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A Generalized Class of Univalent Harmonic Functions Associated with a Multiplier Transformation

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Abstract. We define a new subclass of univalent harmonic mappings using multiplier transformation and investigate various properties like necessary and sufficient conditions, extreme points, starlikeness, radius of convexity. We prove that the class is closed under harmonic convolutions and convex combinations. Finally, we show that this class is invariant under Bernandi-Libera-Livingston integral for harmonic functions.

1. Introduction and Preliminaries

In the open unit disk $\mathbb{E} = \{z \in \mathbb{C} : |z| < 1\}$ a complex valued harmonic function $f = u + iv$ can be decomposed into two parts h, g where h is called the analytic part and g is the co-analytic part of f and thus f can be expressed as $f = h + \overline{g}$. Let $\mathcal{H}$ be the class of complex valued harmonic mappings $f = h + \overline{g}$ defined in $\mathbb{E}$ and normalized by $f(0) = f_z(0) - 1 = 0$. Under the given normalization conditions h, g have Taylor's series representation as

$$h(z) = z + \sum_{k=2}^{\infty} a_k z^k, \qquad g(z) = \sum_{k=1}^{\infty} b_k z^k,$$

and consequently $f = h + \overline{g}$ has the representation

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k + \overline{\sum_{k=1}^{\infty} b_k z^k}. \tag{1.1}$$

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In 1936, Lewy [6] proved that a necessary and sufficient condition for a harmonic function $f = h+\overline{g}$ to be locally univalent and sense-preserving in $\mathbb{E}$ is that its Jacobian $J_f = |h'|^2 - |g'|^2$ is positive in $\mathbb{E}$. We denote by S_H the subclass of $\mathcal{H}$ consisting of all sense-preserving univalent harmonic mappings f and S_H^0 is the subclass of S_H whose members satisfy additional condition $f_{\overline{z}}(0) = 0$, i.e. $g'(0) = 0$. Further $\overline{S}_H$ denote the subclass of $\mathcal{H}$ consisting of functions of the type $f_n = h + \overline{g}_n$, where

$$h(z) = z - \sum_{k=2}^{\infty} |a_k| z^k, \qquad g_n(z) = (-1)^n \sum_{k=1}^{\infty} |b_k| z^k. \tag{1.2}$$

Clunie and Sheil-Small [2] introduced the class S_H and some of its subclasses and investigated their geometric properties. Since then many subclasses of the class S_H were defined and their various properties were studied by the researchers e.g. see ([1, 4, 5, 9] and [10]) and references therein. In 1994, Opoola [7] defined a subclass $T_n^{\beta}(\alpha)$ of normalized analytic functions that satisfy the condition $\Re\left(\dfrac{D^n(f(z)^{\beta})}{\beta^n z^{\beta}}\right) > \alpha, \forall z \in \mathbb{E}, n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, 0 \leq \alpha < 1, \beta > 0$, where

$$f(z)^{\beta} = z^{\beta} + \sum_{k=2}^{\infty} \beta a_k z^{k+\beta-1}$$

and D^n is the Salagean differential operator. Using the modified Salagean operator for harmonic functions, Khalifa Al-Shaqsi et.al. [1] studied a class $H(n, \beta, \alpha)$, consisting of harmonic functions

$$f(z)^{\beta} = h(z)^{\beta} + \overline{g(z)^{\beta}},$$

where

$$h(z)^{\beta} = z^{\beta} + \sum_{k=2}^{\infty} \beta a_k z^{k+\beta-1}, \qquad g(z)^{\beta} = \sum_{k=1}^{\infty} \beta b_k z^{k+\beta-1}, \quad |b_1| < 1. \tag{1.3}$$

For more details about Salagean and modified Salagean operators respectively, we refer to [8] and [4]. In this paper, using a multiplier transformation $I(n, \lambda)$, we define a generalized class $H^{\beta}(n, \lambda, \alpha)$ $(n \in \mathbb{N}_0$, $\beta \geq 1$, $0 \leq \lambda < 1$,) of harmonic functions $f(z)^{\beta} = h(z)^{\beta} + \overline{g(z)^{\beta}}$ that satisfy the condition

$$\Re\left(\frac{I(n+1, \lambda) f(z)^{\beta}}{I(n, \lambda) f(z)^{\beta}}\right) \geq \alpha, \quad 0 \leq \alpha < 1. \tag{1.4}$$

