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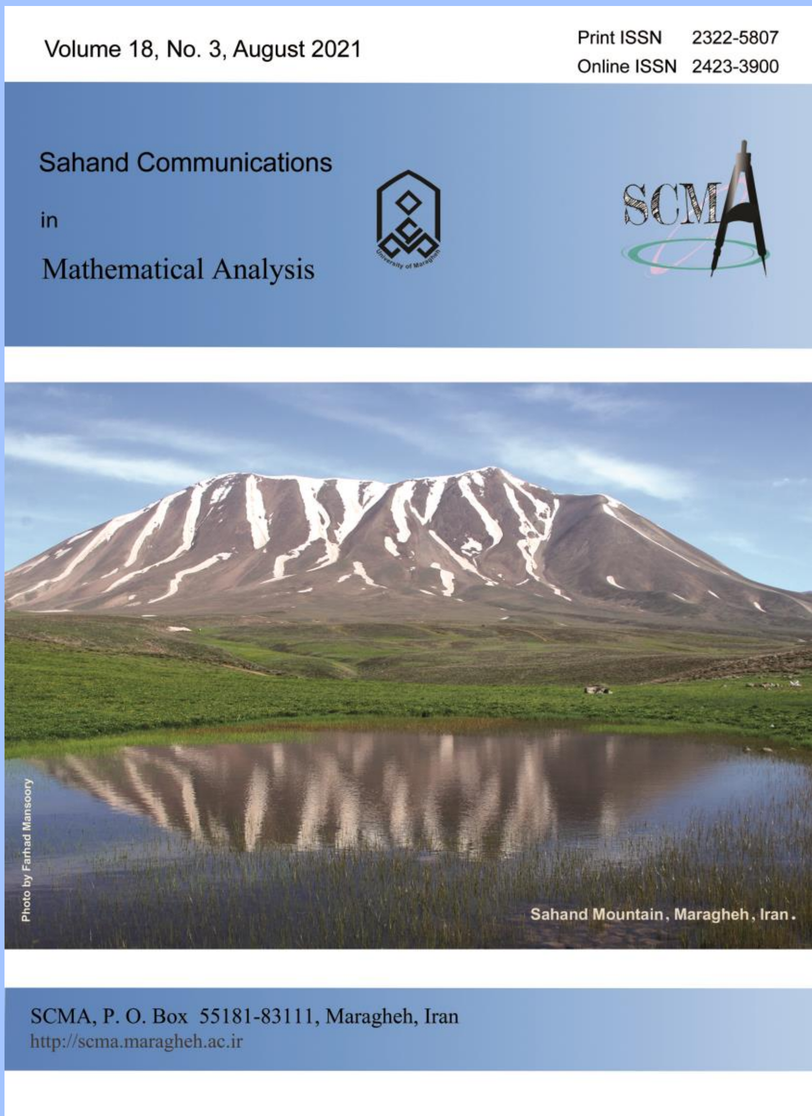

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Fixed Points of p-Hybrid L-Fuzzy Contractions

Mohammed Shehu Shagari[1*], Ibrahim Aliyu Fulatan[2] and Yahaya Sirajo[3]

Abstract. In this paper, the notion of p-hybrid L-fuzzy contractions in the framework of b-metric space is introduced. Sufficient conditions for existence of common L-fuzzy fixed points under such contractions are also investigated. The established ideas are generalizations of many concepts in fuzzy mathematics. In the case where our postulates are reduced to their classical variants, the concept presented herein merges and extends several significant and well-known fixed point theorems in the setting of both single-valued and multi-valued mappings in the corresponding literature of discrete and computational mathematics. A few of these special cases are pointed out and discussed. In support of our main hypotheses, a nontrivial example is provided.

1. Introduction

From the start of creation, man has always been making enormous efforts in comprehending nature and then developing a strong link between life and its requirements. These efforts comprise of three stages, viz, comprehending the surrounding environment, acknowledgement of novelty, and planning for the future. In these struggles, many issues like linguistic interpretation, characterization of interrelated phenomena into proper categories, use of restricted ideas, uncertainty in data analysis, and a host of others, affect the accuracy of results. The above mentioned problems inherent with everyday life can be overcome by availing the notions of fuzzy sets due to their flexibility in nature as compared to crisp sets. After the introduction of fuzzy sets by Zadeh [38], various areas of mathematics, social sciences and engineering were

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exposed to tremendous revolutions. Specifically, fuzzy set is characterized by a membership function which assigns to each of its elements a grade of membership ranging between zero and one. Meanwhile, the basic notions of fuzzy sets have been improved and applied in different directions. In 1981, Heilpern [14] used the idea of fuzzy set to initiate a class of fuzzy set-valued maps and proved a fixed point theorem for fuzzy contraction mappings which is a fuzzy analogue of fixed point theorems due to Nadler [24] and Banach [6]. Subsequently, several authors have studied the existence of fixed points of fuzzy set-valued maps, for example, see [2–4, 23, 29, 35]. A very interesting generalization of fuzzy sets by replacing the interval $[0, 1]$ of range set by a complete distributive lattice was initiated by Goguen [13] and called L-fuzzy sets.

Not long ago, Rashid et al [32] initiated the notion of L-fuzzy mappings and established a common fixed point theorem through β_{FL}-admissible pair of L-fuzzy mappings. As an improvement of the notion of Hausdorff distance and μ_∞-metric for fuzzy sets, Rashid et al [33] defined the concepts of $D_{\alpha L}$ and μ_L^∞ distances for L-fuzzy sets and generalized some known fixed point theorems for fuzzy and multi-valued mappings.

The study of new spaces and their characterizations have been an interesting topic among the mathematical research community. In this context, the concept of b-metric spaces is presently thriving. The idea started with the work of Bakhtin [5] and Bourbaki [9]. Thereafter, Czerwik [10] produced an axiom which is weaker than the usual triangle inequality and formally introduced a b-metric space with the aim of improving the Banach contraction principle. Meanwhile, the idea of b-metric space is gaining enormous generalizations, see for example [16, 17, 30, 31]. For a recent short survey on basic concepts and results in fixed point theory in the setting of b-metric spaces, the interested reader may consult Karapinar [20]. On similar development, one of the active areas of fixed point theory that is also currently attracting the attentions of investigators is the study of hybrid contractions. The notion has been understood in two directions, viz, first, hybrid contraction concerns contractions involving both single-valued and multivalued mappings and the second merges linear and nonlinear contractions. Not long ago, Karapinar and Fulga [19] launched a new notion of b-hybrid contraction in the setting of b-metric space and studied the existence and uniqueness of fixed points for such contractions. Shortly, Alansari et al. [1] presented a multi-valued extension of the results in [19]. Interestingly, hybrid fixed point theory has potential applications in functional inclusions, optimization theory, fractal graphics, discrete dynamics for set-valued operators and other areas of nonlinear functional analysis. For some results in this direction, the reader is referred to [19, 22, 25, 27, 28].

The main aim of this paper is to initiate the idea of p-hybrid L-fuzzy contractions in the framework of b-metric space and then to study the existence of L-fuzzy fixed points for such contractions. We notice that when our results are reduced to their crisp variants, the concept presented herein unifies and generalizes a number of significant fixed point theorems in the setting of both single-valued and multi-valued mappings in the corresponding literature. A few of these particular cases are highlighted and discussed. In support of our main assumptions, a nontrivial example is provided. As far as we know, there is no contribution in the literature of hybrid fixed point theory via the concept of L-fuzzy sets. Thus, the idea of the present paper is new, and it will open up another research directions in the field of classical and fuzzy fixed point theory.

2. Preliminaries

In this section, we collect some important notations, useful definitions and basic results coherent with the literature. Throughout this paper, we denote by $\mathbb{N}$, $\mathbb{R}_+$ and $\mathbb{R}$ the sets of natural numbers, non-negative reals and real numbers, respectively. These preliminary concepts are recorded from [10, 19, 24].

In 1993, Czerwik [10] initiated the idea of a b-metric space as follows:

Definition 2.1 ([10])**.** Let $\mho$ be a nonempty set and $\eta \geq 1$ be a constant. Suppose that the mapping $\mu : \mho \times \mho \to \mathbb{R}_+$ satisfies the following conditions for all $\varsigma, \omega, \xi \in \mho$:

(i) $\mu(\varsigma, \omega) = 0$ if and only if $\varsigma = \omega$ (self-distancy);
(ii) $\mu(\varsigma, \omega) = \mu(\omega, \varsigma)$ (symmetry);
(iii) $\mu(\varsigma, \omega) \leq \eta\,[\mu(\varsigma, \xi) + \mu(\xi, \omega)]$ (weighted triangle inequality).

Then, the triple $(\mho, \mu, \eta)$ is called a b-metric space.

It is noteworthy that every metric is a b-metric with the parameter $\eta = 1$. Also, in general, a b-metric is not a continuous functional. Hence, the class of b-metric is larger than the class of classical metric.

Example 2.2 ([7])**.** Let $\mho = l_p(\mathbb{R})$ with $0 < p < 1$, where

$$l_p(\mathbb{R}) = \left\{ \{\varsigma_n\}_{n\in\mathbb{N}} \subseteq \mathbb{R} : \sum_{n=1}^{\infty} |\varsigma_n|^p < \infty \right\}.$$

Define $\mu : \mho \times \mho \to \mathbb{R}_+$ as

$$\mu(\varsigma, \omega) = \left(\sum_{n=1}^{\infty} |\varsigma_n - \omega_n|^p \right)^{\frac{1}{p}},$$

where $\varsigma = \{\varsigma_n\}_{n\in\mathbb{N}}$ and $\omega = \{\omega_n\}_{n\in\mathbb{N}}$. Then μ is a b-metric with parameter $\eta = 2^{\frac{1}{p}}$ and hence $\left(\mho, \mu, 2^{\frac{1}{p}}\right)$ is a b-metric space.

