n, fx_{n+1})\\
&\le r^n\mu_b(fx_0, fx_1).
\end{aligned}$$

Taking limit as $n \to \infty$ in the above ineqality, we obtain

$$\lim_{n\to\infty}\mu_b(fx_n, fx_n) = 0.$$

As $0 \le M_{bfx_nfx_m} - m_{bfx_nfx_m} = |\mu_b(fx_n, fx_n) - \mu_b(fx_m, fx_m)|$, it follows that

$$\lim_{n,m\to\infty}(M_{bfx_nfx_m} - m_{bfx_nfx_m}) = 0.$$

Thus, (fx_n) is an m_b-Cauchy sequence in $f(X)$. Since $f(X)$ is m_b-complete, there exists $u \in f(X)$ such that $fx_n \to u = ft$ for some $t \in X$.

So it must be the case that

$$\lim_{n\to\infty}(\mu_b(fx_n, u) - m_{bfx_nu}) = 0, \qquad \lim_{n\to\infty}(M_{bfx_nu} - m_{bfx_nu}) = 0.$$

As $\lim\limits_{n\to\infty} m_{bfx_nu} = 0$, it follows that $\lim\limits_{n\to\infty}\mu_b(fx_n, u) = 0$.

Moreover, $M_{bfx_nu} - m_{bfx_nu} = |\mu_b(fx_n, fx_n) - \mu_b(u, u)|$ implies that $\mu_b(u, u) = 0$. Since

$$\frac{1}{2}\left\{\delta_b(Tx_n, Tt) + \delta_b(Tt, Tx_n)\right\} = H_b^+(Tx_n, Tt)$$

$$\leq k\mu_b(fx_n, ft),$$

it follows that

$$\liminf_{n\to\infty} \{\delta_b(Tx_n, Tt) + \delta_b(Tt, Tx_n)\} = 0.$$

Since

$$\begin{aligned}\liminf_{n\to\infty} \delta_b(Tx_n, Tt) &+ \liminf_{n\to\infty} \delta_b(Tt, Tx_n) \\ &\leq \liminf_{n\to\infty} \{\delta_b(Tx_n, Tt) + \delta_b(Tt, Tx_n)\},\end{aligned}$$

we have

$$\liminf_{n\to\infty} \delta_b(Tx_n, Tt) + \liminf_{n\to\infty} \delta_b(Tt, Tx_n) = 0.$$

This implies that

$$\liminf_{n\to\infty} \delta_b(Tx_n, Tt) = 0.$$

By using part (d) of Proposition 4.2, we have

$$\begin{aligned}\mu_b(ft, Tt) - \sup_{x\in Tt} m_{bftx} &\leq s\Bigg[\left(\mu_b(ft, fx_{n+1}) - m_{bftfx_{n+1}}\right) \\ &\quad + \left(\mu_b(fx_{n+1}, Tt) - \inf_{x\in Tt} m_{bfx_{n+1}x}\right)\Bigg] \\ &\leq s\left(\mu_b(ft, fx_{n+1}) - m_{bftfx_{n+1}}\right) \\ &\quad + s\mu_b(fx_{n+1}, Tt) \\ &\leq s\left(\mu_b(ft, fx_{n+1}) - m_{bftfx_{n+1}}\right) \\ &\quad + s\delta_b(Tx_n, Tt).\end{aligned}$$

Since $m_{bftx} = 0$, it follows that

$$\mu_b(ft, Tt) \leq s\left(\mu_b(ft, fx_{n+1}) - m_{bftfx_{n+1}}\right) + s\delta_b(Tx_n, Tt).$$

As $\lim_{n\to\infty}(\mu_b(fx_{n+1}, ft) - m_{bfx_{n+1}ft}) = 0$, we get

$$\begin{aligned}\mu_b(ft, Tt) &\leq \liminf_{n\to\infty} \left[s\left(\mu_b(ft, fx_{n+1}) - m_{bftfx_{n+1}}\right) + s\delta_b(Tx_n, Tt)\right] \\ &= \lim_{n\to\infty} s\left(\mu_b(ft, fx_{n+1}) - m_{bftfx_{n+1}}\right) + \liminf_{n\to\infty} s\delta_b(Tx_n, Tt) \\ &= 0.\end{aligned}$$

This gives that $\mu_b(ft, Tt) = 0$ and hence $u = ft \in Tt$, Tt being a closed subset of X. This shows that u is a point of coincidence of f and T in $f(X)$ with $\mu_b(u, u) = 0$. □

Corollary 4.10. *Let* (X, μ_b) *be a complete* m_b*-metric space with the coefficient* $s \geq 1$. *Suppose that* $T : X \to CB_{\mu_b}(X)$ *satisfies the following conditions:*

$(C1\acute{)}$ *there exists* $k \in [0, \frac{1}{s^2})$ *such that*

$$H_b^+(Tx, Ty) \leq k\mu_b(x, y), \quad \forall x, y \in X,$$

$(C2\acute{)}$ *for every* $x \in X$ *and* $\epsilon > 0$, *there exist* $y \in Tx, z \in Ty$ *such that*

$$\mu_b(z, z) \leq \mu_b(y, z) \leq H_b^+(Tx, Ty) + \epsilon.$$

Then T *has a fixed point* u*(say) in* X *with* $\mu_b(u, u) = 0$.

Proof. The proof follows from Theorem 4.9 by taking $f = I$, the identity map on X. □

Theorem 4.11. *Let* (X, μ_b) *be an* m_b*-metric space with the coefficient* $s \geq 1$ *and the mappings* $T : X \to CB_{\mu_b}(X)$ *and* $f : X \to X$ *be Kannan type* H_b^+*-contraction with the constant* $k \in [0, \frac{1}{(s+1)^2})$. *If* $f(X)$ *is a complete* m_b*-metric subspace of* X, *then* f *and* T *have a point of coincidence* u*(say) in* $f(X)$ *with* $\mu_b(u, u) = 0$.

