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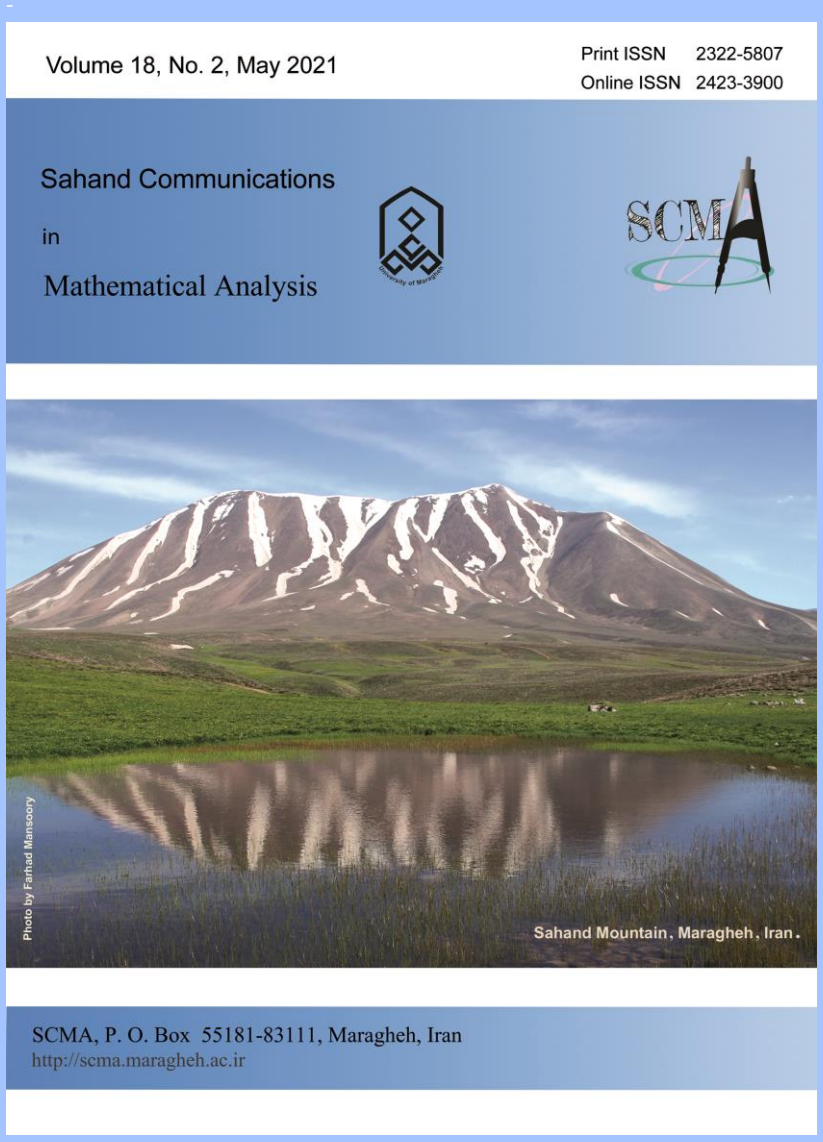

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Some Common Fixed Point Results for Generalized α_*-ψ-contractive Multi-valued Mappings on Ordered Metric Spaces with Application to Initial Value Problem

Sajjad Pahlavany[1], Jalal Hassanzadeh Asl[1,2*] and Shahram Razapour[3]

Abstract. In 2012, Samet, et al. introduced the notion of α-ψ-contractive type mappings. They have been establish some fixed point theorems for the mappings on complete metric spaces. In this paper, we introduce the notion of generalized α_*-ψ-contractive multi-valued mappings and we give some related fixed point results on ordered metric spaces via application to an initial value problem.

1. Introduction

You know, fixed point theory has many applications and was extended by several authors from different views (see for example [1–10]). Recently B. Samet et al introduced the notion of $\alpha-\psi-$contractive type mappings [9]. Denote with Ψ the family of no decreasing functions $\psi : [0,\infty) \to [0,\infty)$ such that $\sum_{n=1}^{\infty} \psi^n(t) < \infty$ for all $t > 0$, where ψ^n is the $n-$th iterate of ψ. It is known that $\psi(t) < t$ for all $t > 0$ and $\psi \in \Psi$ [9]. Let (X, d) be a metric space, T a self-map on X, $\psi \in \Psi$ and $\alpha : X \times X \to [0,\infty)$ a function. Then T is called a α-ψ-contraction mapping whenever $\alpha(x,y)d(Tx,Ty) \leq \psi(d(x,y))$ for all $x, y \in X$. Also, we say that T is α-admissible whenever $\alpha(x,y) \geq 1$ implies $\alpha(Tx,Ty) \geq 1$ [9]. Let (X, d) be a complete metric Space,T a $\alpha-$admissible and α-ψ-contractive Mapping on X. Suppose that there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$. If$T$is continuous, then T has a Fixed point see [9, Theorems (2.1) and (2.2)]. Finally, we Say that X has the property (H)

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whenever for each $x, y \in X$ There exists $z \in X$ such that $\alpha(x, z) \geq 1$ and $\alpha(y, z) \geq 1$. If X has the property (H) in the Theorems 2.1 and 2.2, Then T has a unique fixed point [9, Theorem (2.3)]. It is Considerable that the results of Samet et al. Generalize similar ordered results in the literature (see the results of third section in [9]). The aim of this paper is to introduce the notion of Generalized α_*-ψ-contractive multi-valued mappings, weakly increasing multi-valued mappings and give a common fixed Point Result about the multi-valued mappings on ordered metric spaces with application to initial value problem.

2. Preliminaries

In this section, we list some fundamental definitions that are useful tool in consequent analysis.

Definition 2.1. Let $\digamma$ the family of functions $f : [0, \infty) \to \mathbb{R}$ satisfy:

(i) $f(0) = 0$ and $f(t) > 0$ for all $t \in (0, +\infty)$;
(ii) f is continuous;
(iii) f is no decreasing on $[0, +\infty)$;
(iv) $f(t_1 + t_2) \leq f(t_1) + f(t_2)$ for all $t_1, t_2 \in (0, +\infty)$.

Definition 2.2. Let Ψ the family of functions $\psi : [0, +\infty) \to [0, +\infty)$ satisfy:

(ψ_1) ψ is no decreasing;
(ψ_2) $\sum_{n=1}^{+\infty} \psi^n(t) < \infty$, for all $t \in (0, +\infty)$.