Example 2.3 ([15])**.** Let $\mho = \mathbb{N} \cup \{\infty\}$ and $\mu : \mho \times \mho \to \mathbb{R}_+$ be defined by

$$\mu(\varsigma, \omega) = \begin{cases} 0, & \text{if } \varsigma = \omega, \\ \left|\frac{1}{\varsigma} - \frac{1}{\omega}\right|, & \text{if } \varsigma, \omega \text{ are even or } xy = \infty, \\ 5, & \text{if } \varsigma, \omega \text{ are odd and } \varsigma \neq \omega, \\ 2, & \text{otherwise.} \end{cases}$$

Then $(\mho, \mu)$ is a b-metric space with parameter $\eta = 3$, but μ is not a continuous functional.

Definition 2.4 ([8])**.** Let $(\mho, \mu, \eta)$ be a b-metric space. A sequence $\{\varsigma_n\}_{n\in\mathbb{N}}$ is said to be:

(i) convergent if and only if there exists $\varsigma \in \mho$ such that $\mu(\varsigma_n, \varsigma) \to 0$ as $n \to \infty$, and we write this as $\lim_{n\to\infty} \mu(\varsigma_n, \varsigma) = 0$.
(ii) Cauchy if and only if $\mu(\varsigma_n, \varsigma_m) \to 0$ as $n, m \to \infty$.
(iii) complete if every Cauchy sequence in $\mho$ is convergent.

In a b-metric space, the limit of a sequence is not always unique. However, if a b-metric is continuous, then every convergent sequence has a unique limit.

Definition 2.5 ([8])**.** Let $(\mho, \mu, \eta)$ be a b-metric space. Then, a subset ∇ of $\mho$ is called:

(i) compact if and only if for every sequence of elements of ∇, there exists a subsequence that converges to an element of ∇.
(ii) closed if and only if for every sequence $\{\varsigma_n\}_{n\in\mathbb{N}}$ of elements of ∇ that converges to an element ς, we have $\varsigma \in \nabla$.

Definition 2.6 ([18])**.** A nonempty subset ∇ of $\mho$ is called proximal if, for each $\varsigma \in \mho$, there exists $a \in \nabla$ such that $\mu(\varsigma, a) = \mu(\varsigma, \nabla)$.

Throughput this paper, we shall denote by $CB(\mho)$, $\mathcal{P}^r(\mho)$, $\mathcal{P}^r_b(\mho)$ and $\mathcal{K}(\mho)$, the set of all nonempty closed and bounded subsets of $\mho$, the family of all nonempty proximal subsets of $\mho$, the set of all bounded proximal subsets of $\mho$ and the class of nonempty compact subsets of $\mho$, respectively.

Let $(\mho, \mu, \eta)$ be a b-metric space. For $\nabla, \triangle \in \mathcal{P}^r(\mho)$, the function $\aleph : \mathcal{P}^r(\mho) \times \mathcal{P}^r(\mho) \to \mathbb{R}_+$, defined by

$$\aleph(\nabla, \triangle) = \begin{cases} \max\left\{\sup_{\varsigma\in\nabla} \mu(\varsigma, \triangle), \sup_{\varsigma\in\triangle} \mu(\varsigma, \nabla)\right\}, & \text{if it exists,} \\ \infty, & \text{otherwise,} \end{cases}$$

is called Hausdorff-Pompeiu b-metric on $\mathcal{P}^r(\mho)$ induced by the b-metric μ, where

$$\mu(\varsigma, \nabla) = \inf_{\omega \in \nabla} \mu(\varsigma, \omega).$$

Remark 2.7. Since every compact set is proximal and every proximal set is closed (see [18]), we have the inclusions:

$$\begin{aligned}\mathcal{K}(\mho) &\subseteq \mathcal{P}^r(\mho)\\ &\subseteq CB(\mho)\\ &\subseteq \mathcal{N}(\mho).\end{aligned}$$

In what follows, we recall specific concepts of fuzzy sets and L-fuzzy sets that are needed in the sequel. For these concepts, we follow [13, 32, 38].

Let $\mho$ be a universal set. A fuzzy set in $\mho$ is a function with domain $\mho$ and values in $[0, 1] = I$. If ∇ is a fuzzy set in $\mho$, then the function value $\nabla(\varsigma)$ is called the grade of membership of ς in ∇. The α-level set of a fuzzy set ∇ is denoted by $[\nabla]_\alpha$ and is defined as follows:

$$[\nabla]_\alpha = \begin{cases} \overline{\{\varsigma \in \mho : \nabla(\varsigma) > 0\}}, & \text{if } \alpha = 0, \\ \{\varsigma \in \mho : \nabla(\varsigma) \geq \alpha\}, & \text{if } \alpha \in (0, 1]. \end{cases}$$

where by $\overline{M}$, we mean the closure of the crisp set M. We denote the family of fuzzy sets in $\mho$ by $I^\mho$.

A fuzzy set ∇ in a metric space V is said to be an approximate quantity if and only if $[\nabla]_\alpha$ is compact and convex in V and $\sup_{\varsigma \in V} \nabla(\varsigma) = 1$. Denote the collection of all approximate quantities in V by $W(V)$. If there exists an $\alpha \in [0, 1]$ such that $[\nabla]_\alpha, [\triangle]_\alpha \in \mathcal{P}^r_b(\mho)$, then define

$$\begin{aligned} D_\alpha(\nabla, \triangle) &= \aleph\left([\nabla]_\alpha, [\triangle]_\alpha\right), \\ \mu_\infty(\nabla, \triangle) &= \sup_\alpha D_\alpha(\nabla, \triangle). \end{aligned}$$

Definition 2.8. A relation $\preceq$ on a nonempty set L is called a partial order if it is

(i) Reflexive;
(ii) Antisymmetric;
(iii) Transitive.

A set L together with a partial ordering $\preceq$ is called a partially ordered set (poset, for short) and is denoted by $(L, \preceq_L)$. Recall that partial orderings are used to give an order to sets that may not have a natural one.

Definition 2.9. Let L be a nonempty set and $(L, \preceq)$ be a partially ordered set. Then any two elements $\varsigma, \omega \in L$ are said to be comparable if either $\varsigma \preceq \omega$ or $\omega \preceq \varsigma$.

Definition 2.10. A partially ordered set $(L, \preceq_L)$ is called:

(i) a lattice, if $\varsigma \vee \omega \in L$, $\varsigma \wedge \omega \in L$ for any $\varsigma, \omega \in L$;
(ii) a complete lattice, if $\bigvee \nabla \in L$, $\bigwedge \nabla \in L$ for any $\nabla \subseteq L$;
(iii) distributive lattice if

$$\varsigma \vee (\omega \wedge \xi) = (\varsigma \vee \omega) \wedge (\varsigma \vee \xi), \qquad \varsigma \wedge (\omega \vee \xi) = (\varsigma \wedge \omega) \vee (\varsigma \wedge \xi),$$

for any $\varsigma, \omega, \xi \in L$.

A partially ordered set L is called a complete lattice if for every $\zeta, \omega \in L$, either $\sup\{\varsigma, \omega\} = \varsigma \bigvee \omega$ or $\inf\{\varsigma, \omega\} = \varsigma \bigwedge \omega$ exists.

Definition 2.11. An L-fuzzy set ∇ on a nonempty set $\mho$ is a function with domain $\mho$ whose range lies in a complete distributive lattice L with top and bottom elements 1_L and 0_L, respectively.

Remark 2.12. The class of L-fuzzy sets is larger than the class of fuzzy sets as an L-fuzzy set reduces to a fuzzy set if $L = I = [0, 1]$.

Denote the class of all L-fuzzy sets on a nonempty set $\mho$ by $L^{\mho}$ (to mean a function : $\mho \to L$).

Definition 2.13. The α_L-level set of an L-fuzzy set ∇ is denoted by $[\nabla]_{\alpha L}$ and is defined as follows:

$$[\nabla]_{\alpha L} = \begin{cases} \overline{\{\varsigma \in \mho : 0_L \preceq_L \nabla(\varsigma)\}}, & \text{if } \alpha_L = 0, \\ \{\varsigma \in \mho : \alpha_L \preceq_L \nabla(\varsigma)\}, & \text{if } \alpha_L \in L \setminus \{0_L\}. \end{cases}$$

Definition 2.14. Let $\mho$ be an arbitrary nonempty set and Y a metric space. A mapping $\Psi : \mho \to L^Y$ is called an L-fuzzy mapping. The function value $\Psi(\varsigma)(\omega)$ is called the degree of membership of ω in $\Psi(\varsigma)$. For any two L-fuzzy mappings $\Upsilon, \Psi : \mho \to L^Y$, a point $u \in \mho$ is called an L-fuzzy fixed point of Υ if there exists $\alpha_L \in L \setminus \{0_L\}$ such that $u \in [\Upsilon u]_{\alpha_L}$. A point u is known as a common L-fuzzy fixed point of Υ and Ψ if $u \in [\Upsilon u]_{\alpha_L} \cap [\Psi u]_{\alpha_L}$.

Definition 2.15 ([19, 34])**.** A nondecreasing function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is called:

(i) a c-comparison function if $\varphi^n(t) \to 0$ as $n \to \infty$ for every $t \in \mathbb{R}_+$;
(ii) a b-comparison function if there exist $k_0 \in \mathbb{N}$, $\lambda \in (0, 1)$ and a convergent non-negative series $\sum_{n=1}^{\infty} \varsigma_n$ such that

$$\eta^{k+1}\varphi^{k+1}(t) \leq \lambda \eta^k \varphi^k(t) + \varsigma_k,$$

for $\eta \geq 1$, $k \geq k_0$ and any $t \geq 0$, where φ^n denotes the n-th iterate of φ.

Denote by Ω, the family of functions $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying the following conditions:

(i) φ is a b-comparison function;
(ii) $\varphi(t) = 0$ if and only if $t = 0$;
(iii) φ is continuous.

Remark 2.16 ([19])**.** A b-comparison function is a c-comparison function when $\eta = 1$. Moreover, it can be shown that a c-comparison function is a comparison function, but the converse is not always true.

Lemma 2.17 ([34])**.** *For a comparison function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$, the following properties hold:*

(i) *each iterate φ^n, $n \in \mathbb{N}$, is also a comparison function;*
(iii) *$\varphi(t) < t$ for all $t > 0$.*

Lemma 2.18 ([34])**.** *Let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be a b-comparison function. Then, the series $\sum_{k=0}^{\infty} \eta^k \varphi^k(t)$ converges for every $t \in \mathbb{R}_+$.*

Remark 2.19 ([19])**.** In Lemma 2.18, every b-comparison function is a comparison function and thus, in Lemma 2.17, every b-comparison function satisfies $\varphi(t) < t$.