Proof. Let $\epsilon > 0$ be given and take $x_0 \in X$ to be arbitrary. As in the proof of Theorem 4.9, we can construct a sequence (x_n) in X such that $fx_{n+1} \in Tx_n$ and

$$\mu_b(fx_{n+1}, fx_{n+1}) \leq \mu_b(fx_n, fx_{n+1}) \leq H_b^+(Tx_{n-1}, Tx_n) + \epsilon, \tag{4.4}$$

for all $n \in \mathbb{N}$. At each step, we could choose ϵ depending on x_{n-1} and x_n. Observe that if $H_b^+(Tx_{n-1}, Tx_n) = 0$ for some n, then $Tx_{n-1} = Tx_n$ and the proof is complete. Therefore, we assume that $H_b^+(Tx_{n-1}, Tx_n) > 0$ for all $n \in \mathbb{N}$.

We set $\epsilon = \left(\frac{1}{\sqrt{k}} - 1\right) H_b^+(Tx_{n-1}, Tx_n) > 0$. Then by using conditions (4.4) and $(K1)$, we get

$$\begin{aligned}
\mu_b(fx_n, fx_{n+1}) &\leq H_b^+(Tx_{n-1}, Tx_n) + \left(\frac{1}{\sqrt{k}} - 1\right) H_b^+(Tx_{n-1}, Tx_n) \\
&= \frac{1}{\sqrt{k}} H_b^+(Tx_{n-1}, Tx_n) \\
&\leq \frac{1}{\sqrt{k}} k \left[\mu_b(fx_{n-1}, Tx_{n-1}) + \mu_b(fx_n, Tx_n)\right] \\
&\leq \sqrt{k} \left[\mu_b(fx_{n-1}, fx_n) + \mu_b(fx_n, fx_{n+1})\right],
\end{aligned}$$

for all $n \in \mathbb{N}$. Thus, we have

$$\begin{aligned}
\mu_b(fx_n, fx_{n+1}) &\leq \frac{\sqrt{k}}{1 - \sqrt{k}} \mu_b(fx_{n-1}, fx_n) \\
&= r\mu_b(fx_{n-1}, fx_n),
\end{aligned} \tag{4.5}$$

where $r = \frac{\sqrt{k}}{1-\sqrt{k}} < \frac{1}{s}$, since $k \in \Big[0, \frac{1}{(s+1)^2}\Big)$.

By repeated use of Condition (4.5), we get

$$\begin{aligned} 0 &\le \mu_b(fx_n, fx_{n+1}) \\ &\le r^n \mu_b(fx_0, fx_1). \end{aligned}$$

Taking limit as $n \to \infty$ in the above ineqality, we obtain

$$\lim_{n\to\infty} \mu_b(fx_n, fx_{n+1}) = 0.$$

Proceeding similarly to that of Theorem 4.9, it follows that

$$\lim_{n,m\to\infty} (\mu_b(fx_n, fx_m) - m_{bfx_nfx_m}) = 0.$$

On the other hand, we have

$$\begin{aligned} 0 &\le \mu_b(fx_{n+1}, fx_{n+1}) \\ &\le \mu_b(fx_n, fx_{n+1}) \\ &\le r^n \mu_b(fx_0, fx_1). \end{aligned}$$

Taking limit as $n \to \infty$ in the above ineqality, we obtain

$$\lim_{n\to\infty} \mu_b(fx_n, fx_n) = 0.$$

As $M_{bfx_nfx_m} - m_{bfx_nfx_m} = |\mu_b(fx_n, fx_n) - \mu_b(fx_m, fx_m)|$, we have

$$\lim_{n,m\to\infty} (M_{bfx_nfx_m} - m_{bfx_nfx_m}) = 0.$$

Thus, (fx_n) is an m_b-Cauchy sequence in $f(X)$. Since $f(X)$ is m_b-complete, there exists $u \in f(X)$ such that $fx_n \to u = ft$ for some $t \in X$. So it must be the case that

$$\lim_{n\to\infty} (\mu_b(fx_n, u) - m_{bfx_nu}) = 0, \qquad \lim_{n\to\infty} (M_{bfx_nu} - m_{bfx_nu}) = 0.$$

As $\lim_{n\to\infty} m_{bfx_nu} = 0$, it follows that $\lim_{n\to\infty} \mu_b(fx_n, u) = 0$.

Moreover, $M_{bfx_nu} - m_{bfx_nu} = |\mu_b(fx_n, fx_n) - \mu_b(u, u)|$ implies that $\mu_b(u, u) = 0$.

Since

$$\begin{aligned} \frac{1}{2}\{\delta_b(Tx_n, Tt) + \delta_b(Tt, Tx_n)\} &= H_b^+(Tx_n, Tt) \\ &\le k[\mu_b(fx_n, Tx_n) + \mu_b(ft, Tt)] \\ &\le k[\mu_b(fx_n, fx_{n+1}) + \mu_b(ft, Tt)], \end{aligned}$$

it follows that

$$\liminf_{n\to\infty} \{\delta_b(Tx_n, Tt) + \delta_b(Tt, Tx_n)\} \le 2k\mu_b(ft, Tt).$$

Since

$$\begin{aligned}\liminf_{n\to\infty}\delta_b(Tx_n,Tt) &\le \liminf_{n\to\infty}\delta_b(Tx_n,Tt)+\liminf_{n\to\infty}\delta_b(Tt,Tx_n)\\ &\le \liminf_{n\to\infty}\left\{\delta_b(Tx_n,Tt)+\delta_b(Tt,Tx_n)\right\},\\ &\le 2k\mu_b(ft,Tt),\end{aligned}$$

we have

$$\liminf_{n\to\infty}\delta_b(Tx_n,Tt)\le 2k\mu_b(ft,Tt).$$

By using part (d) of Proposition 4.2, we have

$$\begin{aligned}\mu_b(ft,Tt)-\sup_{x\in Tt}m_{bftx} &\le s\Bigg[\left(\mu_b(ft,fx_{n+1})-m_{bftfx_{n+1}}\right)\\ &\qquad+\left(\mu_b(fx_{n+1},Tt)-\inf_{x\in Tt}m_{bfx_{n+1}x}\right)\Bigg]\\ &\le s\left(\mu_b(ft,fx_{n+1})-m_{bftfx_{n+1}}\right)+s\mu_b(fx_{n+1},Tt)\\ &\le s\left(\mu_b(ft,fx_{n+1})-m_{bftfx_{n+1}}\right)+s\delta_b(Tx_n,Tt).\end{aligned}$$

Since $m_{bftx}=0$, it follows that

$$\mu_b(ft,Tt)\le s\left(\mu_b(ft,fx_{n+1})-m_{bftfx_{n+1}}\right)+s\delta_b(Tx_n,Tt).$$