These functions are known in the literature as (c)-comparison functions. It is easily proved that if ψ is a (c)-comparison function, then $\psi(t) < t$ for all $t > 0$. Let 2^X denote the family of all nonempty subsets of X.

Definition 2.3. Let (X, d) be a metric space and $T, S : X \to 2^X$ with given multi-valued mappings. Let $\alpha : X \times X \to [0, +\infty)$, $\alpha_*(A, B) = \inf\{\alpha(a, b) : a \in A, b \in B\}$, $f \in \digamma$, $\psi \in \Psi$, $D(s, Ts) = \inf\{d(s, z)/z \in Ts\}$, H is the Hausdorff metric and let?

$$m(Tx, Sy) = \max\left\{ d(x, y), D(x, Tx), D(y, Sy), \frac{1}{2}[D(x, Sy) + D(y, Tx)] \right\}, \tag{2.1}$$

One says that T, S are generalized α_*-ψ-contractive multi-valued mappings whenever

$$\alpha_*(Tx, Sy)f(H(Tx, Sy)) \leq \psi(f(m(Tx, Sy))) \tag{2.2}$$

or

$$\alpha_*(Sx, Ty)f(H(Sx, Ty)) \leq \psi(f(m(Sx, Ty))), \tag{2.3}$$

for all $x, y \in X$ and $A, B \subset X$.

Definition 2.4. Let $T, S : X \to 2^X$ and $\alpha : X \times X \to [0, +\infty)$. One says that T, S are α_*-admissible if

$$\alpha(x, y) \geq 1 \to \alpha_*(Tx, Sy) \geq 1 \quad \text{or} \quad \alpha_*(Sx, Ty) \geq 1 \tag{2.4}$$

for all $x, y \in X$.

Definition 2.5. A subset $B \subset X$ is said to be an approximation if for each given $y \in X$, there exists $z \in B$ such that

$$D(B, y) = d(z, y).$$

Definition 2.6. A multi-valued mapping $T : X \longrightarrow 2^X$ is said to have approximate values in X if Tx is an approximation for each $x \in X$.

Definition 2.7 ([5])**.** A multi-valued mapping $T : X \to 2^X$ is called order closed if for monotone sequences $x_n \in X$ and $y_n \in Tx_n$, with $x_n \to x$ and $y_n \to y$, implies $y \in Tx$.

Definition 2.8. Let (X, d) be a metric space. If $T : X \to 2^X$ is a multi-valued mapping, then $x \in X$ is called fixed point for T if and only if $x \in F(x)$ The set

$$Fix(T) := \{x \in X | x \in Tx\}$$

is called the fixed point set of T.

Throughout this paper we always assume that all multi-valued operators have approximate values. We have the following result. finally, we should emphasize that throughout this paper we suppose that all multi-valued mappings on a metric space (X, d) have closed values.

3. Main Result

Now, we are ready to state and prove our main results.

Lemma 3.1. *Let (X, d) be a metric space. Suppose that $T, S : X \to 2^X$ are generalized α_*-ψ-contractive multi-valued mappings and satisfies the following conditions:*

(i) *T, S are α_*-admissible;*

(ii) *there exists* $x_0 \in X$ *such that*

$$\alpha_*(\{x_0\}, Tx_0) \geq 1 \quad or \quad \alpha_*(Tx_0, \{x_0\}) \geq 1.$$

Then $Fix(T) = Fix(S)$.

Proof. We first show that any fixed point of T is also a fixed point of S and conversely. Since $Fix(T) \neq Fix(S)$, we may assume there exists $x^* \in X$ such that $x^* \in Fix(T)$, but $x^* \notin Fix(S)$, since $D(x^*, Sx^*) > 0$. Let $x_0 \in X$ such that $\alpha_*(\{x_0\}, Tx_0) \geq 1$. Define the sequence $\{x_n\}$ in X by $x_{2n+1} \in Tx_{2n}$ and $x_{2n+2} \in Sx_{2n+1}$ for all $n \in \mathbb{N}_0$. If $x_{n_0} = x_{n_0+1}$ for some $n_0 > 1$, then $x^* = x_{n_0}$ are a common fixed point for T, S. So, we can assume that $x_{2n} \notin Tx_{2n}$ and $x_{2n+1} \notin Sx_{2n+1}$ for all $n \in \mathbb{N}_0$. Since T, S are α_*-admissible, we have

$$\begin{aligned}
&\alpha(x_0, x_1) \geq \alpha_*(\{x_0\}, Tx_0) \geq 1 \quad \Rightarrow \quad \alpha_*(Tx_0, Sx_1) \geq 1;\\
&\alpha(x_1, x_2) \geq \alpha_*(Tx_0, Sx_1) \geq 1 \quad \Rightarrow \quad \alpha_*(Sx_1, Tx_2) \geq 1;\\
&\alpha(x_2, x_3) \geq \alpha_*(Sx_1, Tx_2) \geq 1 \quad \Rightarrow \quad \alpha_*(Tx_2, Sx_3) \geq 1.
\end{aligned}$$

Inductively, we have

$$\alpha(x_{2n}, x_{2n+1}) \geq 1 \quad \Rightarrow \quad \alpha_*(Tx_{2n}, Sx_{2n+1}) \geq 1$$

and

$$\alpha(x_{2n+1}, x_{2n+2}) \geq 1 \quad \Rightarrow \quad \alpha_*(Sx_{2n+1}, Tx_{2n+2}) \geq 1$$

for all $n \in \mathbb{N}_0$. We obtain

$$\begin{aligned}
m(Tx^*, Sx^*) &= \max\Big\{d(x^*, x^*), D(Tx^*, x^*), D(x^*, Sx^*)\\
&\qquad , \frac{1}{2}[D(Tx^*, x^*) + D(x^*, Sx^*)]\Big\}\\
&= D(x^*, Sx^*)
\end{aligned} \tag{3.1}$$

and

$$\begin{aligned}
f(D(x^*, Sx^*)) &\leq f(H(Tx^*, Sx^*))\\
&\leq \alpha_*(Tx^*, Sx^*) f(H(Tx^*, Sx^*))\\
&\leq \psi(f(m(Tx^*, Sx^*)))\\
&\leq \psi(f(D(x^*, Sx^*)))\\
&< f(D(x^*, Sx^*)).
\end{aligned} \tag{3.2}$$

This contradiction establishes that $Fix(T) \subseteq Fix(S)$. A similar argument establishes the reverse Containment, and therefore $Fix(T) = Fix(S)$. □

Theorem 3.2. *Let (X, d) be a complete metric space. Suppose that $T, S : X \to 2^X$ are generalized α_*-ψ-contractive multi-valued mappings and satisfies The following conditions:*

(i) *T, S are α_*-admissible;*

(ii) *there exists $x_0 \in X$ such that*

$$\alpha_*(\{x_0\}, Tx_0) \geq 1 \quad or \quad \alpha_*(Tx_0, \{x_0\}) \geq 1;$$

(iii) *T or S is order closed.*

Then T, S have common fixed point $x^* \in X$. Further, for each $x_0 \in X$, the iterated sequence $\{x_n\}$ with $x_{2n+1} \in Tx_{2n}$ and $x_{2n+2} \in Sx_{2n+1}$ converges to the common fixed point of T, S.