Lemma 2.20 ([37])**.** *Let $(\mho, \mu, \eta)$ be a b-metric space. For $\nabla, \triangle \in \mathcal{K}(\mho)$ and $\varsigma, \omega \in \mho$, the following conditions hold:*

(i) $\mu(\varsigma, \triangle) \leq \aleph(\nabla, \triangle)$, *for any $\varsigma \in \nabla$;*
(ii) $\mu(\varsigma, \nabla) \leq \eta\,[\mu(\varsigma, \omega) + \mu(\omega, \nabla)]$;
(iii) $\mu(\varsigma, \nabla) = 0 \Leftrightarrow \varsigma \in \nabla$;
(iv) $\aleph(\nabla, \triangle) = 0 \Leftrightarrow \nabla = \triangle$;
(v) $\aleph(\nabla, \triangle) = \aleph(\triangle, \nabla)$;
(vi) $\aleph(\nabla, \triangle) \leq \eta\,[\aleph_b(\nabla, C) + \aleph_b(C, \triangle)]$.

3. Main Results

We start this section by introducing the following definition of p-hybrid L-fuzzy contraction.

Definition 3.1. Let $(\mho, \mu, \eta)$ be a b-metric space and $\Upsilon, \Psi : \mho \to L^{\mho}$ be L-fuzzy set-valued maps. Then, the pair (Υ, Ψ) is said to form a p-hybrid L-fuzzy contraction, if for all $\varsigma, \omega \in \mho$, there exists $\alpha_L \in L \setminus \{0_L\}$ such that

$$\aleph\left([\Upsilon\varsigma]_{\alpha_L}, [\Psi\omega]_{\alpha_L}\right) \leq \varphi\left(\mathcal{C}^p_{(\Upsilon,\Psi)}(\varsigma, \omega, \alpha_L)\right), \tag{3.1}$$

where $\varphi \in \Omega$, $p \geq 0$, $a_i \geq 0$, $i = 1, 2, 3, 4$ with $\sum_{i=1}^{4} a_i = 1$ and

$$\mathcal{C}^{p}_{(\Upsilon,\Psi)}(\varsigma,\omega,\alpha_L) \tag{3.2}$$
$$= \begin{cases} \Bigg[a_1(\mu(\varsigma,\omega))^p + a_2\Big(\mu(\varsigma,[\Upsilon\varsigma]_{\alpha_L}\Big)^p \\ \quad +a_3\Big(\mu\Big(\omega,[\Psi\omega]_{\alpha_L}\Big)\Big)^p \\ \quad +a_4\left(\frac{\mu\left(\omega,[\Upsilon\varsigma]_{\alpha_L}\right)+\mu\left(\varsigma,[\Psi\omega]_{\alpha_L}\right)}{2\eta}\right)^p\Bigg]^{\frac{1}{p}}, & \text{for } p>0, \varsigma,\omega\in\mho, \\ (\mu(\varsigma,\omega))^{a_1}\Big(\mu\Big(\varsigma,[\Upsilon\varsigma]_{\alpha_L}\Big)\Big)^{a_2} \\ \quad \times\Big(\mu\Big(\omega,[\Psi\omega]_{\alpha_L}\Big)\Big)^{a_3} \\ \quad \times\left(\frac{\mu\left(\varsigma,[\Psi\omega]_{\alpha_L}\right)+\mu\left(\omega,[\Upsilon\varsigma]_{\alpha_L}\right)}{2\eta}\right)^{a_4}, & \text{for } p=0, \varsigma,\omega\in\backslash\mathcal{F}_{ix}(\Upsilon,\Psi), \end{cases}$$

where $\mathcal{F}_{ix}(\Upsilon,\Psi)=\{\varsigma,\omega\in\mho:\varsigma\in[\Upsilon\varsigma]_{\alpha_L},\ \omega\in[\Psi\omega]_{\alpha_L}\}$.

In particular, if 3.1 holds for $p=0$, then we say that the pair (Υ,Ψ) forms a 0-hybrid L-fuzzy contraction.

Our main result is presented as follows.

Theorem 3.2. *Let $(\mho,\mu,\eta)$ be a complete b-metric space and $\Upsilon,\Psi:\mho\to L^{\mho}$ be L-fuzzy set-valued maps. Suppose that for each $\varsigma\in\mho$, there exists $\alpha_L\in L\setminus\{0_L\}$ such that $[\Upsilon\varsigma]_{\alpha_L}$ and $[\Psi\varsigma]_{\alpha_L}$ are nonempty bounded proximal subsets of $\mho$. If the pair (Υ,Ψ) forms a p-hybrid L-fuzzy contraction, then Υ and Ψ have a common L-fuzzy fixed point in $\mho$.*

Proof. Let $\varsigma_0\in\mho$, then, by hypotheses, there exists $\alpha_L\in L\setminus\{0_L\}$ such that $[\Upsilon\varsigma_0]_{\alpha_L}\in\mathcal{P}^r_b(\mho)$. Take $\varsigma_1\in[\Upsilon\varsigma_0]_{\alpha_L}$ such that

$$\mu(\varsigma_0,\varsigma_1)=\mu\left(\varsigma_0,[\Upsilon\varsigma_0]_{\alpha_L}\right).$$

Similarly, $[\Psi\varsigma_1]_{\alpha_L}\in\mathcal{P}^r_b(\mho)$, by hypothesis. So, we can find $\varsigma_2\in[\Psi\varsigma_1]_{\alpha_L}$ such that by proximality of Ψ, $\mu(\varsigma_1,\varsigma_2)=\mu(\varsigma_1,[\Psi\varsigma_1]_{\alpha_L})$. Continuing in this direction, we can construct a sequence $\{\varsigma_n\}_{n\in\mathbb{N}}$ of elements of $\mho$ such that

$$\varsigma_{2k+1}\in[\Upsilon\varsigma_{2k}]_{\alpha_L},\qquad \varsigma_{2k+2}\in[\Psi\varsigma_{2k+1}]_{\alpha_L},$$

and

$$\mu(\varsigma_{2k},\varsigma_{2k+1})=\mu\left(\varsigma_{2k},[\Upsilon\varsigma_{2k}]_{\alpha_L}\right),$$
$$\mu(\varsigma_{2k+1},\varsigma_{2k+2})=\mu(\varsigma_{2k+1},[\Psi\varsigma_{2k+1}]_{\alpha_L}),\quad k\in\mathbb{N}.$$

By Lemma 2.20 and the above relations, we have

$$\mu(\varsigma_{2k},\varsigma_{2k+1})\leq\aleph\left([\Upsilon\varsigma_{2k}]_{\alpha_L},[\Psi\varsigma_{2k-1}]_{\alpha_L}\right). \tag{3.3}$$
$$\mu(\varsigma_{2k+1},\varsigma_{2k+2})\leq\aleph\left([\Upsilon\varsigma_{2k}]_{\alpha_L},[\Psi\varsigma_{2k+1}]_{\alpha_L}\right). \tag{3.4}$$

Suppose that $\varsigma_{2k}=\varsigma_{2k+1}$, for some $k\in\mathbb{N}$ and $p>0$. Then, from (3.2), we have

$$\begin{aligned}
&\mathcal{C}^p_{(\Upsilon,\Psi)}(\varsigma_{2k}, \varsigma_{2k+1}, , \alpha_L)\\
&\quad = \Bigg[a_1 \left(\mu(\varsigma_{2k}, \varsigma_{2k+1})\right)^r + a_2 \left(\mu(\varsigma_{2k}, [\Upsilon\varsigma_{2k}]_{\alpha_L})\right)^p\\
&\qquad + a_3 \left(\mu(\varsigma_{2k+1}, [\Psi\varsigma_{2k+1}]_{\alpha_L})\right)^r\\
&\qquad + a_4 \left(\frac{\mu(\varsigma_{2k+1}, [\Upsilon\varsigma_{2k}]_{\alpha_L}) + \mu(\varsigma_{2k}, [\Psi\varsigma_{2k+1}]_{\alpha_L})}{2\eta}\right)^p\Bigg]^{\frac{1}{p}}\\
&\quad = \Bigg[a_1(\mu(\varsigma_{2k}, \varsigma_{2k+1}))^p + a_2(\mu(\varsigma_{2k}, \varsigma_{2k+1}))^p + a_3(\mu(\varsigma_{2k+1}, \varsigma_{2k+2}))^p\\
&\qquad + a_4 \left(\frac{\mu(\varsigma_{2k+1}, \varsigma_{2k+1}) + \mu(\varsigma_{2k}, \varsigma_{2k+2})}{2\eta}\right)^p\Bigg]^{\frac{1}{p}}\\
&\quad \leq \left[a_3(\mu(\varsigma_{2k+1}, \varsigma_{2k+2}))^p + a_4 \left(\eta\left(\frac{\mu(\varsigma_{2k}, \varsigma_{2k+1}) + \mu(\varsigma_{2k+1}, \varsigma_{2k+2})}{2\eta}\right)\right)^p\right]^{\frac{1}{p}}\\
&\quad \leq [a_3(\mu(\varsigma_{2k+1}, \varsigma_{2k+2}))^p + a_4(\mu(\varsigma_{2k+1}, \varsigma_{2k+2}))^p]^{\frac{1}{p}}\\
&\quad = (a_3 + a_4)^{\frac{1}{p}} \mu(\varsigma_{2k+1}, \varsigma_{2k+2})\\
&\quad = \mu(\varsigma_{2k+1}, \varsigma_{2k+2}), \qquad \text{as } p \to \infty.
\end{aligned}$$

Therefore, using Lemma 2.17, we have

$$\begin{aligned}
\mu(\varsigma_{2k+1}, \varsigma_{2k+2}) &\leq \aleph\left([\Upsilon\varsigma_{2k}]_{\alpha_L}, [\Psi\varsigma_{2k+1}]_{\alpha_L}\right)\\
&\leq \varphi(\mu(\varsigma_{2k+1}, \varsigma_{2k+2}))\\
&< \mu(\varsigma_{2k+1}, \varsigma_{2k+2}),
\end{aligned}$$

a contradiction. It follows that for all $k \in \mathbb{N}$,

$$\varsigma_{2k} = \varsigma_{2k+1} \in [\Upsilon\varsigma_{2k}]_{\alpha_L},$$

and

$$\begin{aligned}
\varsigma_{2k} &= \varsigma_{2k+1}\\
&= \varsigma_{2k+2} \in [\Psi\varsigma_{2k+1}]_{\alpha_L} = [\Psi\varsigma_{2k}]_{\alpha_L}.
\end{aligned}$$

It follows that ς_{2k} is the common fixed point of Υ and Ψ.