Here $I(n, \lambda)$ is the multiplier transformation defined as

$$I(n, \lambda) f(z)^{\beta} = I(n, \lambda) h(z)^{\beta} + (-1)^n \overline{I(n, \lambda) g(z)^{\beta}},$$

where

$$I(n,\lambda)h(z)^{\beta}=z^{\beta}+\sum_{k=2}^{\infty}\beta\left(\frac{k+\lambda}{1+\lambda}\right)^{n}a_k z^{\beta+k-1},$$

and

$$I(n,\lambda)g(z)^{\beta}=\sum_{k=1}^{\infty}\beta\left(\frac{k+\lambda}{1+\lambda}\right)^{n}b_k z^{\beta+k-1}.$$

It is worth to mention here that the class $H^{\beta}(n,\lambda,\alpha)$ is a more generalized class and it includes a variety of classes. For different choices of parameters n,β,λ we obtain different well known subclasses of S_H. For example, $H^{1}(0,0,\alpha)=S_H^{*}(\alpha)$ and $H^{1}(1,0,\alpha)=K_H(\alpha)$, where $S_H^{*}(\alpha)$ and $K_H(\alpha)$, introduced by Jahangiri [3], are well known subclasses of S_H consisting of harmonic starlike functions of order α and convex of order α, respectively. Moreover, $H^{\beta}(n,0,\alpha)=H(n,\beta,\alpha)$ is the class studied by Khalifa et. al. [1]. We denote by $\overline{H^{\beta}}(n,\lambda,\alpha)$ the class consisting of functions of the type $f_n^{\beta}=h^{\beta}+\overline{g_n^{\beta}}$ satisfying condition (1.4) where

$$h(z)^{\beta}=z^{\beta}-\sum_{k=2}^{\infty}\beta|a_k|z^{k+\beta-1}, \tag{1.5}$$

$$g_n(z)^{\beta}=(-1)^{n}\sum_{k=1}^{\infty}\beta|b_k|z^{k+\beta-1}.$$

In Section 2, we obtain sufficient condition for harmonic functions to be in $H^{\beta}(n,\lambda,\alpha)$ and then we prove that this condition is also necessary for the functions in the class $\overline{H^{\beta}}(n,\lambda,\alpha)$. We also investigate various properties like starlikeness, radius of convexity, Bernandi-Libera-Livingston integral, distortion bounds, convex combinations for the functions in the class $\overline{H^{\beta}}(n,\lambda,\alpha)$.

2. Characterization Properties

Theorem 2.1. *Let $f^{\beta}=h^{\beta}+\overline{g^{\beta}}$, where h^{β} and g^{β} are given by (1.3). If*

(2.1)
$$\sum_{k=1}^{\infty}\left(\frac{k+\lambda}{1+\lambda}\right)^{n}\left(\frac{k-\alpha+\lambda(1-\alpha)}{1+\lambda}\beta|a_k|+\frac{k+\alpha+\lambda(1+\alpha)}{1+\lambda}\beta|b_k|\right) \le (1+\beta)(1-\alpha),$$

where $a_1=1$, $n\in\mathbb{N}_0$, $\beta\ge 1$, $0\le\lambda<1$, $0\le\alpha<1$, then f^{β} is harmonic univalent and sense-preserving in $\mathbb{E}$ and $f^{\beta}\in H^{\beta}(n,\lambda,\alpha)$.