Proof. By lemma 3.1, we have $Fix(T) = Fix(S)$ and

$$\begin{aligned} f(d(x_{2n+1}, x_{2n+2})) &\leq f(H(Tx_{2n}, Sx_{2n+1})) \\ &\leq \alpha_*(Tx_{2n}, Sx_{2n+1}) f(H(Tx_{2n}, Sx_{2n+1})) \\ &\leq \psi(f(m(Tx_{2n}, Sx_{2n+1}))). \end{aligned}$$

Now,

$$\begin{aligned} m(Tx_{2n}, Sx_{2n+1}) &= \max \Bigg\{ d(x_{2n}, x_{2n+1}), D(x_{2n}, Tx_{2n}), D(x_{2n+1}, Sx_{2n+1}) \\ &\quad , \frac{1}{2} \left[D(x_{2n}, Sx_{2n+1}) + D(x_{2n+1}, Tx_{2n}) \right] \Bigg\} \\ &= \max \Bigg\{ d(x_{2n}, x_{2n+1}), d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2}) \\ &\quad , \frac{1}{2} \left[d(x_{2n}, x_{2n+2}) + d(x_{2n+1}, x_{2n+1}) \right] \Bigg\} \\ &\leq \max \Bigg\{ d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2}) \\ &\quad , \frac{1}{2} \left[d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2}) + 0 \right] \Bigg\} \\ &= \max\{ d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2}) \}. \end{aligned}$$

If

$$\max\{ d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2}) \} = d(x_{2n+1}, x_{2n+2}).$$

So, in general,

$$\begin{aligned} f(d(x_{2n+1}, x_{2n+2})) &\leq \psi \left(f(d(x_{2n+1}, x_{2n+2})) \right) \\ &< f(d(x_{2n+1}, x_{2n+2})), \end{aligned}$$

Which is contradiction since $d(x_{2n+1}, x_{2n+2}) > 0$. Thus

$$f(d(x_{2n+1}, x_{2n+2})) \leq \psi(f(d(x_{2n}, x_{2n+1}))).$$

Similarly,

$$f(d(x_{2n}, x_{2n+1})) \leq \psi(f(d(x_{2n-1}, x_{2n}))),$$

We have

$$f(d(x_n, x_{n+1})) \leq \psi(f(d(x_{n-1}, x_n))),$$

For all $n \in \mathbb{N}_0$. By induction, we get

$$\begin{aligned} f(d(x_n, x_{n+1})) &\leq \psi(f(d(x_{n-1}, x_n))) \\ &\vdots \\ &\leq \psi^n(f(d(x_0, x_1))), \end{aligned}$$

for all $n \in \mathbb{N}_0$. Fix $\epsilon > 0$ and let $n(\epsilon) \in \mathbb{N}_0$ such that

$$\sum_{n=n(\epsilon)}^{\infty} \psi^n(f(d(x_0, x_1))) < \epsilon.$$

Let $n, m \in \mathbb{N}_0$ with $m > n > n(\epsilon)$. Using the triangle inequality, we obtain

$$\begin{aligned} f(d(x_n, x_m)) &\leq \sum_{k=n}^{m-1} f(d(x_k, x_{k+1})) \\ &\leq \sum_{k=n}^{m-1} \psi^k(f(d(x_0, x_1))) \\ &\leq \sum_{n=n(\epsilon)}^{\infty} \psi^n(f(d(x_0, x_1))) \\ &< \epsilon. \end{aligned}$$

Thus we proved that $\{x_n\}$ is a Cauchy sequence in the metric space (X, d). Since (X, d) is complete metric space, there exists $x^* \in X$ such that $x_n \to x^*$ as $n \to +\infty$. From the order closed of T, it follows that $x_{2n+1} \in Tx_{2n} \to Tx^*$ as $n \to +\infty$, then $x^* \in Tx^*$. Similarly if S is order closed, we have $x^* \in Sx^*$. □

Example 3.3. Let $X = [0, \infty), d(x, y) = |x - y|$ and define $T, S : X \to 2^X$ by $Tx = [0, \frac{1}{3}x]$ and $Sx = [0, \frac{1}{2}x]$ for all $x \in X$ and $\alpha : X \times X \to [0, \infty)$ by $\alpha(x, y) = 1$ whenever $x, y \in [0, 1]$ And $\alpha(x, y) = 0$ whenever $x \notin [0, 1]$ or $y \notin [0, 1]$. Now, we show that T, S are α_*-admissible. If $\alpha(x, y) \geq 1$, then $x, y \in [0, 1]$ and so Tx and Sy are subsets of $[0, 1]$. Thus $a, b \in [0, 1]$ for all $a \in Tx$ and $b \in Sy$. Hence, $\alpha(a, b) = 1$ for all $a \in Tx$ and $b \in Sy$. This implies that $\alpha_*(Tx, Sy) = \inf\{\alpha(a, b) : a \in$

$Tx, b \in Sy\} = 1$. Therefore, T, S are coupled α_*-admissible. Now, we show that T, S are generalized α_*-ψ-contractive multi-valued mappings, where $\psi(t) = \frac{1}{6}t$ for all $t \geq 0$. If $x \notin [0,6]$ or $y \notin [0,6]$, then an easy calculation shows us that $\alpha_*(Tx, Sy) = 0$. If $0 \leq x, y \leq 6$, then $\alpha_*(Tx, Sy) = 1$. By using the definition of the Hausdorff metric, it is easy to see that $f(H(Tx, Sy)) \leq \frac{1}{6}f(m(Tx, Sy))$, $f(x) = x$, for $x, y \in [0,6]$. Thus, $\alpha_*(Tx, Sy)f(H(Tx, Ty)) \leq \psi(f(m(Tx, Sy)))$ for $x, y \in X$. Therefore, T, S are generalized α_*-ψ-contractive multi-valued mappings.

Now, we prove the following result for self-maps.