Again, for $p = 0$ and $\varsigma_{2k} = \varsigma_{2k+1}$, for some $k \in \mathbb{N}$, we get

$$\mathcal{C}^p_{(\Upsilon,\Psi)}(\varsigma_{2k}, \varsigma_{2k+1}) = 0, \quad \forall k \in \mathbb{N}.$$

Therefore, by property (ii) of Ω, one obtains $\mu(\varsigma_{2k+1}, \varsigma_{2k+2}) = 0$, for all $k \in \mathbb{N}$; from which, on similar arguments as above, the same conclusion follows that $\varsigma_{2k} \in [\Upsilon\varsigma_{2k}]_{\alpha_L} \cap [\Psi\varsigma_{2k}]_{\alpha_L}$. Hereafter, we assume that for all

$k \in \mathbb{N}$, $\varsigma_{k+1} \neq \varsigma_k$ if and only if $\mu(\varsigma_{k+1}, \varsigma_k) > 0$.

Now, in view of (3.2), setting $\varsigma = \varsigma_{2k}$ and $\omega = \varsigma_{2k-1}$, we have

$$\mathcal{C}^p_{(\Upsilon,\Psi)}(\varsigma_{2k}, \varsigma_{2k-1},, \alpha_L)$$
$$= \begin{cases} \Big[a_1 \left(\mu(\varsigma_{2k}, \varsigma_{2k-1})\right)^p + a_2 \left(\mu\left(\varsigma_{2k}, [\Upsilon \varsigma_{2k}]_{\alpha_L}\right)\right)^p \\ \quad +a_3 \left(\mu\left(\varsigma_{2k-1}, [\Psi \varsigma_{2k-1}]_{\alpha_L}\right)\right)^p & \text{for } p > 0, \\ \quad +a_4 \left(\frac{\mu(\varsigma_{2k-1}, [\Upsilon \varsigma_{2k}]_{\alpha_L}) + \mu(\varsigma_{2k}, [\Psi \varsigma_{2k-1}]_{\alpha_L})}{2\eta}\right)^p\Big]^{\frac{1}{p}}, \\ \left(\mu\left(\varsigma_{2k}, \varsigma_{2k-1}\right)\right)^{a_1} \left(\mu\left(\varsigma_{2k}, [\Upsilon \varsigma_{2k}]_{\alpha_L}\right)\right)^{a_2} \\ \quad \times \left(\mu\left(\varsigma_{2k-1}, [\Psi \varsigma_{2k-1}]_{\alpha_L}\right)\right)^{a_3} & \text{for } p = 0. \\ \quad \times \left(\frac{\mu(\varsigma_{2k}, [\Psi \varsigma_{2k-1}]_{\alpha_L}) + \mu(\varsigma_{2k-1}, [\Upsilon \varsigma_{2k}]_{\alpha_L})}{2\eta}\right)^{a_4}, \end{cases}$$

That is,

$$\mathcal{C}^p_{(\Upsilon,\Psi)}(\varsigma_{2k}, \varsigma_{2k-1}), \alpha_L \tag{3.5}$$
$$= \begin{cases} \Big[a_1 \left(\mu\left(\varsigma_{2k}, \varsigma_{2k-1}\right)\right)^p + a_2 \left(\mu\left(\varsigma_{2k}, \varsigma_{2k+1}\right)\right)^p \\ \quad +a_3 \left(\mu\left(\varsigma_{2k-1}, \varsigma_{2k}\right)\right)^p & \text{for } p > 0, \\ \quad +a_4 \left(\frac{\mu(\varsigma_{2k-1}, \varsigma_{2k+1}) + \mu(\varsigma_{2k}, \varsigma_{2k})}{2\eta}\right)^p\Big]^{\frac{1}{p}}, \\ \left(\mu\left(\varsigma_{2k}, \varsigma_{2k-1}\right)\right)^{a_1} \left(\mu\left(\varsigma_{2k}, \varsigma_{2k+1}\right)\right)^{a_2} \\ \quad \times \left(\mu\left(\varsigma_{2k-1}, \varsigma_{2k}\right)\right)^{a_3} & \text{for } p = 0. \\ \quad \times \left(\frac{\mu(\varsigma_{2k}, \varsigma_{2k}) + \mu(\varsigma_{2k-1}, \varsigma_{2k+1})}{2\eta}\right)^{a_4}, \end{cases}$$

Now, we consider the following two cases:

Case 1: $p > 0$. Suppose that $\mu(\varsigma_{2k}, \varsigma_{2k+1}) \geq \mu(\varsigma_{2k-1}, \varsigma_{2k})$, then from (3.5), we have

(3.6)

$$\mathcal{C}^p_{(\Upsilon,\Psi)}(\varsigma_{2k}, \varsigma_{2k-1}, \alpha_L)$$
$$\leq \Big[a_1 \left(\mu\left(\varsigma_{2k+1}, \varsigma_{2k}\right)\right)^p + a_2 \left(\mu\left(\varsigma_{2k+1}, \varsigma_{2k}\right)\right)^p + a_3 \left(\mu\left(\varsigma_{2k+1}, \varsigma_{2k}\right)\right)^p$$
$$+ a_4 \left(\eta \left(\frac{\mu(\varsigma_{2k+1}, \varsigma_{2k}) + \mu(\varsigma_{2k}, \varsigma_{2k-1})}{2\eta}\right)\right)^p\Big]^{\frac{1}{p}}$$

$$
\begin{aligned}
&\leq \Bigg[a_1\left(\mu\left(\varsigma_{2k+1},\varsigma_{2k}\right)\right)^p + a_2\left(\mu\left(\varsigma_{2k+1},\varsigma_{2k}\right)\right)^p + a_3\left(\mu\left(\varsigma_{2k+1},\varsigma_{2k}\right)\right)^p \\
&\quad + a_4\left(\eta\left(\frac{\mu(\varsigma_{2k+1},\varsigma_{2k})+\mu(\varsigma_{2k+1},\varsigma_{2k})}{2\eta}\right)^p\right)\Bigg]^{\frac{1}{p}} \\
&\leq \left[a_1\left(\mu\left(\varsigma_{2k+1},\varsigma_{2k}\right)\right)^p + a_2\left(\mu\left(\varsigma_{2k+1},\varsigma_{2k}\right)\right)^p + a_3\left(\mu\left(\varsigma_{2k+1},\varsigma_{2k}\right)\right)^p \right. \\
&\quad \left. + a_4\left(\mu\left(\varsigma_{2k+1},\varsigma_{2k}\right)\right)^p\right]^{\frac{1}{p}} \\
&= \left[(a_1+a_2+a_3+a_4)\mu(\varsigma_{2k+1},\varsigma_{2k})^p\right]^{\frac{1}{p}} \\
&= \mu(\varsigma_{2k+1},\varsigma_{2k})\left(\sum_{i=1}^{4} a_i\right)^{\frac{1}{p}} \\
&= \mu(\varsigma_{2k+1},\varsigma_{2k}).
\end{aligned}
$$

Hence, from (3.1) and (3.6), we have

$$\mu(\varsigma_{2k+1},\varsigma_{2k}) \leq \varphi\left(\mu(\varsigma_{2k+1},\varsigma_{2k})\right). \tag{3.7}$$

Given that φ is a b-comparison function, (3.7) implies

$$\mu(\varsigma_{2k+1},\varsigma_{2k}) < \mu(\varsigma_{2k+1},\varsigma_{2k}),$$

which is a contradiction. Consequently, it follows that

$$\mu(\varsigma_{2k+1},\varsigma_{2k}) \leq \mu(\varsigma_{2k},\varsigma_{2k-1}).$$

Thus, from (3.7), we obtain

$$\mu(\varsigma_{2k+1},\varsigma_{2k}) \leq \varphi(\mu(\varsigma_{2k},\varsigma_{2k-1})). \tag{3.8}$$

Setting $n = 2k \in \mathbb{N}$ in (3.8), yields

$$
\begin{aligned}
\mu(\varsigma_{n+1},\varsigma_n) &\leq \varphi(\mu(\varsigma_n,\varsigma_{n-1})) \\
&\leq \varphi^2(\mu(\varsigma_{n-1},\varsigma_{n-2})) \\
&\leq \varphi^3(\mu(\varsigma_{n-2},\varsigma_{n-3})) \\
&\ \ \vdots \\
&\leq \varphi^n(\mu(\varsigma_1,\varsigma_0)).
\end{aligned} \tag{3.9}
$$

From (3.9), by triangle inequality on $(\mho,\mu,\eta)$, for all $k \geq 1$, we have

$$
\begin{aligned}
\mu(\varsigma_{n+k},\varsigma_n) &\leq \eta\left(\mu(\varsigma_{n+k},\varsigma_{n+1}) + \mu(\varsigma_{n+1},\varsigma_n)\right) \\
&\leq \frac{1}{\eta^{n-1}}\sum_{i=n}^{n+k-1}\eta^k\mu(\varsigma_i,\varsigma_{i+1})
\end{aligned} \tag{3.10}
$$

$$\leq \frac{1}{\eta^{n-1}} \sum_{i=n}^{n+k-1} \eta^k \varphi^k(\mu(\varsigma_1, \varsigma_0))$$
$$\leq \frac{1}{\eta^{n-1}} \sum_{i=n}^{\infty} \eta^i \varphi^i(\mu(\varsigma_1, \varsigma_0)).$$