As $\lim_{n\to\infty}(\mu_b(fx_{n+1},ft)-m_{bfx_{n+1}ft})=0$, we get

$$\begin{aligned}\mu_b(ft,Tt) &\le \liminf_{n\to\infty}\left[s\left(\mu_b(ft,fx_{n+1})-m_{bftfx_{n+1}}\right)+s\delta_b(Tx_n,Tt)\right]\\ &= \lim_{n\to\infty}s\left(\mu_b(ft,fx_{n+1})-m_{bftfx_{n+1}}\right)+\liminf_{n\to\infty}s\delta_b(Tx_n,Tt)\\ &\le 2ks\mu_b(ft,Tt).\end{aligned}$$

Therefore,

$$\mu_b(ft,Tt)\le 2ks\mu_b(ft,Tt).$$

If $\mu_b(ft,Tt)>0$, then we have $1\le 2ks$, which is a contradiction as $k\in\left[0,\frac{1}{(s+1)^2}\right)$ implies that $ks<\frac{s}{s+1}.\frac{1}{s+1}<\frac{1}{s+1}\le\frac{1}{2}$. So, $\mu_b(ft,Tt)=0$. Thus, by Lemma 3.12, $u=ft\in\overline{Tt}=Tt$. This shows that u is a point of coincidence of f and T in $f(X)$ with $\mu_b(u,u)=0$. □

Corollary 4.12. *Let (X,μ_b) be a complete m_b-metric space with the coefficient $s\ge 1$. Suppose that $T:X\to CB_{\mu_b}(X)$ satisfies the conditions $(C2)'$ and the following:*

$(K1)'$ *there exists $k\in[0,\frac{1}{(s+1)^2})$ such that*

$$H_b^+(Tx,Ty)\le k[\mu_b(x,Tx)+\mu_b(y,Ty)],\quad \forall x,y\in X.$$

Then T has a fixed point u(say) in X with $\mu_b(u,u)=0$.

Proof. The proof follows from Theorem 4.11 by taking $f = I$. □

Theorem 4.13. *Let (X, μ_b) be an m_b-metric space with the coefficient $s \geq 1$ and the mappings $T : X \to CB_{\mu_b}(X)$ and $f : X \to X$ be Fisher type H_b^+-contraction with the constant $k \geq 0$ satisfying $(s^2 + 2s)\sqrt{k} < 1$. If $f(X)$ is a complete m_b-metric subspace of X, then f and T have a point of coincidence u(say) in $f(X)$ with $\mu_b(u, u) = 0$.*

Proof. Let $\epsilon > 0$ be given and take $x_0 \in X$ to be arbitrary. As in the proof of Theorem 4.9, we can construct a sequence (x_n) in X such that $fx_{n+1} \in Tx_n$ and

$$\mu_b(fx_{n+1}, fx_{n+1}) \leq \mu_b(fx_n, fx_{n+1}) \leq H_b^+(Tx_{n-1}, Tx_n) + \epsilon, \tag{4.6}$$

for all $n \in \mathbb{N}$. At each step, we could choose ϵ depending on x_{n-1} and x_n. Observe that if $H_b^+(Tx_{n-1}, Tx_n) = 0$ for some n, then $Tx_{n-1} = Tx_n$ and the proof is complete. Therefore, we assume that $H_b^+(Tx_{n-1}, Tx_n) > 0$ for all $n \in \mathbb{N}$.

We set $\epsilon = \left(\frac{1}{\sqrt{k}} - 1\right) H_b^+(Tx_{n-1}, Tx_n) > 0$. Then by using conditions (4.6) and $(F1)$, we get

(4.7)
$$\begin{aligned}
\mu_b(fx_n, fx_{n+1}) &\leq H_b^+(Tx_{n-1}, Tx_n) + \left(\frac{1}{\sqrt{k}} - 1\right) H_b^+(Tx_{n-1}, Tx_n) \\
&= \frac{1}{\sqrt{k}} H_b^+(Tx_{n-1}, Tx_n) \\
&\leq \frac{1}{\sqrt{k}} k[\mu_b(fx_{n-1}, Tx_n) + \mu_b(fx_n, Tx_{n-1})] \\
&\leq \sqrt{k}[\mu_b(fx_{n-1}, fx_{n+1}) + \mu_b(fx_n, fx_n)] \\
&\leq \sqrt{k}\Big[s(\mu_b(fx_{n-1}, fx_n) - m_{bfx_{n-1}fx_n}) \\
&\quad + s(\mu_b(fx_n, fx_{n+1}) - m_{bfx_nfx_{n+1}}) \\
&\quad + m_{bfx_{n-1}fx_{n+1}} + \mu_b(fx_n, fx_n)\Big] \\
&\leq r[s(\mu_b(fx_{n-1}, fx_n) + \mu_b(fx_n, fx_{n+1})) + A_n],
\end{aligned}$$

where $r = \sqrt{k}$ and

$$A_n = \mu_b(fx_n, fx_n) - sm_{bfx_{n-1}fx_n} - sm_{bfx_nfx_{n+1}} + m_{bfx_{n-1}fx_{n+1}}.$$

Moreover, we have

$$\begin{aligned}
0 &\leq \mu_b(fx_{n+1}, fx_{n+1}) \\
&\leq \mu_b(fx_n, fx_{n+1}).
\end{aligned} \tag{4.8}$$

Now, we consider the following two cases:

Case-I: If $\mu_b(fx_n, fx_n) \leq \mu_b(fx_{n-1}, fx_{n-1})$ or

$$\mu_b(fx_n, fx_n) \leq \mu_b(fx_{n+1}, fx_{n+1}),$$

then

$$\begin{aligned} A_n &= \mu_b(fx_n, fx_n) - sm_{bfx_{n-1}fx_n} - sm_{bfx_nfx_{n+1}} + m_{bfx_{n-1}fx_{n+1}} \\ &\leq m_{bfx_{n-1}fx_{n+1}} \\ &\leq \mu_b(fx_{n+1}, fx_{n+1}). \end{aligned} \tag{4.9}$$

Combining conditions (4.7), (4.8) and (4.9), we obtain

$$\begin{aligned} \mu_b(fx_n, fx_{n+1}) &\leq r\Big[s\mu_b(fx_{n-1}, fx_n) + s\mu_b(fx_n, fx_{n+1}) \\ &\quad + \mu_b(fx_{n+1}, fx_{n+1})\Big] \\ &\leq r\Big[s\mu_b(fx_{n-1}, fx_n) + s\mu_b(fx_n, fx_{n+1}) \\ &\quad + \mu_b(fx_n, fx_{n+1})\Big]. \end{aligned}$$

This gives that

$$\mu_b(fx_n, fx_{n+1}) \leq \frac{rs}{1 - rs - r}\mu_b(fx_{n-1}, fx_n), \tag{4.10}$$

where $0 \leq r_1 = \frac{rs}{1-rs-r} < \frac{1}{s}$, since $0 \leq (s^2 + 2s)\sqrt{k} < 1$.