Corollary 3.4. *Let (X, d) be a complete metric space. Suppose that $T, S : X \to X$ are generalized α-ψ-contractive functions and satisfies the following conditions:*

(i) *T, S areα-admissible;*
(ii) *there exists $x_0 \in X$ such that*

$$\alpha(x_0, Tx_0) \geq 1 \quad or \quad \alpha(Tx_0, x_0) \geq 1;$$

(iii) *T or S is continuous.*

Then T, S have common fixed point $x^* \in X$. Further, for each $x_0 \in X$, the iterated sequence $\{x_n\}$ with $x_{2n+1} = Tx_{2n}$ and $x_{2n+2} = Sx_{2n+1}$ converges to the common fixed point of T, S. ([7])

Corollary 3.5. *Let (X, d) be a complete metric space and $T : X \to X$ be a given generalized α-ψ-contractive mapping. Suppose that there exist $\psi \in \Psi$, a function $\alpha : X \times X \to [0, +\infty)$, and satisfies the following conditions:*

(i) *T is α-admissible;*
(ii) *there exists $x_0 \in X$ such that*

$$\alpha(x_0, Tx_0) \geq 1 \quad or \quad \alpha(Tx_0, x_0) \geq 1;$$

(iii) *T is continuous.*

Then T has fixed point $x^* \in X$. Further, for each $x_0 \in X$, the iterated sequence $\{x_n\}$ with $x_{n+1} = Tx_n$ converges to the fixed point of T.([7])

Example 3.6. Let $X = [0, \infty)$ and $d(x, y) = |x - y|$. Define the self map T on X by $Tx = 2x - \frac{3}{2}$ for $x > 1$, $Tx = \frac{x}{2}$ for $0 \leq x \leq 1$ and $\alpha : X \times X \to [0, \infty)$ by $\alpha(x, y) = 1$ whenever $x, y \in [0, 1]$ and $\alpha(x, y) = 0$ whenever $x \notin [0, 1]$ or $y \notin [0, 1]$. Then it is easy to check that T is a α-admissible and $\alpha(x, y)d(Tx, Ty) \leq \psi(m(x, y))$ for all $x, y \in X$, where $\psi(t) = \frac{t}{2}$ for all $t \geq 0$. Also, $\alpha(1, T1) = 1$ and T is a continuous. Note that T has two fixed points.

4. Fixed Point Theorems for Weakly Increasing and Order Closed Multi-Valued Mappings

In the following we provide multi-valued versions of the preceding theorem. The results are related to those in ([5]).
Let X be a topological space and $\preceq$ be a partial order on X.

Definition 4.1. Let A, B be two nonempty subsets of X, the relations between A and B are defines follows:

- (r_1) If for every $a \in A$, there exists $b \in B$ such that $a \preceq b$, then $A \prec_1 B$.
- (r_2) If for every $b \in B$ there exists $a \in A$, such that $a \preceq b$, then $A \prec_2 B$.
- (r_3) If $A \prec_1 B$ and $A \prec_2 B$, then $A \prec B$.

Remark 4.2. $\prec_1$ and $\prec_2$ are different relations between A and B. For example, let $X = \mathbb{R}, A = [\frac{1}{4}, 1], B = [0, 1], \preceq$ be usual order on X, then $A \prec_1 B$ but $A \not\prec_2 B$; if $A = [0, 1], B = [0, \frac{1}{4}]$, then $A \prec_2 B$ while $A \not\prec_1 B$.

Remark 4.3. $\prec_1, \prec_2$ and $\prec$ are reflexive and transitive, But are not ant symmetric. For instance, let $X = \mathbb{R}, A = [0, 4], B = [0, 1] \cup [2, 4], \preceq$ be usual order on X, then $A \prec B$ and $B \prec A$, but $A \neq B$. Hence, they are not partial orders On 2^X.

Definition 4.4 ([3, 4])**.** Let $(X, \preceq)$ be a partially ordered set. Two mappings $f, g : X \to X$ are said to be weakly increasing if $fx \preceq gfx$ and $gx \preceq fgx$ hold for all $x \in X$.

Note that, two weakly increasing mappings need not be no decreasing.

Example 4.5. Let $X = \mathbb{R}^+$ endowed with usual ordering. Let $f, g : X \to X$ defined by

$$f(x) = \begin{cases} x & 0 \leq x \leq 1, \\ 0 & 1 < x < \infty \end{cases} \quad \text{and} \quad g(x) = \begin{cases} \sqrt{x} & 0 \leq x \leq 1, \\ 0 & 1 < x < \infty \end{cases}$$

Then it is obvious that $fx \leq gfx$ and $gx \leq fgx$ for all $x \in X$. Thus f and g are weakly increasing mappings. Note that both f and g are not no decreasing.

Definition 4.6 ([1])**.** Let $(X, \preceq)$ be a partially ordered set. Two mapping $F, G : X \to 2^X$ are said to be weakly increasing with respect to $\prec_1$ if for any $x \in X$ we have $Fx \prec_1 Gy$ for all $y \in Fx$ and $Gx \prec_1 Fy$ for all $y \in Gx$. Similarly two maps $F, G : X \to 2^X$ are said to be weakly increasing with respect to $\prec_2$ if for any $x \in X$ we have $Gy \prec_2 Fx$ for all $y \in Fx$ and $Fy \prec_2 Gx$ for all $y \in Gx$.

Now we give some examples.

Example 4.7 ([1])**.** Let $X = [1, \infty)$ and $\leq$ be usual order on X. Consider two mappings $F, G : X \to 2^X$ defined by $Fx = [1, x^2]$ and $Gx = [1, 2x]$ for all $x \in X$. Then the pair of mappings F and G are weakly increasing with respect to $\prec_2$ but not $\prec_1$. Indeed, since

$$Gy = [1, 2y] \prec_2 [1, x^2] = Fx \quad \text{for all } y \in Fx$$

and

$$Fy = [1, y^2] \prec_2 [1, 2x] = Gx \quad \text{for all } y \in Gx$$

So F and G are weakly increasing with respect to $\prec_2$ but $F2 = [1, 4] \not\prec_1 [1, 2] = G1$ for $1 \in F2$, so F and G are not weakly increasing with respect to $\prec_1$.