Letting $n \to \infty$ in (3.10) and applying Lemma 2.18, we find that $\lim_{n\to\infty} \mu(\varsigma_{n+k}, \varsigma_n) = 0$. Therefore, $\{\varsigma_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence of points of $(\mho, \mu, \eta)$. The completeness of this space implies that there exists $u \in \mho$ such that

$$\lim_{n\to\infty} \mu(\varsigma_n, u) = 0. \tag{3.11}$$

Now, we show that u is the expected common fixed point of Υ and Ψ. First, assume that $u \notin [\Upsilon u]_{\alpha_L}$ so that $\mu(u, [\Upsilon u]_{\alpha_L}) > 0$. Then, by Lemma 2.20 and the case $p > 0$ in the contractive inequality (3.1), we have

(3.12)

$$\begin{aligned}
\mu(u, [\Upsilon u]_{\alpha_L}) &\leq \eta\mu(u, \varsigma_n) + \eta\mu(\varsigma_n, [\Upsilon u]_{\alpha_L}) \\
&\leq \eta\mu(u, \varsigma_n) + \eta\aleph\left([\Upsilon u]_{\alpha_L}, [\Psi\varsigma_{n-1}]_{\alpha_L}\right) \\
&\leq \eta\mu(u, \varsigma_n) + \eta\varphi\left(\mathcal{C}^p_{(\Upsilon,\Psi)}(u, \varsigma_{n-1})\right) \\
&= \eta\mu(u, \varsigma_n) + \eta\varphi\Bigg(\Bigg[a_1\left(\mu\left(u, \varsigma_{n-1}\right)\right)^p + a_2\left(\mu\left(u, [\Upsilon u]_{\alpha_L}\right)\right)^p \\
&\quad + a_3\left(\mu\left(\varsigma_{n-1}, [\Psi\varsigma_{n-1}]_{\alpha_L}\right)\right)^p \\
&\quad + a_4\left(\frac{\mu\left(\varsigma_{n-1}, [\Upsilon u]_{\alpha_L}\right) + \mu(u, [\Psi\varsigma_{n-1}]_{\alpha_L})}{2\eta}\right)^p\Bigg]^{\frac{1}{p}}\Bigg) \\
&= \eta\mu(u, \varsigma_n) + \eta\varphi\Bigg(\Bigg[a_1(\mu(u, \varsigma_{n-1}))^p + a_2(\mu(u, [\Upsilon u]_{\alpha_L}))^p \\
&\quad + a_3(\mu(\varsigma_{n-1}, \varsigma_n))^p \\
&\quad + a_4\left(\frac{\mu(\varsigma_{n-1}, [\Upsilon u]_{\alpha_L}) + \mu(u, \varsigma_n)}{2\eta}\right)^p\Bigg]^{\frac{1}{p}}\Bigg).
\end{aligned}$$

Letting $n \to \infty$ in (3.12), and using the properties of $\varphi \in \Omega$, gives

$$\mu(u, [\Upsilon u]_{\alpha_L}) < \eta\mu(u, [\Upsilon u]_{\alpha_L})(a_2 + a_4)^{\frac{1}{p}},$$

and as $p \to \infty$,

$$\mu(u, [\Upsilon u]_{\alpha_L}) < \eta\mu(u, [\Upsilon u]_{\alpha_L}). \tag{3.13}$$

We note that putting $\eta = 1$ in (3.13) yields a contradiction. Thus, $\mu(u, [\Upsilon u]_{\alpha_L}) = 0$, which further implies that $u \in [\Upsilon u]_{\alpha_L}$. On similar steps, by assuming that u is not a fixed point of Ψ, and considering

$$\begin{aligned}\mu(u, [\Psi u]_{\alpha_L}) &\leq \eta\mu(u, \varsigma_n) + \eta\mu(\varsigma_n, [\Psi u]_{\alpha_L}) \\ &\leq \eta\mu(u, \varsigma_n) + \eta\aleph\left([\Upsilon\varsigma_{n-1}]_{\alpha_L}, [\Psi u]_{\alpha_L}\right) \\ &\leq \eta\mu(u, \varsigma_n) + \eta\varphi\left(\mathcal{C}^p_{(\Upsilon,\Psi)}(\varsigma_{n-1}, u)\right),\end{aligned}$$

we can show that $u \in [\Psi u]_{\alpha_L}$. Consequently, for $p > 0$, there exists $u \in \mho$ such that $u \in [\Upsilon u]_{\alpha_L} \cap [\Psi u]_{\alpha_L}$.

Case 2: $p = 0$. Applying the inequality (3.5) on account of b-comparison of φ, we have

(3.14)

$$\begin{aligned}\mu(\varsigma_{2k}, \varsigma_{2k-1}) &\leq \aleph\left([\Upsilon\varsigma_{2k-1}]_{\alpha_L}, [\Psi\varsigma_{2k-2}]_{\alpha_L}\right) \\ &\leq \varphi\left(\mathcal{C}^p_{(\Upsilon,\Psi)}(\varsigma_{2k-1}, \varsigma_{2k-2})\right) \\ &< (\mu(\varsigma_{2k-1}, \varsigma_{2k-2}))^{a_1}(\varsigma_{2k-1}, [\Upsilon\varsigma_{2k-1}]_{\alpha_L})^{a_2} \\ &\quad \times (\mu(\varsigma_{2k-2}, [\Psi\varsigma_{2k-2}]_{\alpha_L}))^{a_3} \\ &\quad \times \left(\frac{\mu(\varsigma_{2k-1}, [\Psi\varsigma_{2k-2}]_{\alpha_L}) + \mu(\varsigma_{2k-2}, [\Upsilon\varsigma_{2k-1}]_{\alpha_L})}{2\eta}\right)^{a_4} \\ &= (\mu(\varsigma_{2k-1}, \varsigma_{2k-2}))^{a_1}(\mu(\varsigma_{2k-1}, \varsigma_{2k}))^{a_2}(\mu(\varsigma_{2k-2}, \varsigma_{2k-1}))^{a_3} \\ &\quad \times \left(\frac{\mu(\varsigma_{2k-1}, \varsigma_{2k-1}) + \mu(\varsigma_{2k-2}, \varsigma_{2k})}{2\eta}\right)^{a_4} \\ &\leq (\mu(\varsigma_{2k-1}, \varsigma_{2k-2}))^{a_1}(\mu(\varsigma_{2k-1}, \varsigma_{2k}))^{a_2}(\mu(\varsigma_{2k-2}, \varsigma_{2k-1}))^{a_3} \\ &\quad \times \left(\frac{\mu(\varsigma_{2k}, \varsigma_{2k-1}) + \mu(\varsigma_{2k-1}, \varsigma_{2k-2})}{2}\right)^{a_4} \\ &= (\mu(\varsigma_{2k-1}, \varsigma_{2k-2}))^{a_1+a_3}(\mu(\varsigma_{2k-1}, \varsigma_{2k}))^{a_2} \\ &\quad \times \left(\frac{\mu(\varsigma_{2k}, \varsigma_{2k-1}) + \mu(\varsigma_{2k-1}, \varsigma_{2k-2})}{2}\right)^{1-a_1-a_2-a_3}.\end{aligned}$$

Assume that $\mu(\varsigma_{2k-1}, \varsigma_{2k-2}) \leq \mu(\varsigma_{2k}, \varsigma_{2k-1})$, then (3.14) gives

$$\begin{aligned}\mu(\varsigma_{2k}, \varsigma_{2k-1}) &\leq \varphi\left(\mathcal{C}^p_{(\Upsilon,\Psi)}(\varsigma_{2k-1}, \varsigma_{2k-2})\right) \\ &< (\mu(\varsigma_{2k}, \varsigma_{2k-1}))^{a_1+a_2+a_3}(\mu(\varsigma_{2k}, \varsigma_{2k-1}))^{1-a_1-a_2-a_3} \\ &= \mu(\varsigma_{2k}, \varsigma_{2k-1}),\end{aligned}$$

a contradiction. Therefore,

$$\mu(\varsigma_{2k}, \varsigma_{2k-1}) \leq \mu(\varsigma_{2k-1}, \varsigma_{2k-2}). \tag{3.15}$$

Using (3.14) and (3.15), we obtain

$$\mu(\varsigma_{2k}, \varsigma_{2k-1}) \leq \varphi(\mu(\varsigma_{2k-1}, \varsigma_{2k-2})). \tag{3.16}$$

Notice that (3.16) is equivalent to (3.9). So, on similar steps, we deduce that the sequence $\{\varsigma_n\}_{n\in\mathbb{N}}$ is Cauchy in $(\mho, \mu, \eta)$. Thus, the completeness of this space guarantees that $\mu(\varsigma_n, u) \to 0$ as $n \to \infty$, for some $u \in \mho$.
To see that u is a common fixed point of Ψ and Υ, we employ Lemma 2.20 and inequality (3.5) as follows:

$$\begin{aligned} \mu(u, [\Psi u]_{\alpha_L}) &\leq \eta\mu(u, \varsigma_n) + \eta\mu(\varsigma_n, [\Psi u]_{\alpha_L}) \\ &\leq \eta\mu(u, \varsigma_n) + \eta\aleph([\Upsilon\varsigma_{n-1}]_{\alpha_L}, [\Psi u]_{\alpha_L}) \\ &\leq \eta\mu(u, \varsigma_n) + \eta\varphi\left(\mathcal{C}^p_{(\Upsilon,\Psi)}(\varsigma_{n-1}, u)\right), \end{aligned} \tag{3.17}$$

where

$$\begin{aligned} \mathcal{C}^p_{(\Upsilon,\Psi)}(\varsigma_{n-1}, u, \alpha_L) &= (\mu(\varsigma_{n-1}, u))^{a_1}(\mu(\varsigma_{n-1}, [\Upsilon\varsigma_{n-1}]_{\alpha_L}))^{a_2}(\mu(u, [\Psi u]_{\alpha_L}))^{a_3} \\ &\quad\times \left(\frac{\mu(\varsigma_{n-1}, [\Psi u]_{\alpha_L}) + \mu(u, [\Upsilon\varsigma_{n-1}]_{\alpha_L})}{2\eta}\right)^{a_4} \\ &= (\mu(\varsigma_{n-1}, u))^{a_1}(\mu(\varsigma_{n-1}, \varsigma_n))^{a_2}(\mu(u, [\Psi u]_{\alpha_L}))^{a_3} \\ &\quad\times \left(\frac{\mu(\varsigma_{n-1}, [\Psi u]_{\alpha_L}) + \mu(u, \varsigma_n)}{2\eta}\right)^{a_4}. \end{aligned}$$

We see that $\lim_{n\to\infty} \mathcal{C}^p_{(\Upsilon,\Psi)}(\varsigma_{n-1}, u) = 0$. Hence, under this limiting case, (3.17) becomes

$$\mu(u, \Psi u) \leq \eta\varphi(0). \tag{3.18}$$

By condition (ii) of φ, (3.18) implies that $\mu(u, [\Psi u]_{\alpha_L}) = 0$. Therefore, $u \in [\Psi u]_{\alpha_L}$. On similar steps, we can show that $u \in [\Upsilon u]_{\alpha_L}$. Consequently, there exists $\alpha_L \in L \setminus \{0_L\}$ such that $u \in [\Upsilon u]_{\alpha_L} \cap [\Psi u]_{\alpha_L}$. $\square$

From Case 2 in the Proof of Theorem 3.2, we have also proved the next theorem.