Case-II: $\mu_b(fx_n, fx_n) \geq \mu_b(fx_{n-1}, fx_{n-1})$ or

$$\mu_b(fx_n, fx_n) \geq \mu_b(fx_{n+1}, fx_{n+1}).$$

If $\mu_b(fx_n, fx_n) \geq \mu_b(fx_{n-1}, fx_{n-1})$, then

$$m_{bfx_{n-1}fx_n} = \mu_b(fx_{n-1}, fx_{n-1}).$$

Therefore,

$$\begin{aligned} A_n &= \mu_b(fx_n, fx_n) - sm_{bfx_{n-1}fx_n} - sm_{bfx_nfx_{n+1}} + m_{bfx_{n-1}fx_{n+1}} \\ &= \mu_b(fx_n, fx_n) - s\mu_b(fx_{n-1}, fx_{n-1}) - sm_{bfx_nfx_{n+1}} \\ &\quad + m_{bfx_{n-1}fx_{n+1}} \\ &\leq \mu_b(fx_n, fx_n) - s\mu_b(fx_{n-1}, fx_{n-1}) - sm_{bfx_nfx_{n+1}} \\ &\quad + \mu_b(fx_{n-1}, fx_{n-1}) \\ &\leq \mu_b(fx_n, fx_n). \end{aligned}$$

If $\mu_b(fx_n, fx_n) \geq \mu_b(fx_{n+1}, fx_{n+1})$, then

$$m_{bfx_nfx_{n+1}} = \mu_b(fx_{n+1}, fx_{n+1}).$$

Therefore,

$$A_n = \mu_b(fx_n, fx_n) - sm_{bfx_{n-1}fx_n} - sm_{bfx_nfx_{n+1}} + m_{bfx_{n-1}fx_{n+1}}$$

$$= \mu_b(fx_n, fx_n) - sm_{bfx_{n-1}fx_n} - s\mu_b(fx_{n+1}, fx_{n+1}) \\ + m_{bfx_{n-1}fx_{n+1}} \\ \leq \mu_b(fx_n, fx_n) - sm_{bfx_{n-1}fx_n} - s\mu_b(fx_{n+1}, fx_{n+1}) \\ + \mu_b(fx_{n+1}, fx_{n+1}) \\ \leq \mu_b(fx_n, fx_n).$$

Thus, in this case

(4.11)

$$A_n = \mu_b(fx_n, fx_n) - sm_{bfx_{n-1}fx_n} - sm_{bfx_nfx_{n+1}} + m_{bfx_{n-1}fx_{n+1}} \\ \leq \mu_b(fx_n, fx_n).$$

Combining conditions (4.7), (4.8) and (4.11), we obtain

$$\mu_b(fx_n, fx_{n+1}) \leq r[s\mu_b(fx_{n-1}, fx_n) + s\mu_b(fx_n, fx_{n+1}) + \mu_b(fx_n, fx_n)] \\ \leq r[s\mu_b(fx_{n-1}, fx_n) + s\mu_b(fx_n, fx_{n+1}) + \mu_b(fx_{n-1}, fx_n)].$$

This gives that

$$\mu_b(fx_n, fx_{n+1}) \leq \frac{rs + r}{1 - rs}\mu_b(fx_{n-1}, fx_n), \tag{4.12}$$

where $0 \leq r_2 = \frac{rs+r}{1-rs} < \frac{1}{s}$, since $0 \leq (s^2 + 2s)\sqrt{k} < 1$.

Let $\alpha = \max\{r_1, r_2\}$. Then $0 \leq \alpha < \frac{1}{s}$ and we get from conditions (4.10) and (4.12) that

$$\mu_b(fx_n, fx_{n+1}) \leq \alpha\mu_b(fx_{n-1}, fx_n). \tag{4.13}$$

By repeated use of Condition (4.13), we get

$$0 \leq \mu_b(fx_n, fx_{n+1}) \\ \leq \alpha^n\mu_b(fx_0, fx_1),$$

for all $n \in \mathbb{N}$.

Taking limit as $n \to \infty$ in the above ineqality, we obtain

$$\lim_{n\to\infty} \mu_b(fx_n, fx_{n+1}) = 0.$$

Proceeding similarly to that of Theorem 4.9, it follows that

$$\lim_{n,m\to\infty} (\mu_b(fx_n, fx_m) - m_{bfx_nfx_m}) = 0.$$

Moreover, it follows from Condition (4.8) that

$$\lim_{n\to\infty} \mu_b(fx_n, fx_n) = 0. \tag{4.14}$$

By an argument similar to that used in Theorem 4.11, we can prove that

$$\lim_{n,m\to\infty} (M_{bfx_nfx_m} - m_{bfx_nfx_m}) = 0.$$

Thus, (fx_n) is an m_b-Cauchy sequence in $f(X)$. Since $f(X)$ is m_b-complete, there exists $u \in f(X)$ such that $fx_n \to u = ft$ for some $t \in X$.

So it must be the case that

$$\lim_{n\to\infty}(\mu_b(fx_n,u) - m_{bfx_nu}) = 0, \qquad \lim_{n\to\infty}(M_{bfx_nu} - m_{bfx_nu}) = 0.$$

As $\lim_{n\to\infty} m_{bfx_nu} = 0$, it follows that $\lim_{n\to\infty} \mu_b(fx_n,u) = 0$.

Moreover, $M_{bfx_nu} - m_{bfx_nu} = |\mu_b(fx_n,fx_n) - \mu_b(u,u)|$ implies that $\mu_b(u,u) = 0$.