Example 4.8 ([1])**.** Let $X = [1, \infty)$ and $\leq$ be usual order on X. Consider two mappings $F, G : X \to 2^X$ defined by $Fx = \{0, 1\}$ and $Gx = [x, 1]$ for all $x \in X$. Then the pair of mappings F and G are weakly increasing with respect to $\prec_1$ but not $\prec_2$. Indeed, since

$$Fx = \{0, 1\} \prec_1 [y, 1] = Gy \quad \text{for all } y \in Fx$$

and

$$Gx = [x, 1] \prec_1 \{0, 1\} = Fy \quad \text{for all } y \in Gx$$

So F and G are weakly increasing with respect to $\prec_1$ but $G1 = 1 \not\prec_2 \{0, 1\} = F1$ for $1 \in F1$, so F and G are not weakly increasing with respect to $\prec_2$.

Theorem 4.9. *Let $(X, \preceq, d)$ be a partially ordered complete matric space. Suppose that $F, G : X \to 2^X$ are generalized α_*-ψ-contractive multi-valued mappings and satisfies The following conditions:*

(i) *F and G be a weakly increasing pair on X w.r.t $\prec_1$;*
(ii) *there exists $x_0 \in X$ such that $\alpha_*(\{x_0\}, Fx_0) \geq 1$;*
(iii) *F or G is order closed,*

Then F, G have common fixed point $x^* \in X$. Further, for each $x_0 \in X$, the iterated sequence $\{x_n\}$ with $x_{2n+1} \in Fx_{2n}$ and $x_{2n+2} \in Gx_{2n+1}$ converges to the common fixed point of F, G.

Proof. By Lemma 3.1, we have $Fix(F) = Fix(G)$. Define the sequence x_n in X by $x_{2n+1} \in Fx_{2n}$ and $x_{2n+2} \in Gx_{2n+1}$ For all $n \in \mathbb{N}_0$. If $x_n = x_{n+1}$ for some $n \in \mathbb{N}_0$, then $x^* = x_n$ is a common fixed point for F and G. Using that the pair of multi-valued mappings F and G Is weakly increasing and by define

$$\alpha(x, y) = \begin{cases} 1 & \text{if } x \preceq y \\ 0 & \text{if } x \succ y \end{cases}$$

It can be easily shown that the sequence x_n is non decreasing w.r.t, $\preceq$ i.e.; and

$$\begin{aligned} \alpha_*(\{x_0\}, Fx_0) \geq 1 \quad &\Rightarrow \quad \exists x_1 \in Fx_0, \\ \alpha(x_0, x_1) \geq 1 \quad &\Rightarrow \quad x_0 \preceq x_1; \end{aligned}$$

Now since F and G are weakly increasing with respect to $\prec_1$, we have $x_1 \in Fx_0 \prec_1 Gx_1$. Thus there exist some $x_2 \in Gx_1$ such that $x_1 \preceq x_2$. Again since F and G are weakly increasing with respect to $\prec_1$,we have $x_2 \in Gx_1 \prec_1 Fx_2$. Thus there exist some $x_3 \in Fx_2$ such that $x_2 \preceq x_3$. Continue this process, we will get a no decreasing sequence $\{x_n\}_{n=1}^{\infty}$ which satisfies

$$x_{2n+1} \in Fx_{2n} \quad \text{and} \quad x_{2n+2} \in Gx_{2n+1}, \quad \text{for all } n \in \mathbb{N}_0.$$

We get

$$x_0 \preceq x_1 \preceq x_2 \preceq \cdots \preceq x_{2n} \preceq x_{2n+1} \preceq x_{2n+2} \preceq \cdots .$$

In particular x_{2n}, x_{2n+1} and x_{2n+2} are comparable.

We get

$$\alpha(x_{2n}, x_{2n+1}) \geq 1 \quad \text{and} \quad \alpha(x_{2n+1}, x_{2n+2}) \geq 1 \quad \text{for all } n \in \mathbb{N}_0.$$

Following the proof of Theorem 3.2, we know that $\{x_n\}$ is a Cauchy sequence in the partially ordered complete matric space $(X, \preceq, d)$. There exists $x^* \in X$ such that $x_n \to x^*$ as $n \to +\infty$. In the case, suppose that, for example, F is an order closed multi-valued mappings then we have that $Fx_n \to Fx^*$, which (taking n even) implies that $x^* \in Fx^*$. The proof is similar when G is an order closed multi-valued mappings. Then x^* is a common fixed point of F, G. □

Theorem 4.10. *Let $(X, \preceq, d)$ be a partially ordered complete matric space. Suppose that $F, G : X \to 2^X$ are generalized coupled α_*-ψ-contractive multi-valued mappings and satisfies the following conditions:*

(i) *F and G be a weakly increasing pair on X w.r.t $\prec_2$;*
(ii) *there exists $x_0 \in X$ such that$\alpha_*(Fx_0, \{x_0\}) \geq 1$;*
(iii) *F or G is order closed,*

Then F, G have common fixed point $x^* \in X$. Further, for each $x_0 \in X$, the iterated sequence $\{x_n\}$ with $x_{2n+1} \in Fx_{2n}$ and $x_{2n+2} \in Gx_{2n+1}$ converges to the common fixed point of F, G.

Proof. By Lemma 3.1, we have $Fix(F) = Fix(G)$. Define the sequence x_n in X by $x_{2n+1} \in Fx_{2n}$ and $x_{2n+2} \in Gx_{2n+1}$ For all $n \in \mathbb{N}_0$. If $x_n = x_{n+1}$ for some $n \in \mathbb{N}_0$, then $x^* = x_n$ is a common fixed point for F, G. Using that the pair of multi-valued mappings F and G is weakly

increasing and by define

$$\alpha(x,y)=\begin{cases}1 & \text{if } x\preceq y\\ 0 & \text{if } x\succ y\end{cases}$$

It can be easily shown that the sequence x_n is non-increasing w.r.t, $\preceq$ i.e; and

$$\begin{aligned}&\alpha_*(Fx_0,\{x_0\})\geq 1 \quad\Rightarrow\quad \exists x_1\in Fx_0,\\ &\alpha(x_1,x_0)\geq 1 \quad\Rightarrow\quad x_1\preceq x_0;\end{aligned}$$

Now since F and G are weakly increasing with respect to $\prec_2$, we have $Gx_1\prec_2 Fx_0$. Thus there exist some $x_2\in Gx_1$ such that $x_2\preceq x_1$. Again since F and G are weakly increasing with respect to $\prec_2$, we have $Fx_2\preceq_2 Gx_1$. Thus there exist some $x_3\in Fx_2$ such that $x_3\preceq x_2$. Continue this process, we will get a no increasing sequence $\{x_n\}_{n=1}^{\infty}$ which satisfies

$$x_{2n+1}\in Fx_{2n}\quad\text{and}\quad x_{2n+2}\in Gx_{2n+1},\quad\text{for all } n\in\mathbb{N}_0.$$