Theorem 3.3. *Let $(\mho, \mu, \eta)$ be a complete b-metric space and $\Upsilon, \Psi : \mho \to L^{\mho}$ be L-fuzzy set-valued maps. Suppose that for each $\varsigma \in \mho$, there exists $\alpha_L \in L \setminus \{0_L\}$ such that $[\Upsilon\varsigma]_{\alpha_L}$ and $[\Psi\varsigma]_{\alpha_L}$ are nonempty bounded proximal subsets of $\mho$. If the pair (Υ, Ψ) forms a 0-hybrid L-fuzzy contraction, then Υ and Ψ have a common L-fuzzy fixed point in $\mho$.*

The following example is provided to support the hypotheses of Theorem 3.2.

Example 3.4. Let $L = \{a, b, c, g, s, m, n, v\}$ be such that $a \preceq_L s \preceq_L c \preceq_L v$, $a \preceq_L g \preceq_L b \preceq_L v$, $s \preceq_L m \preceq_L v$, $g \preceq_L m \preceq_L v$, $n \preceq_L b \preceq_L v$; and each elements of the pairs $\{c, m\}, \{m, b\}, \{s, n\}, \{n, g\}$ are not comparable. It follows that $(L, \preceq_L)$ is a complete distributive lattice. Let $\mho = [0, \infty)$ and $\mu(\varsigma, \omega) = |\varsigma - \omega|^2$ for all $\varsigma, \omega \in \mho$. Then, $(\mho, \mu, \eta = 2)$ is a complete b-metric space. We notice that $(\mho, \mu, \eta = 2)$ is not a metric space, since for $\varsigma = 1,\ \omega = 5$ and $\xi = 3$,

$$\begin{aligned}\mu(\varsigma, \omega) &= 16\\ &> 8\\ &= \mu(\varsigma, \xi) + \mu(\xi, \omega).\end{aligned}$$

Let $\alpha_L : \mho \to L \setminus \{0_L\}$ be a mapping. For each $\varsigma \in \mho$, consider two L-fuzzy set-valued maps $\Upsilon(\varsigma), \Psi(\varsigma) : \mho \to L$ defined as follows:

If $\varsigma = 0$,

$$\begin{aligned}\Upsilon(\varsigma)(t) &= \Psi(\varsigma)(t)\\ &= \begin{cases} v, & \text{if } t = 0,\\ a, & \text{if } t \neq 0,\end{cases}\end{aligned}$$

if $\varsigma \in (0, 1]$,

$$\Upsilon(\varsigma)(t) = \begin{cases} v, & \text{if } 0 \leq t \leq \varsigma - \frac{\varsigma^2}{6},\\ b, & \text{if } \varsigma - \frac{\varsigma^2}{6} < t \leq \varsigma - \frac{\varsigma^2}{3},\\ s, & \text{if } \varsigma - \frac{\varsigma^2}{3} < t < \infty,\end{cases}$$

$$\Psi(\varsigma)(t) = \begin{cases} v, & \text{if } 0 \leq t \leq \varsigma - \frac{\varsigma^2}{6},\\ c, & \text{if } \varsigma - \frac{\varsigma^2}{6} < t \leq \varsigma - \frac{\varsigma^2}{2},\\ m, & \text{if } \varsigma - \frac{\varsigma^2}{2} < t < \infty,\end{cases}$$

if $\varsigma > 1$,

$$\begin{aligned}\Upsilon(\varsigma)(t) &= \Psi(\varsigma)(t)\\ &= \begin{cases} v, & \text{if } 0 \leq t \leq 6,\\ a, & \text{if } t > 6.\end{cases}\end{aligned}$$

Define the function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ as:

$$\varphi(t) = \begin{cases} t - \frac{t^2}{6}, & \text{if } t \in [0, 1],\\ \frac{1}{6}, & \text{if } t > 0.\end{cases}$$

Clearly, $\varphi(t) < t$ for all $t > 0$. Suppose that $\alpha_L(\varsigma) := \alpha_L = v$, for each $\varsigma \in \mho$. Then, there exists $v \in L \setminus \{0_L\}$ such that $[\Upsilon\varsigma]_v$ and $[\Psi\varsigma]_v$ are nonempty bounded proximal subsets of $\mho$. Now, to verify inequality 3.1, consider the following cases:

Case 1. If $\varsigma=\omega=0$, $p=0$, then for all $a_i\geq 0$ $(i=1,2,3,4)$, we have $[\Upsilon\varsigma]_v=[\Psi\omega]_v$ and so,

$$\begin{aligned}\aleph\left([\Upsilon\varsigma]_v,[\Psi\omega]_v\right)&=0\\&\leq\varphi\left(\mathcal{C}^p_{(\Upsilon,\Psi)}(\varsigma,\omega,v)\right).\end{aligned}$$

Case 2. If $\varsigma=0,\omega\in(0,1]$, $p=1$, $a_1=1$ and $a_2=a_3=a_4=0$, we have

$$[\Upsilon 0]_v=\{0\},\qquad [\Psi\omega]_v=\left[0,\omega-\frac{\omega^2}{6}\right].$$

Therefore,

$$\begin{aligned}\aleph([\Upsilon 0]_v,[\Psi\omega]_v)&=\left|\omega-\frac{\omega^2}{6}\right|^2\\&=\varphi(|\omega-0|^2)\\&\leq\varphi\left(\mathcal{C}^p_{(\Upsilon,\Psi)}(\varsigma,\omega,v)\right).\end{aligned}$$

Case 3. If $\varsigma,\omega\in(0,1]$, $p=1$, $a_1=1$ and $a_2=a_3=a_4=0$, we have

$$\begin{aligned}\aleph([\Upsilon\varsigma]_v,[\Psi\omega]_v)&=\aleph\left(\left[0,\varsigma-\frac{\varsigma^2}{6}\right],\left[0,\omega-\frac{\omega^2}{6}\right]\right)\\&=\left|\varsigma-\frac{\varsigma^2}{6}-\omega+\frac{\omega^2}{6}\right|^2\\&=\left|(\varsigma-\omega)-\frac{1}{6}\left(\varsigma^2-\omega^2\right)\right|^2\\&=\left|(\varsigma-\omega)\left(1-\frac{|\varsigma+\omega|}{6}\right)\right|^2\\&\leq\left||\varsigma-\omega|\left(1-\frac{|\varsigma-\omega|}{6}\right)\right|^2\\&=\left||\varsigma-\omega|-\frac{|\varsigma-\omega|^2}{6}\right|^2\\&=\varphi(|\varsigma-\omega|^2)\\&\leq\varphi\left(\mathcal{C}^p_{(\Upsilon,\Psi)(\varsigma,\omega,v)}\right).\end{aligned}$$

Case 4. If $\varsigma,\omega\in(1,\infty)$, then for all $a_i\geq 0$ $(i=1,2,3,4)$, $p>0$, we have $[\Upsilon\varsigma]_v=[\Psi\omega]_v$ and

$$\begin{aligned}\aleph([\Upsilon\varsigma]_v,[\Psi\omega]_v)&=0\\&\leq\varphi\left(\mathcal{C}^p_{(\Upsilon,\Psi)}(\varsigma,\omega,v)\right).\end{aligned}$$

Hence, all the hypotheses of Theorem 3.2 are satisfied. Consequently, Υ and Ψ have a common L-fuzzy fixed point in $\mho$.

In what follows, we study the concept of p-hybrid L-fuzzy contractions via μ_L^{∞}-distance for L-fuzzy sets. It is noteworthy that the study of fixed points of fuzzy set-valued maps in connection with μ_{∞}-metric is very significant in evaluating Hausdorff dimensions. These dimensions help us to understand the notions of ε^{∞} -space which is of tremendous importance in higher energy physics (see, e.g. [11, 12]). Consistent with Rashid et al. [32, 33], we define some needed auxiliary concepts in the framework of b-metric space as follows. Let $(\mho, \mu, \eta)$ be a b-metric space and take $\alpha_L \in L \setminus \{0_L\}$ such that $[\nabla]_{\alpha_L}, [\triangle]_{\alpha_L} \in \mathcal{P}_b^r(\mho)$. Then, define

$$\begin{aligned} p_{\alpha_L}(\nabla, \triangle) &= \inf_{\varsigma \in [\nabla]_{\alpha_L}, \omega \in [\triangle]_{\alpha_L}} \mu(\varsigma, \omega), \\ D_{\alpha_L}(\nabla, \triangle) &= \aleph([\nabla]_{\alpha_L}, [\triangle]_{\alpha_L}), \\ p(\nabla, \triangle) &= \sup_{\alpha_L} p_{\alpha_L}(\nabla, \triangle), \\ \mu_L^{\infty}(\nabla, \triangle) &= \sup_{\alpha_L} D_{\alpha_L}(\nabla, \triangle). \end{aligned}$$

Note that μ_L^{∞} is a metric on $\mathcal{P}_b^r(\mho)$ (induced by the Hausdorff metric $\aleph$) and the completeness of $(\mho, \mu, \eta)$ implies the completeness of the corresponding metric space $(\mathcal{K}_{\mathcal{F}}(\mho), \mu_L^{\infty})$. Furthermore, $(\mho, \mu, \eta) \longmapsto (\mathcal{P}_b^r(\mho), \aleph) \longmapsto (\mathcal{K}_{\mathcal{F}}(\mho), \mu_L^{\infty}, \eta)$, are isometric embeddings via the relations $\varsigma \to \{\varsigma\}$ and $M \to \chi_M$, respectively; where

$$\mathcal{K}_{\mathcal{F}}(\mho) = \left\{\nabla \in L^{\mho} : [\nabla]_{\alpha} \in \mathcal{P}_b^r(\mho), \text{ for each } \alpha_L \in L \setminus \{0_L\}\right\},$$

and χ_M is the characteristic function of M.