We obtain by using conditions (4.14), $(F1)$ and part (d) of Proposition 4.2 that

$$\begin{aligned}
\frac{1}{2}\{\delta_b(Tx_n,Tt) + \delta_b(Tt,Tx_n)\} &= H_b^+(Tx_n,Tt) \\
&\le k[\mu_b(fx_n,Tt) + \mu_b(ft,Tx_n)] \\
&\le k[\mu_b(fx_n,Tt) + \mu_b(ft,fx_{n+1})] \\
&\le k\Big[s(\mu_b(fx_n,ft) - m_{bfx_nft}) \\
&\quad + s(\mu_b(ft,Tt) - \inf_{x\in Tt} m_{bftx}) \\
&\quad + \sup_{x\in Tt} m_{bfx_nx} + \mu_b(ft,fx_{n+1})\Big] \\
&\le k\Big[s(\mu_b(fx_n,ft) + \mu_b(ft,Tt)) \\
&\quad + \sup_{x\in Tt} m_{bfx_nx} + \mu_b(ft,fx_{n+1})\Big] \\
&\le k\Big[s(\mu_b(fx_n,ft) + \mu_b(ft,Tt)) \\
&\quad + \mu_b(fx_n,fx_n) + \mu_b(ft,fx_{n+1})\Big].
\end{aligned}$$

This implies that

$$\liminf_{n\to\infty}\{\delta_b(Tx_n,Tt) + \delta_b(Tt,Tx_n)\} \le 2ks\mu_b(ft,Tt).$$

Since

$$\begin{aligned}
\liminf_{n\to\infty}\delta_b(Tx_n,Tt) &\le \liminf_{n\to\infty}\delta_b(Tx_n,Tt) + \liminf_{n\to\infty}\delta_b(Tt,Tx_n) \\
&\le \liminf_{n\to\infty}\{\delta_b(Tx_n,Tt) + \delta_b(Tt,Tx_n)\}, \\
&\le 2ks\mu_b(ft,Tt),
\end{aligned}$$

we have

$$\liminf_{n\to\infty}\delta_b(Tx_n,Tt) \le 2ks\mu_b(ft,Tt).$$

Again, by using part (d) of Proposition 4.2, we have

$$\begin{aligned}\mu_b(ft,Tt)-\sup_{x\in Tt} m_{bftx} &\leq s\Big[(\mu_b(ft,fx_{n+1})-m_{bftfx_{n+1}})\\ &\qquad +(\mu_b(fx_{n+1},Tt)-\inf_{x\in Tt} m_{bfx_{n+1}x})\Big]\\ &\leq s\left(\mu_b(ft,fx_{n+1})-m_{bftfx_{n+1}}\right)+s\mu_b(fx_{n+1},Tt)\\ &\leq s\left(\mu_b(ft,fx_{n+1})-m_{bftfx_{n+1}}\right)+s\delta_b(Tx_n,Tt).\end{aligned}$$

Since $m_{bftx}=0$, it follows that

$$\mu_b(ft,Tt)\leq s\left(\mu_b(ft,fx_{n+1})-m_{bftfx_{n+1}}\right)+s\delta_b(Tx_n,Tt).$$

As $\lim_{n\to\infty}(\mu_b(fx_{n+1},ft)-m_{bfx_{n+1}ft})=0$, we get

$$\begin{aligned}\mu_b(ft,Tt) &\leq \liminf_{n\to\infty}\left[s\left(\mu_b(ft,fx_{n+1})-m_{bftfx_{n+1}}\right)+s\delta_b(Tx_n,Tt)\right]\\ &=\lim_{n\to\infty} s\left(\mu_b(ft,fx_{n+1})-m_{bftfx_{n+1}}\right)+\liminf_{n\to\infty} s\delta_b(Tx_n,Tt)\\ &\leq 2ks^2\mu_b(ft,Tt).\end{aligned}$$

Therefore,

$$\mu_b(ft,Tt)\leq 2ks^2\mu_b(ft,Tt).$$

If $\mu_b(ft,Tt)>0$, then we have $1\leq 2ks^2$, which is a contradiction since $0\leq (s^2+2s)\sqrt{k}<1$ implies that $2ks^2=2s\sqrt{k}.\sqrt{k}s<2s\sqrt{k}<1$. So, $\mu_b(ft,Tt)=0$.

Thus, by Lemma 3.12, $u=ft\in\overline{Tt}=Tt$. This shows that u is a point of coincidence of f and T in $f(X)$. □

Corollary 4.14. *Let (X,μ_b) be a complete m_b-metric space with the coefficient $s\geq 1$. Suppose that $T:X\to CB_{\mu_b}(X)$ satisfies the conditions $(C2)'$ and the following:*

$(F1)'$ *there exists $k\geq 0$ satisfying $(s^2+2s)\sqrt{k}<1$ such that*

$$H_b^+(Tx,Ty)\leq k[\mu_b(x,Ty)+\mu_b(y,Tx)],\quad \forall x,y\in X.$$

Then T has a fixed point u(say) in X with $\mu_b(u,u)=0$.

Proof. The proof follows from Theorem 4.13 by taking $f=I$. □

Remark 4.15. As a particular case of this study, we can obtain various important fixed point results for multi-valued and single-valued mappings in m-metric and b-metric spaces.

The following examples support our main results.

Example 4.16. Let $X=\{0,1,5\}$ and $\mu_b(x,y)=|x-y|^2+\min\{x^2,y^2\}$ on X. Then (X,μ_b) is a complete m_b-metric space with the coefficient $s=\frac{3}{2}$. Let $T:X\to CB_{\mu_b}(X)$ be defined by $T0=T1=\{0\},T5=$

$\{0,1\}$ and $f : X \to X$ be defined by $f0 = 1, f1 = 0, f5 = 5$. Then $f(X)(= X)$ is a complete m_b-metric space. It is easy to verify that each Tx is a closed and bounded subset of X. We now verify condition $(C1)$ and consider the following possible cases:

Case-I: $x, y \in \{0,1\}$.

In this case, $H_b^+(Tx,Ty) = H_b^+(\{0\},\{0\}) = \mu_b(0,0) = 0 \leq \frac{1}{25}\mu_b(fx,fy)$.

Case-II: $x \in \{0,1\}, y = 5$.