We get

$$x_0\succeq x_1\succeq x_2\succeq\cdots\succeq x_{2n}\succeq x_{2n+1}\succeq x_{2n+2}\succeq\cdots.$$

In particular x_{2n}, x_{2n+1} and x_{2n+2} are comparable.
We get

$$\alpha(x_{2n+1},x_{2n})\geq 1\quad\text{and}\quad\alpha(x_{2n+2},x_{2n+1})\geq 1\quad\text{for all } n\in\mathbb{N}_0.$$

Following the proof of Theorem 3.2, we know that $\{x_n\}$ is a Cauchy sequence in the partially ordered complete matric space $(X,\preceq,d)$. There exists $x^*\in X$ such that $x_n\to x^*$ as $n\to+\infty$. In the case, suppose that, for example, F is an order closed multi-valued mappings then we have that $Fx_n\to Fx^*$, which (taking n even) implies that $x^*\in Fx^*$. The proof is similar when G is an order closed multi-valued mappings. Then x^* is a common fixed point of F, G. □

Corollary 4.11. *Let $(X,\preceq,d)$ be a partially ordered complete matric space. Suppose that $F: X\to 2^X$ is generalized α_*-ψ-contractive multi-valued mapping and satisfy The following conditions:*

(i) *F and i_x be a weakly increasing pair on X w.r.t $\prec_1$ or $\prec_2$;*
(ii) *there exists $x_0\in X$ such that*

$$\alpha_*(\{x_0\},Fx_0)\geq 1\quad or\quad\alpha_*(Fx_0,\{x_0\})\geq 1$$

(iii) *F is order closed,*

Then F has fixed point $x^*\in X$. Further, for each $x_0\in X$, the iterated sequence $\{x_n\}$ with $x_{n+1}\in Fx_n$ converge to the fixed point of F.

Now, we prove the following result for self-maps.

Corollary 4.12. *Let $(X, \preceq, d)$ be a partially ordered complete matric space. Suppose that $f, g : X \to X$ are generalized α-ψ-contractive mappings and satisfies the following conditions:*

(i) *f and g be a weakly increasing pair on X w.r.t $\preceq$;*
(ii) *there exists $x_0 \in X$ such that*

$$\alpha(x_0, fx_0) \geq 1 \quad or \quad \alpha(fx_0, x_0) \geq 1$$

(iii) *f or g is continuous,*

Then f, g have common fixed point $x^* \in X$. Further, for each $x_0 \in X$, the iterated sequence $\{x_n\}$ with $x_{2n+1} = fx_{2n}$ and $x_{2n+2} = gx_{2n+1}$ converges to the common fixed point of f, g.

Corollary 4.13. *Let $(X, \preceq, d)$ be a partially ordered complete matric space. Suppose that $f : X \to X$ are generalized α-ψ-contractive mapping and satisfy The following conditions:*

(i) *f and i_x be a weakly increasing pair on X w.r.t $\preceq$;*
(ii) *there exists $x_0 \in X$ such that*

$$\alpha(x_0, fx_0) \geq 1 \quad or \quad \alpha(fx_0, x_0) \geq 1$$

(iii) *f is continuous.*

Then f has fixed point $x^* \in X$. Further, for each $x_0 \in X$, the iterated sequence $\{x_n\}$ with $x_{n+1} = fx_n$ converge to the fixed point of f.

5. Coupled Fixed Point Theorem

Recall that a function $\eta : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be super-additive if $\eta(s) + \eta(t) \leq \eta(s + t)$ for all $s, t \in \mathbb{R}_+$.

It is well-known that every no decreasing, convex function $\eta : \mathbb{R}_+ \to \mathbb{R}_+$ with $\eta(0) = 0$ is Super-additive; cf.Theorem in [2].

Definition 5.1 ([6])**.** Let $F : X \times X \to X$ be a mapping, where (X, d) is a metric space. We say that $(x, y) \in X \times X$ is a coupled fixed point of F if

$$x = F(x, y), \qquad y = F(y, x).$$

Note that if (x, y) is a coupled fixed point of F then (y, x) are coupled fixed points of F too. Our results are based on the following simple lemma.

Lemma 5.2 ([9])**.** *Let $F : X \times X \to X$ be a given mapping. Define the mapping $T_F : X \times X \to X \times X$ by $T_F(x, y) = (F(x, y), F(y, x))$ for all $(x, y) \in X \times X$. Then, (x, y) is a coupled fixed point of F if and only if (x, y) is a fixed point of T_F.*

Theorem 5.3. *Let (X,d) be a complete metric space and $F: X\times X\to X$ be a given continuous mapping. Assume there are exist no decreasing functions $\psi_i : [0,+\infty)\to[0,+\infty)$, $i=1,2$, such that $\psi=\psi_1+\psi_2$ is convex, $\psi(0)=0$, $\lim_{n\to+\infty}\psi^n(t)=0$ for all $t>0$, a function $\alpha: X^2\times X^2\to[0,+\infty)$ and satisfies the following conditions:*

(i) *for all (x,y), $(u,v)\in X\times X$,*

$$\alpha((x,y),(u,v))d(F(x,y),F(u,v))\le\psi_1(d(x,u))+\psi_2(d(y,v));$$

(ii) *if for all (x,y), $(u,v)\in X\times X$,*

$$\alpha((x,y),(u,v))\ge 1 \quad\Rightarrow\quad \alpha(T_F(x,y),T_F(u,v))\ge 1;$$

(iii) *there exists $(x_0,y_0)\in X\times X$ such that*

$$\alpha((x_0,y_0),T_F(x_0,y_0))\ge 1 \quad \textit{and} \quad \alpha((x_0,y_0),T_F^2(x_0,y_0))\ge 1; \quad \textit{or}$$

(iii)* *there exists $(x_0,y_0)\in X\times X$ such that*

$$\alpha(T_F(x_0,y_0),(x_0,y_0))\ge 1 \quad \textit{and} \quad \alpha(T_F^2(x_0,y_0),(x_0,y_0))\ge 1.$$

Then, F has a coupled fixed point, that is, there exists $(x^*,y^*)\in X\times X$ such that $x^*=F(x^*,y^*)$ and $y^*=F(y^*,x^*)$.