Theorem 3.5. *Let $(\mho, \mu, \eta)$ be a complete b-metric space and $\Upsilon, \Psi : \mho \to \mathcal{K}_{\mathcal{F}}(\mho)$ be L-fuzzy set-valued maps such that*

$$\mu_L^{\infty}(\Upsilon(\varsigma), \Psi(\omega)) \leq \varphi\left(\mathcal{G}^p_{(\Upsilon,\Psi)}(\varsigma, \omega)\right), \tag{3.19}$$

where $\varphi \in \Omega$, $p \geq 0$, $a_i \geq 0$, $i = 1, 2, 3, 4$ with $\sum_{i=1}^{4} a_i = 1$ and

$$\begin{aligned} &\mathcal{G}^p_{(\Upsilon,\Psi)}(\varsigma, \omega) \\ &= \begin{cases} \left[\begin{array}{l} a_1(\mu(\varsigma,\omega))^p + a_2(p(\varsigma,\Upsilon(\varsigma)))^p \\ +a_3(p(\omega,\Psi(\omega)))^p \\ +a_4\left(\frac{p(\omega,\Upsilon(\varsigma))+p(\varsigma,\Psi(\omega))}{2\eta}\right)^p \end{array}\right]^{\frac{1}{p}}, & \text{for } p > 0, \varsigma, \omega \in \mho, \\ \begin{array}{l} (p(\varsigma,\omega))^{a_1}(p(\varsigma,\Upsilon(\varsigma)))^{a_2} \\ \times(p(\omega,\Psi(\omega)))^{a_3} \\ \times\left(\frac{p(\varsigma,\Psi(\omega))+p(\omega,\Upsilon(\varsigma))}{2\eta}\right)^{a_4}, \end{array} & \text{for } p = 0, \varsigma, \omega \in \mho \setminus \mathcal{F}^*{}_{ix}(\Upsilon, \Psi), \end{cases} \end{aligned} \tag{3.20}$$

where

$$\mathcal{F}^*{}_{ix}(\Upsilon, \Psi) = \{\varsigma, \omega \in \mho : \{\varsigma\} \subset \Upsilon(\varsigma),\ \{\omega\} \subset \Psi(\omega)\}.$$

Then, there exists $u \in \mho$ *such that* $\{u\} \subset \Upsilon(u) \cap \Psi(u)$.

Proof. Choose $\varsigma \in \mho$ and define a function $\alpha_L : \mho \to L \setminus \{0_L\}$ by $\alpha_L(\varsigma) := \alpha_L = 1_L$. Then, by hypothesis, $[\Upsilon\varsigma]_{1_L}$ and $[\Psi\varsigma]_{1_L}$ are nonempty bounded proximal subsets of $\mho$. Now, for all $\varsigma, \omega \in \mho$,

$$\begin{aligned} D_{1_L}(\Upsilon(\varsigma), \Psi(\omega)) &\leq \mu_L^{\infty}(\Upsilon(\varsigma), \Psi(\omega)) \\ &\leq \varphi\left(\mathcal{G}^p_{(\Upsilon,\Psi)}(\varsigma, \omega)\right). \end{aligned}$$

Since $[\Upsilon\varsigma]_{1_L} \subseteq [\Upsilon\varsigma]_{\alpha_L} \in \mathcal{P}_b^r(\mho)$ for each $\alpha_L \in L\setminus\{0_L\}$, then $\mu(\varsigma, [\Upsilon\varsigma]_{\alpha_L}) \leq \mu(\varsigma, [\Upsilon\varsigma]_{1_L})$ for each $\alpha_L \in L \setminus \{0_L\}$. So, $p(\varsigma, \Upsilon(\varsigma)) \leq d(\varsigma, [\Upsilon\varsigma]_{1_L})$. This further implies that

$$\aleph([\Upsilon\varsigma]_{1_L}, [\Psi\omega]_{1_L}) \leq \varphi\left(\mathcal{C}^p_{(\Upsilon,\Psi)}(\varsigma, \omega, 1_L)\right).$$

Therefore, Theorem 3.2 can be applied to locate some $u \in \mho$ such that $u \in [\Upsilon u]_{1_L} \cap [\Psi u]_{1_L}$. □

Remark 3.6. By putting $\eta = 1$ and $p = 1$, Theorem 3.5 can easily be applied to obtain the results of [3, Theorems 10 and 11] as special cases. Also, Theorem 3.5 is a proper generalization of the main results of [21] and [26].

Corollary 3.7. *Let* $(\mho, \mu, \eta)$ *be a complete b-metric space and* $\Upsilon : \mho \to L^{\mho}$ *be an L-fuzzy set-valued map. Suppose that for each* $\varsigma \in \mho$*, there exists an* $\alpha_L \in L\setminus\{0_L\}$ *such that* $[\Upsilon\varsigma]_{\alpha_L}$ *is a nonempty bounded proximal subsets of* $\mho$*. If*

$$\aleph([\Upsilon\varsigma]_{\alpha_L}, [\Upsilon\omega]_{\alpha_L}) \leq \varphi\left(\frac{1}{4}\mathcal{C}^p_{(\Upsilon)}(\varsigma, \omega)\right),$$

for all $\varsigma, \omega \in \mho$*, where* $\varphi \in \Omega$ *and*

$$\mathcal{C}^p_{(\Upsilon)} = \mu(\varsigma, \omega) + \mu(\varsigma, [\Upsilon\varsigma]_{\alpha_L}) + \mu(\omega, [\Upsilon\omega]_{\alpha_L}) + \frac{\mu(\omega, [\Upsilon\varsigma]_{\alpha_L}) + \mu(\varsigma, [\Upsilon\omega]_{\alpha_L})}{2\eta},$$

then, there exists $u \in \mho$ *such that* $u \in [\Upsilon u]_{\alpha_L}$.

Proof. Take $\Upsilon = \Psi$, $p = 1$ and $a_1 = a_2 = a_3 = a_4 = \frac{1}{4}$ in Theorem 3.2. □

Corollary 3.8. *Let* $(\mho, \mu, \eta)$ *be a complete b-metric space and* $\Upsilon, \Psi : \mho \to L^{\mho}$ *be L-fuzzy set-valued maps. Suppose that for each* $\varsigma \in \mho$*, there exists an* $\alpha_L \in L \setminus \{0_L\}$ *such that* $[Sx]_{\alpha_L}$ *and* $[Tx]_{\alpha_L}$ *are nonempty bounded proximal subsets of* $\mho$*. If there exists* $\lambda \in [0, 1)$ *such that*

$\aleph([\Upsilon\varsigma]_{\alpha_L}, [\Psi\omega]_{\alpha_L})$

$$\leq \lambda\left(\sqrt[4]{(\mu(\varsigma,\omega))\mu(\varsigma,[\Upsilon\varsigma]_{\alpha_L})(\mu(\omega,[\Psi\omega]_{\alpha_L}))\left(\frac{\mu(\varsigma,[\Psi\omega]_{\alpha_L})+\mu(\omega,[\Upsilon\varsigma]_{\alpha_L})}{2\eta}\right)}\right),$$

then Υ and Ψ have a common L-fuzzy fixed point in $\mho$.

Proof. Take $a_1 = a_2 = a_3 = a_4 = \frac{1}{4}$, $\varphi(t) = \lambda t$ for all $t \geq 0$ and $p = 0$ in Theorem 3.2. □

Corollary 3.9. *Let $(\mho, \mu, \eta)$ be a complete b-metric space and $\Upsilon, \Psi : \mho \to L^{\mho}$ be L-fuzzy set-valued maps. Suppose that for each $\varsigma \in \mho$, there exists $\alpha_L \in L \setminus \{0_L\}$ such that $[\Upsilon\varsigma]_{\alpha_L}$ is a nonempty bounded proximal subsets of $\mho$. If for all $\varsigma, \omega \in \mho$,*

$$\aleph([\Upsilon\varsigma]_{\alpha_L}, [\Psi\omega]_{\alpha_L}) \leq \varphi\Bigg(\max\Bigg\{\mu(\varsigma,\omega), \mu(\varsigma,[\Upsilon\varsigma]_{\alpha_L}), \mu(\omega,[\Psi\omega]_{\alpha_L}), \frac{1}{2}[\mu(\varsigma,[\Psi\omega]_{\alpha_L})+\mu(\omega,[\Upsilon\varsigma]_{\alpha_L})]\Bigg\}\Bigg),$$

then, there exists $u \in \mho$ such that $u \in [\Upsilon u]_{\alpha_L} \cap [\Psi u]_{\alpha_L}$.

Corollary 3.10. *Let $(\mho, \mu, \eta)$ be a complete b-metric space and $\Upsilon : \mho \to L^{\mho}$ be an L-fuzzy set-valued map. Suppose that for each $\varsigma \in \mho$, there exists $\alpha_L \in L \setminus \{0_L\}$ such that $[\Upsilon\varsigma]_{\alpha_L}$ is a nonempty bounded proximal subsets of $\mho$. If there exists $\lambda \in [0, 1)$ such that*

$$\aleph([\Upsilon\varsigma]_{\alpha_L}, [\Upsilon\omega]_{\alpha_L}) \leq \lambda\mu(\varsigma,\omega),$$

then, there exists $u \in \mho$ such that $u \in [Su]_{\alpha_L}$.

Proof. Put $\Upsilon = \Psi$, $a_1 = p = 1$, $a_2 = a_3 = a_4 = 0$ and $\varphi(t) = \lambda t$, $t \geq 0$ in Theorem 3.2. □

In line with the proof of Theorem 3.5, the next result can easily be established by applying Corollary 3.10.