Then,

$$\begin{aligned} H_b^+(Tx,Ty) &= H_b^+(\{0\},\{0,1\}) \\ &= \frac{1}{2}[\delta_b(\{0\},\{0,1\}) + \delta_b(\{0,1\},\{0\})] \\ &= \frac{1}{2}[0+1] \\ &= \frac{1}{2}. \end{aligned}$$

If $x = 0, y = 5$, then $\mu_b(fx,fy) = \mu_b(1,5) = 17$ and so

$$\begin{aligned} H_b^+(Tx,Ty) &= \frac{1}{2} \\ &= \frac{1}{34}\mu_b(fx,fy) \\ &< \frac{1}{25}\mu_b(fx,fy). \end{aligned}$$

If $x = 1, y = 5$, then $\mu_b(fx,fy) = \mu_b(0,5) = 25$ and so

$$\begin{aligned} H_b^+(Tx,Ty) &= \frac{1}{2} \\ &= \frac{1}{50}\mu_b(fx,fy) \\ &< \frac{1}{25}\mu_b(fx,fy). \end{aligned}$$

Case-III: $x = y = 5$.

Then,

$$\begin{aligned} H_b^+(Tx,Ty) &= \delta_b(T5,T5) \\ &= \sup\{\mu_b(z,T5) : z \in T5\} \\ &= \max\{\mu_b(0,T5), \mu_b(1,T5)\} \\ &= 1 \\ &= \frac{1}{25}\mu_b(fx,fy). \end{aligned}$$

Thus, we have

$$H_b^+(Tx,Ty) \leq k\mu_b(fx,fy),$$

for all $x, y \in X$ with $k = \frac{1}{25} \in [0, \frac{1}{s^2})$.

We now verify condition $(C2)$.

For $x \in X, \epsilon > 0$, there exist $fy = f1 \in Tx, fz = f1 \in Ty$ such that

$$\begin{aligned} \mu_b(fz, fz) &= \mu_b(fy, fz) \\ &= 0 \\ &< H_b^+(Tx, Ty) + \epsilon. \end{aligned}$$

Therefore, all the conditions of Theorem 4.9 are fulfilled and 0 is a point of coincidence of f and T in $f(X)$ with $\mu_b(0, 0) = 0$.

Example 4.17. Let $X = \left\{\frac{1}{3^n} : n \in \mathbb{N}\right\} \cup \{0, 1\}$ and $\mu_b(x, y) = |x - y|^2 + \min\left\{x^2, y^2\right\}$ on X. Then (X, μ_b) is a complete m_b-metric space with the coefficient $s = 2$. Let $T : X \to CB_{\mu_b}(X)$ be defined by

$$Tx = \begin{cases} \left\{0, \frac{1}{3^{n+3}}\right\}, x = \frac{1}{3^n}, n \in \mathbb{N} \cup \{0\}, \\ \\ \{0\}, x = 0, \end{cases}$$

and $fx = \frac{x}{3}$ for all $x \in X$. Then $f(X) = X \setminus \{1\}$ is a complete m_b-metric subspace of (X, μ_b) and each Tx is a closed and bounded subset of X. We now verify condition $(K1)$ and consider the following possible cases:

Case-I: $x = \frac{1}{3^n}, n \in \mathbb{N} \cup \{0\}, y = 0$.

Then, $fx = \frac{1}{3^{n+1}}, fy = 0, Tx = \left\{0, \frac{1}{3^{n+3}}\right\}, Ty = \{0\}$ and

$$\begin{aligned} H_b^+(Tx, Ty) &= H_b^+\left(\left\{0, \frac{1}{3^{n+3}}\right\}, \{0\}\right) \\ &= \frac{1}{2}\left[\delta_b\left(\left\{0, \frac{1}{3^{n+3}}\right\}, \{0\}\right) + \delta_b\left(\{0\}, \left\{0, \frac{1}{3^{n+3}}\right\}\right)\right] \\ &= \frac{1}{2}\left[\frac{1}{3^{2n+6}} + 0\right] \\ &= \frac{1}{2} \cdot \frac{1}{3^{2n+6}}. \end{aligned}$$

Moreover,

$$\begin{aligned} \mu_b(fx, Tx) &= \mu_b\left(\frac{1}{3^{n+1}}, \left\{0, \frac{1}{3^{n+3}}\right\}\right) \\ &= \min\left\{\frac{1}{3^{2(n+1)}}, \left|\frac{1}{3^{n+1}} - \frac{1}{3^{n+3}}\right|^2 + \frac{1}{3^{2(n+3)}}\right\} \\ &= \min\left\{\frac{1}{3^{2(n+1)}}, \frac{65}{3^{2(n+1)}.3^4}\right\} \\ &= \frac{65}{3^{2(n+1)}.3^4} \end{aligned}$$

$$= \frac{65}{3^{2n+6}},$$

and $\mu_b(fy, Ty) = \mu_b(0, \{0\}) = 0$.

Therefore,

$$\begin{aligned} H_b^+(Tx, Ty) &= \frac{1}{2}.\frac{1}{3^{2n+6}} \\ &= \frac{1}{130}[\mu_b(fx, Tx) + \mu_b(fy, Ty)]. \end{aligned}$$

Case-II: $x = y = \frac{1}{3^n}, n \in \mathbb{N} \cup \{0\}$.

Then, $fx = fy = \frac{1}{3^{n+1}}, Tx = Ty = \left\{0, \frac{1}{3^{n+3}}\right\}$ and

$$\begin{aligned} H_b^+(Tx, Ty) &= H_b^+\left(\left\{0, \frac{1}{3^{n+3}}\right\}, \left\{0, \frac{1}{3^{n+3}}\right\}\right) \\ &= \delta_b\left(\left\{0, \frac{1}{3^{n+3}}\right\}, \left\{0, \frac{1}{3^{n+3}}\right\}\right) \\ &= \frac{1}{3^{2n+6}}, \\ \mu_b(fx, Tx) &= \mu_b(fy, Ty) \\ &= \mu_b\left(\frac{1}{3^{n+1}}, \left\{0, \frac{1}{3^{n+3}}\right\}\right) \\ &= \frac{65}{3^{2n+6}}. \end{aligned}$$

Thus,

$$\begin{aligned} H_b^+(Tx, Ty) &= \frac{1}{3^{2n+6}} \\ &= \frac{1}{130}[\mu_b(fx, Tx) + \mu_b(fy, Ty)]. \end{aligned}$$

If $x = y = 0$, then

$$\begin{aligned} H_b^+(Tx, Ty) &= H_b^+(\{0\}, \{0\}) \\ &= 0 \\ &= \frac{1}{130}[\mu_b(fx, Tx) + \mu_b(fy, Ty)]. \end{aligned}$$

Case-III: $x = \frac{1}{3^n}, y = \frac{1}{3^m}, m, n \in \mathbb{N} \cup \{0\}, m > n$.