Proof. The idea consists in transporting the problem to the complete metric space (Y,δ), where $Y=X\times X$ and $\delta((x,y),(u,v))=d(x,u)+d(y,v)$, for all (x,y), $(u,v)\in X\times X$. from condition (i), we have

$$\alpha((x,y),(u,v))d(F(x,y),F(u,v))\le\psi_1(d(x,u))+\psi_2(d(y,v)) \tag{5.1}$$

and

$$\alpha((v,u),(y,x))d(F(v,u),F(y,x))\le\psi_1(d(v,y))+\psi_2(d(u,x)) \tag{5.2}$$

for all $x,y,u,v\in X$. Adding (5.1) to (5.2), we get (note that ψ is super-additive)

$$\begin{aligned}\beta(\xi,\eta)\delta(T_F\xi,T_F\eta) &\le \psi_1(d(\xi_1,\eta_1))+\psi_2(d(\xi_2,\eta_2))\\ &\quad+\psi_1(d(\eta_2,\xi_2))+\psi_2(d(\eta_1,\xi_1))\\ &\le \psi_1(d(\xi_1,\eta_1)+d(\eta_2,\xi_2))\\ &\quad+\psi_2(d(\xi_2,\eta_2)+d(\eta_1,\xi_1))\\ &= \psi(d(\xi_1,\eta_1)+d(\eta_2,\xi_2))\\ &= \psi(\delta(\xi,\eta))\end{aligned} \tag{5.3}$$

For all $\xi=(\xi_1,\xi_2)$, $\eta=(\eta_1,\eta_2)\in Y$, where $\beta: Y\times Y\to[0,+\infty)$ is the function defined by

$$\beta((\xi_1,\xi_2),(\eta_1,\eta_2))=\min\{\alpha((\xi_1,\xi_2),(\eta_1,\eta_2)),\alpha((\eta_2,\eta_1),(\xi_2,\xi_1))\} \tag{5.4}$$

and $T_F : Y \to Y$ is given by Lemma 5.2. Then T_F is continuous and $\beta - (\psi_1 + \psi_2)-$contractive mapping. Let $\xi = (\xi_1, \xi_2)$, $\eta = (\eta_1, \eta_2) \in Y$, $\beta(\xi, \eta) \geq 1$. Using condition (ii), we obtain immediately that

$$\beta(T_F\xi, T_F\eta) \geq 1.$$

Then T_F is $\beta-$admissible. Moreover, from condition (iii), we know that there exists $(x_0, y_0) \in Y$ such that $\beta((x_0, y_0), T_F(x_0, y_0)) \geq 1$. All the hypotheses of Theorem 3.2 are satisfied, and so we deduce the existence of a fixed point of T_F that gives us the existence of a coupled fixed point of F. □

6. Coupled Fixed Point Theorem on Partially Ordered Set

Let $(X, \preceq)$ be a partially ordered set. Consider on the product space $X \times X \to X$, the following partial order;

$$(x, y) \preceq_r (u, v) \quad \Leftrightarrow \quad x \preceq u \text{ and } y \preceq v.$$

for all (x, y), $(u, v) \in X \times X$.

Definition 6.1. Let $(X, \preceq)$ be a partially ordered set and $F : X \times X \to X$ Be a given mapping. We say that F has no decreasing in each of its variables if for all $x, y \in X$, we have

$$\begin{aligned} x_1 \preceq x_2 \quad &\Rightarrow \quad F(x_1, y) \preceq_r F(x_2, y), \quad \text{for all } x_1, x_2 \in X, \\ y_1 \preceq y_2 \quad &\Rightarrow \quad F(x, y_1) \preceq_r F(x, y_2) \quad \text{for all } y_1, y_2 \in X. \end{aligned}$$

Theorem 6.2. *Let $(X, \preceq)$ be a partially ordered set and d be a metric on X such that (X, d) is a complete metric space. Assume there are exist no decreasing functions $\psi_i : [0, +\infty) \to [0, +\infty)$, $i = 1, 2$, such that $\psi = \psi_1 + \psi_2$ is convex, $\psi(0) = 0$ and $\lim_{n\to+\infty} \psi^n(t) = 0$ for all $t > 0$. Let $F : X \times X \to X$ be a continuous mapping which is no decreasing in each of its variables and satisfying;*

(i) *$d(F(x, y), F(u, v)) \leq \psi_1(d(x, u)) + \psi_2(d(y, v))$ for each $x \succeq u$, $y \succeq v$;*
(ii) *there exists x_0, $y_0 \in X$ such that $x_0 \succeq F(x_0, y_0)$ and $y_0 \succeq F(y_0, x_0)$.*

Then, F has a coupled fixed point, that is, there exists $(x^*, y^*) \in X \times X$ such that $x^* = F(x^*, y^*)$ and $y^* = F(y^*, x^*)$.

Proof. Define the mapping $\alpha : X^2 \times X^2 \to [0, +\infty)$ by

$$\alpha((x, y), (u, v)) = 1 \quad \text{for all } x, y, u, v \in X \tag{6.1}$$

It follows from (i), that

$$\alpha((x, y), (u, v))d(F(x, y), F(u, v)) \leq \psi_1(d(x, u)) + \psi_2(d(y, v)) \tag{6.2}$$

for all $x, y, u, v \in X$. Let $(x, y), (u, v) \in X \times X$ such that $\alpha((x, y), (u, v)) \geq 1$. By the definition of α and F satisfies the no decreasing in each of its variables property, this implies that $x \succeq u \Rightarrow F(x, y) \succeq_r F(u, y)$ and $y \succeq v \Rightarrow F(u, y) \succeq_r F(u, v)$. We have $F(x, y) \succeq_r F(u, v)$ and similarly $F(y, x) \succeq_r F(v, u)$. Then we get $\alpha((F(x, y), F(y, x)), (F(u, v), F(v, u))) \geq 1$, implies $\alpha(T_F(x, y), T_F(u, v)) \geq 1$. By hypothesis, we know that there exists $(x_0, y_0) \in X \times X$ such that $x_0 \succeq F(x_0, y_0)$ and $y_0 \succeq F(y_0, x_0)$. Now, by the definition of α, this implies that

$$\alpha((x_0, y_0), T_F(x_0, y_0)) \geq 1 \quad \text{and} \quad \alpha(T_F(y_0, x_0), (y_0, x_0)) \geq 1.$$

All the hypotheses of Theorem 3.2 are satisfied, then F has a coupled fixed point. □

Theorem 6.3. *Let $(X, \preceq)$ be a partially ordered set and suppose there is a metric d on X such that (X, d) is a complete metric space. Let $F : X \times X \to X$ be a continuous mapping which is no decreasing in each of its variables on X. Assume there exists $k \in [0, 1)$ with*

$$d(F(x, y), F(u, v)) \leq \frac{k}{2}[(d(x, u)) + (d(y, v))]. \tag{6.3}$$

For all $x, y, u, v \in X$ such that $x \succeq u$ and $y \succeq v$. If there exists $(x_0, y_0) \in X \times X$ such that $x_0 \preceq F(x_0), y_0))$ and $y_0 \preceq F(y_0, x_0)$. Then F has a coupled fixed point.