Corollary 3.11. *Let $(\mho, \mu, \eta)$ be a complete b-metric space and $\Upsilon : \mho \to \mathcal{K}_{\mathcal{F}}(\mho)$ be an L-fuzzy set-valued map. Suppose that for each $\varsigma, \omega \in \mho$, there exists $\lambda \in [0, 1)$ such that*

$$\mu_L^{\infty}(\Upsilon(\varsigma), \Upsilon(\omega)) \leq \lambda\mu(\varsigma,\omega),$$

then, there exists $u \in \mho$ such that $\{u\} \subset \Upsilon(u)$.

Remark 3.12.
- (i) If we take $\eta = 1$ and $L = [0, 1]$, then Corollary 3.11 is a proper generalization of the main result of Heilpern [14], since $W(\mho) \subset \mathcal{K}_{\mathcal{F}}(\mho)$.
- (ii) By setting $\eta = 1$, Corollary 3.9 reduces to [3, Theorem 6].

4. Applications to Multi-Valued and Single-Valued Mappings

In this section, we apply the results of the previous section to deduce some crisp fixed point theorems of multi-valued and single-valued mappings.

Definition 4.1. Let $\mho$ be a nonempty set and $\mathcal{N}(\mho)$ denotes the family of nonempty subsets of $\mho$. A set-valued mapping $\Psi : \mho \to \mathcal{N}(\mho)$ is called a multi-valued mapping. A point $u \in \mho$ is said to be a fixed point of Ψ if $u \in \Psi u$.

Corollary 4.2. *Let $(\mho, \mu, \eta)$ be a complete b-metric space and $\Theta, \Lambda : \mho \to \mathcal{K}(\mho)$ be multivalued mappings. Assume that for all $\varsigma, \omega \in \mho$,*

$$\aleph(\Theta\varsigma, \Lambda\omega) \leq \varphi\Bigg(\max\Bigg\{\mu(\varsigma, \omega), \mu(\varsigma, \Theta\varsigma), \mu(\omega, \Lambda\omega), \frac{1}{2}[\mu(\varsigma, \Lambda\omega) + \mu(\omega, \Theta\varsigma)]\Bigg\}\Bigg),$$

then, there exists $u \in \mho$ such that $u \in \Theta u \cap \Lambda u$.

Proof. Let $(L, \preceq_L)$ be a complete distributive lattice and $\alpha_{L_\Theta}, \alpha_{L_\Lambda} : \mho \to L \setminus \{0_L\}$ be arbitrary mappings. For each $\varsigma \in \mho$, consider two L-fuzzy set-valued maps $\Upsilon(\varsigma), \Psi(\varsigma) : \mho \to L$ defined by

$$\Upsilon(\varsigma)(t) = \begin{cases} \alpha_{L_\Theta}(\varsigma), & \text{if } t \in \Theta(\varsigma), \\ 0_L, & \text{otherwise,} \end{cases}$$

and

$$\Psi(\varsigma)(t) = \begin{cases} \alpha_{L_\Lambda}(\varsigma), & \text{if } t \in \Lambda(\varsigma), \\ 0_L, & \text{otherwise.} \end{cases}$$

Suppose that $\alpha_{L_\Theta}(\varsigma) := \alpha_L$, then, there exists $\alpha_L \in L \setminus \{0_L\}$ such that

$$\begin{aligned} [\Upsilon(\varsigma)]_{\alpha_L} &= \{t \in \mho : \alpha_{L_g}(\varsigma) \preceq_L \Upsilon(\varsigma)\} \\ &= \Theta\varsigma. \end{aligned}$$

Similarly, $[\Psi\varsigma]_{\alpha_L} = \Lambda\varsigma$. Therefore, Corollary 3.9 can be applied to find $u \in \mho$ such that

$$u \in [\Upsilon u]_{\alpha_L} \cap [\Psi u]_{\alpha_L} = \Theta u \cap \Lambda u.$$

□

Following the proof of Corollary 4.2, we can easily derive the next result by applying Corollary 3.7.

Corollary 4.3 ([1])**.** *Let $(\mho, \mu, \eta)$ be a complete b-metric space and $\Theta : \mho \to \mathcal{P}_b^r(\mho)$ be a multi-valued mapping such that for each $\varsigma, \omega \in \mho$,*

$$\aleph(\Theta\varsigma, \Theta\omega) \leq \varphi\left(\frac{1}{4}\mathcal{C}^p_{(\Theta)}(\varsigma, \omega)\right),$$

for all $\varsigma, \omega \in \mho$, where $\varphi \in \Omega$ and

$$\mathcal{C}^p_{(\Theta)} = \mu(\varsigma, \omega) + \mu(\varsigma, \Theta\varsigma) + \mu(\omega, \Theta\omega) + \frac{\mu(\omega, \Theta\varsigma) + \mu(\varsigma, \Theta\omega)}{2\eta}.$$

Then, there exists $u \in \mho$ such that $u \in \Upsilon u$.

Corollary 4.4 ([19, Theorem 1])**.** *Let $(\mho, \mu, \eta)$ be a complete b-metric space and $g : \mho \to \mho$ be a single-valued mapping. If*

$$\mu(gx, gy) \leq \varphi\left(\mathcal{C}^p_g(\varsigma, \omega)\right),$$

for all $\varsigma, \omega \in \mho$, where $\varphi \in \Omega$, $p \geq 0$, $a_i \geq 0$, $i = 1, 2, 3, 4$ with $\sum_{i=1}^{4} a_i = 1$ and

$$\mathcal{C}^p_g(\varsigma, \omega) = \begin{cases} \Big[a_1(\mu(\varsigma, \omega))^p + a_2(\mu(\varsigma, gx))^p \\ \quad + a_3(\mu(\omega, gy))^p \\ \quad + a_4\left(\frac{\mu(\omega, gx) + \mu(\varsigma, gy)}{2\eta}\right)^p\Big]^{\frac{1}{p}}, & \text{for } p > 0,\ \varsigma, \omega \in \mho, \\ (\mu(\varsigma, \omega))^{a_1}(\mu(\varsigma, gx))^{a_2} \\ \quad \times (\mu(\omega, gy))^{a_3} \\ \quad \times \left(\frac{\mu(\varsigma, gy) + \mu(\omega, gx)}{2\eta}\right)^{a_4}, & \text{for } p = 0, \varsigma, \omega \in \mho \setminus \mathcal{F}_{ix}(g), \end{cases} \tag{4.1}$$

where

$$\mathcal{F}_{ix}(g) = \{\varsigma \in \mho : \varsigma = gx\}.$$

Then there exists $u \in \mho$ such that $u = gu$.

Proof. Let $(L, \preceq_L)$ be a complete distributive lattice and $\alpha_{L_g} : \mho \to L \setminus \{0_L\}$ be an arbitrary mapping. For each $\varsigma \in \mho$, consider an L-fuzzy set-valued map $\Upsilon(\varsigma) : \mho \to L$ defined by

$$\Upsilon(\varsigma)(t) = \begin{cases} \alpha_{L_g}(\varsigma), & \text{if } t = g(\varsigma), \\ 0_L, & \text{otherwise.} \end{cases}$$

Suppose that $\alpha_{L_g}(\varsigma) := \alpha_L$, then, there exists $\alpha_L \in L \setminus \{0_L\}$ such that

$$\begin{aligned} [\Upsilon(\varsigma)]_{\alpha_L} &= \{t \in \mho : \alpha_{L_g}(\varsigma) \preceq_L \Upsilon(\varsigma)\} \\ &= \{g(\varsigma)\}. \end{aligned}$$

Clearly, $\{g(\varsigma)\} \in \mathcal{P}_b^r(\mho)$ for each $\varsigma \in \mho$. Note that in this case, for all $\varsigma, \omega \in \mho$,

$$\aleph([\Upsilon\varsigma]_{\alpha_L}, [\Upsilon\omega]_{\alpha_L}) = \mu(g(\varsigma), g(\omega)).$$

Therefore, by Theorem 3.2, there exists $u \in \mho$ such that $u \in [\Upsilon u]_{\alpha_L} = \{g(u)\}$; which further implies that $g(u) = u$. $\square$

By using the method of proving Corollary 4.4, we can deduce the next fixed point theorem due to Czerwik [10] by applying Corollary 3.11.

Corollary 4.5 ([10])**.** *Let* $(\mho, \mu, \eta)$ *be a complete b-metric space and* $g : \mho \to \mho$ *be a single-valued mapping. If there exists* $\lambda \in (0,1)$ *such that for all* $\varsigma, \omega \in \mho$,

$$\mu(g(\varsigma), g(\omega)) \leq \lambda \mu(\varsigma, \omega),$$

then, there exists $u \in \mho$ *such that* $g(u) = u$.

Remark 4.6.

(i) For $\eta = 1$ in Corollary 4.2, we obtain the result of [3, Theorem 7].

(ii) It is clear that if we take $\eta = 1$ in all the above results, we can deduce their analogues in the setting of metric space. Also, several independent consequences of our results can be pointed, but we skip listing out many of such special cases due to the length of the paper.

Conclusion

The idea of p-hybrid L-fuzzy contractions in the framework of b-metric space is initiated in this paper. Thereafter, sufficient conditions for existence of common L-fuzzy fixed points for a pair of L-fuzzy set-valued maps have been established. We noticed that in the case where our results are reduced to their corresponding crisp counterparts, the concept presented herein combines and generalizes a few well-known fixed point theorems in the setting of both single-valued and multi-valued mappings in the corresponding literature. A few of these special cases have been highlighted and discussed. To authenticate the hypotheses of our main result, an example was provided. To the best of our knowledge, there is no contribution in the existing literature of hybrid fixed point theory via the concept of L-fuzzy sets. Thereby, justifying the motivation and novelty of the present work. It is interesting to note that the current idea herein can be improved upon when presented in other models such as intuitionistic fuzzy sets, soft sets, neutrosophic soft sets, N-soft sets, rough sets, and so on. Also, its metric space component can be any of quasi or pseudo-metric space such as F-metric space, G-metric space, modular metric space, to mention a few. Moreover, the established contractive inequalities in this work can be used together with suitable hypotheses to discuss existence criteria for solutions of several classes of differential and integral inclusions.

Conflict of Interests

The authors declare that they have no competing interests.

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