Then, $fx = \frac{1}{3^{n+1}}, fy = \frac{1}{3^{m+1}}, Tx = \left\{0, \frac{1}{3^{n+3}}\right\}, Ty = \left\{0, \frac{1}{3^{m+3}}\right\}$ and

$$\begin{aligned} \delta_b(Tx, Ty) &= \delta_b\left(\left\{0, \frac{1}{3^{n+3}}\right\}, \left\{0, \frac{1}{3^{m+3}}\right\}\right) \\ &= \sup\left\{\mu_b\left(t, \left\{0, \frac{1}{3^{m+3}}\right\}\right) : t \in \left\{0, \frac{1}{3^{n+3}}\right\}\right\} \end{aligned}$$

$$= \max\left\{\mu_b\left(0,\left\{0,\frac{1}{3^{m+3}}\right\}\right),\mu_b\left(\frac{1}{3^{n+3}},\left\{0,\frac{1}{3^{m+3}}\right\}\right)\right\}$$
$$= \max\left\{0,\mu_b\left(\frac{1}{3^{n+3}},\left\{0,\frac{1}{3^{m+3}}\right\}\right)\right\}$$
$$= \mu_b\left(\frac{1}{3^{n+3}},\left\{0,\frac{1}{3^{m+3}}\right\}\right)$$
$$= \min\left\{\frac{1}{3^{2n+6}},\left|\frac{1}{3^{n+3}}-\frac{1}{3^{m+3}}\right|^2+\frac{1}{3^{2m+6}}\right\}$$
$$= \left|\frac{1}{3^{n+3}}-\frac{1}{3^{m+3}}\right|^2+\frac{1}{3^{2m+6}},$$

since
(4.15)

$$\begin{aligned}\left|\frac{1}{3^{n+3}}-\frac{1}{3^{m+3}}\right|^2+\frac{1}{3^{2m+6}} &= \frac{1}{3^{2n+6}}\left|1-\frac{1}{3^{m-n}}\right|^2+\frac{1}{3^{2m+6}}\\ &= \frac{1}{3^{2n+6}}\left[\left(1-\frac{1}{3^{m-n}}\right)^2+\frac{1}{3^{2(m-n)}}\right]\\ &= \frac{1}{3^{2n+6}}\left[(1-p)^2+p^2\right],\ \text{where } p=\frac{1}{3^{m-n}}<1\\ &< \frac{1}{3^{2n+6}}[1-2p+2p]\\ &= \frac{1}{3^{2n+6}}.\end{aligned}$$

Furthermore,

$$\begin{aligned}\delta_b(Ty,Tx) &= \delta_b\left(\left\{0,\frac{1}{3^{m+3}}\right\},\left\{0,\frac{1}{3^{n+3}}\right\}\right)\\ &= \sup\left\{\mu_b\left(t,\left\{0,\frac{1}{3^{n+3}}\right\}\right):t\in\left\{0,\frac{1}{3^{m+3}}\right\}\right\}\\ &= \max\left\{\mu_b\left(0,\left\{0,\frac{1}{3^{n+3}}\right\}\right),\mu_b\left(\frac{1}{3^{m+3}},\left\{0,\frac{1}{3^{n+3}}\right\}\right)\right\}\\ &= \max\left\{0,\frac{1}{3^{2m+6}}\right\}\\ &= \frac{1}{3^{2m+6}},\end{aligned}$$

since $\mu_b\left(\frac{1}{3^{m+3}},\left\{0,\frac{1}{3^{n+3}}\right\}\right)=\min\left\{\frac{1}{3^{2m+6}},\left|\frac{1}{3^{n+3}}-\frac{1}{3^{m+3}}\right|^2+\frac{1}{3^{2m+6}}\right\}=\frac{1}{3^{2m+6}}$.

$$\mu_b(fx,Tx)=\mu_b\left(\frac{1}{3^{n+1}},\left\{0,\frac{1}{3^{n+3}}\right\}\right)$$

$$= \frac{65}{3^{2n+6}},$$

$$\mu_b(fy,Ty) = \mu_b\left(\frac{1}{3^{m+1}}, \left\{0, \frac{1}{3^{m+3}}\right\}\right)$$
$$= \frac{65}{3^{2m+6}}.$$

By using condition (4.15), we get

$$\begin{aligned} H_b^+(Tx,Ty) &= \frac{1}{2}[\delta_b(Tx,Ty) + \delta_b(Ty,Tx)] \\ &= \frac{1}{2}\left[\left|\frac{1}{3^{n+3}} - \frac{1}{3^{m+3}}\right|^2 + \frac{2}{3^{2m+6}}\right] \\ &< \frac{1}{2}\left[\frac{1}{3^{2n+6}} + \frac{1}{3^{2m+6}}\right] \\ &= \frac{1}{130}[\mu_b(fx,Tx) + \mu_b(fy,Ty)]. \end{aligned}$$

Thus, we have

$$H_b^+(Tx,Ty) \leq k[\mu_b(fx,Tx) + \mu_b(fy,Ty)],$$

for all $x, y \in X$ with $k = \frac{1}{130} \in \left[0, \frac{1}{(s+1)^2}\right)$.

We now verify condition $(C2)$.

For $x = \frac{1}{3^n}, n \in \mathbb{N} \cup \{0\}, \epsilon > 0$, there exist $fy = f0 \in Tx, fz = f0 \in Ty$ such that

$$\begin{aligned} \mu_b(fz,fz) &= \mu_b(fy,fz) \\ &= 0 \\ &< H_b^+(Tx,Ty) + \epsilon. \end{aligned}$$

The case $x = 0$ can be treated similarly. Thus, all the hypotheses of Theorem 4.11 hold true and 0 is a point of coincidence of f and T in $f(X)$ with $\mu_b(0,0) = 0$.