Proof. Define the mapping $\alpha : X^2 \times X^2 \to [0, +\infty)$ by

$$\alpha((x, y), (u, v)) = 1. \tag{6.4}$$

It follows from 6.3, that

$$\alpha((x, y), (u, v))d(F(x, y), F(u, v)) \leq \frac{k}{2}[(d(x, u)) + (d(y, v))]$$

For all $x, y, u, v \in X$. It implies that

$$\alpha((x, y), (u, v))d(F(x, y), F(u, v)) \leq \psi_1(d(x, u)) + \psi_2(d(y, v)),$$

with $\psi_i(t) = \frac{kt}{2}$ for $i = 1, 2$ and $t \geq 0$. All the hypothesis of Theorem 3.2 are satisfied, then F has a coupled fixed point. □

7. Application

In this section, we study the existence of a unique solution to an initial value problem, as an application to the coupled fixed point theorem. Consider the initial value problem

$$\begin{aligned} x' &= f(t, x(t), y(t)), \quad t \in I = [0, 1], \quad x(0) = x_0 \\ y' &= g(t, y(t), x(t)), \quad t \in I = [0, 1], \quad y(0) = y_0 \end{aligned} \tag{7.1}$$

where $f, g : I \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $x_0, y_0 \in \mathbb{R}$.

Definition 7.1. Let $I = [0, 1]$. An element $(\beta, \gamma) \in C(I, \mathbb{R}) \times C(I, \mathbb{R})$ is called a coupled lower of the initial value problem (7.1) if

$$\beta'(t) \leq f(t, \beta(t), \gamma(t))$$

and

$$\gamma'(t) \leq g(t, \gamma(t), \beta(t)),$$

for each $t \in I$ together with the initial conditions

$$\beta(0) = \gamma(0) = x_0.$$

Theorem 7.2. *Consider the initial value problem* (7.1) *with* $f, g \in C(I \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ *which is no decreasing in the second and third variables. Suppose that for* $x \geq u$ *and* $y \geq v$, *we have*

$$\begin{aligned} 0 &\leq f(t, x, y) - g(t, u, v) \\ &\leq \psi_1(x - u) + \psi_2(y - v), \end{aligned}$$

where $\psi_i : [0, +\infty) \to [0, +\infty)$, $i = 1, 2$, *are no decreasing functions such that* $\psi = \psi_1 + \psi_2$ *is convex,* $\psi(0) = 0$, *and* $\lim_{n\to\infty} \psi^n(t) = 0$ *for each* $t > 0$. *Then the existence of a coupled lower solution for* (7.1) *provides the existence of a unique solution of* (7.1) *in* $C(I, \mathbb{R})$.

Proof. The initial value problem (7.1) is equivalent to the integral equation

$$x(t) = x_0 + \int_0^t f(s, x(s), y(s))ds, \quad t \in I$$

$$y(t) = y_0 + \int_0^t g(s, y(s), x(s))ds, \quad t \in I. \tag{7.2}$$

Let $X = C(I, \mathbb{R})$ is partially ordered if we define the following order relation in X:

$$x, y \in X, \quad x \leq y \quad \Leftrightarrow \quad x(t) \leq y(t), \quad \text{for all } t \in [a, b].$$

It is well-known that (X, d) is a complete metric space with the metric

$$d(x, y) = \sup_{t\in I} |x(t) - y(t)|, \quad x, y \in C(I, \mathbb{R}).$$

Suppose $\{x_n\}$ is a no decreasing sequence in X that converges to $x \in X$. Then for every $t \in I$, the sequence of the real numbers

$$x_1(t) \leq x_2(t) \leq \cdots \leq x_n(t) \leq \cdots,$$

converges to $x(t)$. Therefore, for all $t \in I$ and $n \in N$, we have $x_n(t) \leq x(t)$. Hence $x_n \leq x$, for all $n \in N$. Also, $X \times X$ is a partially ordered set if we define the following order relation in $X \times X$:

$$(x, y) \leq_r (u, v) \quad \Leftrightarrow \quad x(t) \leq u(t) \quad \text{and} \quad y(t) \leq v(t), \quad \text{for all } t \in I,$$

for all (x, y), $(u, v) \in X \times X$. for any $x, y \in X$, $\max\{x(t), u(t)\}$ for all $t \in I$ is in X and is the upper bound of x, u. Therefore, for every (x, y) and $(u, v) \in X \times X$ $\max\{x(t), u(t)\}$, $\max\{y(t), v(t)\}$, in $X \times X$ for all $t \in I$ is comparable to (x, y) and (u, v).
Define $F, G : X \times X \to X$ by

$$F(x, y)(t) = x_0 + \int_0^t f(s, x(s), y(s))ds, \quad \text{for all } t \in I,$$

$$G(x, y)(t) = y_0 + \int_0^t g(s, y(s), x(s))ds, \quad \text{for all } t \in I.$$

Since f, g is no decreasing in the second and third of its variables then F, G are no decreasing in each of its variables.

Now, for $x \geq u$, $y \geq v$, that is, $x(t) \geq u(t)$, $y(t) \geq v(t)$ for all $t \in I$. we have

$$\begin{aligned} d(F(x, y), G(u, v)) &= \sup_{t \in I} |F(x, y)(t) - G(u, v)(t)| \\ &= \sup_{t \in I} \left\{ \int_0^t [f(s, x(s), y(s)) - g(s, u(s), v(s))]\, ds \right\} \\ &\leq \sup_{t \in I} \left\{ \int_0^t [\psi_1(x(s) - u(s)) + \psi_2(y(s) - v(s))]\, ds \right\} \\ &\leq \psi_1(d(x, u)) + \psi_2(d(y, v)). \end{aligned}$$

Thus F, G satisfies the condition of Theorem 6.2. Now, let (β, γ) be a coupled lower solution of the initial value problem 7.1 then we have $\beta \leq F(\beta, \gamma)$ and $\gamma \leq G(\gamma, \beta)$. Then, Theorem 6.2 gives that F, G have a unique common coupled fixed point (x, y) with $x = y$. Then $x(t)$ and $y(t)$ are the unique solution of the integral equation (7.2). □

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