Example 4.18. Let $X = \{0, 1, 7\}$ and $\mu_b(x,y) = |x-y|^2 + \min\{x^2, y^2\}$ on X. Then (X, μ_b) is a complete m_b-metric space with the coefficient $s = \frac{3}{2}$. Let $T : X \to CB_{\mu_b}(X)$ be defined by $T0 = T1 = \{0\}, T7 = \{0, 1\}$ and $f : X \to X$ be defined by $f0 = f1 = 0, f7 = 7$. Then $f(X)$ is a complete m_b-metric subspace of (X, μ_b) and each Tx is a closed and bounded subset of X. By an argument similar to that used in Example 4.16, we can show that condition $(C2)$ holds and

$$H_b^+(Tx,Ty) \leq k[\mu_b(fx,Ty) + \mu_b(fy,Tx)],$$

for all $x, y \in X$ with $k = \frac{1}{74} \in \left[0, \frac{1}{(s^2+2s)^2}\right)$.

Thus, we have all the conditions of Theorem 4.13. We find that $0, 1$ are coincidence points and 0 is a point of coincidence of f and T in $f(X)$ with $\mu_b(0,0) = 0$.

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References

1. S.M.A. Aleomraninejad, Sh. Rezapour and N. Shahzad, *Convergence of an iterative scheme for multifunctions*, J. Fixed Point Theory Appl., 12 (2012), pp. 239-246.
2. S.M.A. Aleomraninejad, Sh. Rezapour and N. Shahzad, *Fixed points of hemi-convex multifunctions*, Topo. Metd. Nonlinear Anal., 37 (2011), pp. 383-389.
3. S.M.A. Aleomraninejad and N. Shahzad, *On fixed point generalizations of Suzuki's method*, Appl. Math. Lett., 24 (2011), pp. 1037-1040.
4. S.M.A. Aleomraninejad and N. Shahzad, *Some fixed point results on a metric space with a graph*, Topo. Appl., 159 (2012), pp. 659-663.
5. M. Asadi, E. Karapinar and P. Salimi, *New extension of p-metric spaces with some fixed-point results on M-metric spaces*, J. Inequal. Appl., 2014 (2014), pp. 1-9.
6. I.A. Bakhtin, *The contraction mapping principle in almost metric spaces*, Funct. Anal.,Gos. Ped. Inst. Unianowsk, 30 (1989), pp. 26-37.
7. I. Beg and A. Azam, *Fixed points of multivalued locally contractive mappings*, Boll. Unione Matematica Italiana, 7 (1990), pp. 227-233.
8. I. Beg and A.R. Butt, *Fixed point for set valued mappings satisfying an implicit relation in partially ordered metric spaces*, Nonlinear Analysis: Theory, Methods and Appl., 71 (2009), pp. 3699-3704.
9. S. Czerwik, *Contraction mappings in b-metric spaces*, Acta Math. Inform. Univ. Ostrav, 1 (1993), pp. 5-11.
10. B. Damjanovic, B. Samet and C. Vetro, *Common fixed point theorems for multi-valued maps*, Acta Math. Sci. Ser. B Engl. Ed., 32 (2012), pp. 818-824.
11. R.H. Haghi and Sh. Rezapour, *Fixed points of multifunctions on regular cone metric spaces*, Expositiones Mathematicae, 28 (2010), pp. 71-77.

12. R.H. Haghi, Sh. Rezapour and N. Shahzad, *Be caraful on partial metric fixed point results*, Topo. Appl., 160 (2013), pp. 450-454.
13. R.H. Haghi, Sh. Rezapour and N. Shahzad, *On fixed points of quasi-contraction type multifunctions*, Appl. Math. Lett., 25(5) (2012), pp. 843-846.
14. R.H. Haghi, Sh. Rezapour and N. Shahzad, *Some fixed point generalizations are not real generalizations*, Nonlinear Analysis: Theory, Methods and Appl., 74(5) (2011), pp. 1799-1803.
15. L.-G. Huang and X. Zhang, *Cone metric spaces and fixed point theorems of contractive mappings*, J. Math. Anal. Appl., 332 (2007), pp. 1468-1476.
16. E. Karapinar, *A note on common fixed point theorems in partial metric spaces*, Miskolc Math. Notes, 12 (2011), pp. 185-191.
17. E. Karapinar, *Generalizations of Caristi Kirk's theorem on partial metric spaces*, Fixed Point Theory Appl., 2011 (2011), pp. 1-7.
18. E. Karapinar and S. Romaguera, *Nonunique fixed point theorems in partial metric spaces*, Filomat, 27 (2013), pp. 1305-1314.
19. Z. Ma and L. Jiang, *C^*-algebra-valued b-metric spaces and related fixed point theorems*, Fixed Point Theory Appl., 2015 (2015), pp. 1-12.
20. Z. Ma, L. Jiang and H. Sun, *C^*-algebra-valued metric spaces and related fixed point theorems*, Fixed Point Theory Appl., 2014 (2014), pp. 1-11.
21. S. Matthews, *Partial metric topology*, Ann. N. Y. Acad. Sci., 728 (1994), pp. 183-197.
22. S.K. Mohanta and S. Mohanta, *A common fixed point theorem in G-metric spaces*, Cubo, A Mathematical Journal, 14 (2012), pp. 85-101.
23. S.K. Mohanta and S. Mohanta, *Some fixed point results for mappings in G-metric spaces*, Demonstratio Mathematica, 47 (2014), pp. 179-191.
24. S.K. Mohanta and S. Patra, *Coincidence points and common fixed points for hybrid pair of mappings in b-metric spaces endowed with a graph*, J. Linear. Topological. Algebra., 6 (2017), pp. 301-321.
25. Z. Mustafa and B. Sims, *Fixed point theorems for contractive mappings in complete G-metric spaces*, Fixed Point Theory Appl., 2009 (2009), pp. 1-10.
26. S.B. Nadler, *Multivalued contraction mappings*, Pacific J. Math., 30 (1969), pp. 475-488.
27. Sh. Rezapour and R.H. Haghi, *Two results about fixed point of multifunctions*, Bull. Iranian Math. Soc., 36 (2010), pp. 279-287.

28. Sh. Rezapour, R.H. Haghi and N. Shahzad, *Some notes on fixed points of quasi-contraction maps*, Appl. Math. Lett., 23 (2010), pp. 498-502.
29. H. Şahin, I. Altun and D. Türkoğlu, *Fixed point results for mixed multivalued mappings of Feng-Liu type on M_b-metric spaces*, Mathematical Methods in Engineering, Nonlinear Systems and Complexity, 23 (2019), pp. 67-80.

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