Proof. First we shall prove that f^β is sense-preserving in $\mathbb{E}$. Since $z \in \mathbb{E}$ therefore,

$$\begin{aligned}
|h'(z)^\beta| &\geq \beta \left(|z|^{\beta-1} - \sum_{k=2}^{\infty} (k+\beta-1)|a_k||z|^{k+\beta-2} \right) \\
&> \beta \left(1 - \sum_{k=2}^{\infty} \frac{k-\alpha+\lambda(1-\alpha)}{(1+\lambda)(1-\alpha)} \left(\frac{k+\lambda}{1+\lambda} \right)^n |a_k| \right) \\
&> \beta \left(\sum_{k=1}^{\infty} \left(\frac{k+\lambda}{1+\lambda} \right)^n \left(\frac{k+\alpha+\lambda(1+\alpha)}{(1+\lambda)(1-\alpha)} \right) |b_k| \right) \\
&> \sum_{k=1}^{\infty} \beta(k+\beta-1)|b_k||z|^{k+\beta-2} \\
&> |g'(z)^\beta|.
\end{aligned}$$

Thus f^β is sense-preserving in $\mathbb{E}$. Next we will establish the univalence of f^β. For $z_1 \neq z_2 \in \mathbb{E}$,

$$\begin{aligned}
\left| \frac{f(z_1)^\beta - f(z_2)^\beta}{h(z_1)^\beta - h(z_2)^\beta} \right| &\geq 1 - \left| \frac{g(z_1)^\beta - g(z_2)^\beta}{h(z_1)^\beta - h(z_2)^\beta} \right| \\
&> 1 - \left| \frac{\sum\limits_{k=1}^{\infty} \beta b_k \left(z_1^{k+\beta-1} - z_2^{k+\beta-1} \right)}{z_1^\beta - z_2^\beta + \sum\limits_{k=2}^{\infty} \beta a_k \left(z_1^{k+\beta-1} - z_2^{k+\beta-1} \right)} \right| \\
&> 1 - \frac{\sum\limits_{k=1}^{\infty} (k+\beta-1)|b_k|}{1 - \sum\limits_{k=2}^{\infty} (k+\beta-1)|a_k|} \\
&> 1 - \frac{\sum\limits_{k=1}^{\infty} \left(\frac{k+\lambda}{1+\lambda} \right)^n \frac{k+\alpha+\lambda(1+\alpha)}{(1+\lambda)(1-\alpha)} \beta |b_k|}{1 - \sum\limits_{k=2}^{\infty} \left(\frac{k+\lambda}{1+\lambda} \right)^n \frac{k-\alpha+\lambda(1-\alpha)}{(1+\lambda)(1-\alpha)} \beta |a_k|} \\
&\geq 0.
\end{aligned}$$

We shall now prove that under condition (2.1), $f^\beta \in H^\beta(n, \lambda, \alpha)$ i.e.,

$$\Re \left(\frac{I(n+1,\lambda) f^\beta}{I(n,\lambda) f^\beta} \right) \geq \alpha.$$

We know that $\Re(w) > \alpha$ if and only if $|1 - \alpha + w| > |1 + \alpha - w|$. Thus

$$\left| I(n+1,\lambda) f^\beta + (1-\alpha) I(n,\lambda) f^\beta \right| - \left| I(n+1,\lambda) f^\beta - (1+\alpha) I(n,\lambda) f^\beta \right|$$

$$
\begin{aligned}
&= \Bigg| z^{\beta} + \sum_{k=2}^{\infty} \left(\frac{k+\lambda}{1+\lambda}\right)^{n+1} \beta a_k z^{k+\beta-1} + (-1)^{n+1} \sum_{k=1}^{\infty} \left(\frac{k+\lambda}{1+\lambda}\right)^{n+1} \beta \overline{b_k} \overline{z}^{k+\beta-1} \\
&\quad + (1-\alpha)\left[z^{\beta} + \sum_{k=2}^{\infty} \left(\frac{k+\lambda}{1+\lambda}\right)^{n} \beta a_k z^{k+\beta-1} + (-1)^{n} \sum_{k=1} \left(\frac{k+\lambda}{1+\lambda}\right)^{n} \beta \overline{b_k} \overline{z}^{k+\beta-1} \right] \Bigg| \\
&\quad - \Bigg| z^{\beta} + \sum_{k=2}^{\infty} \left(\frac{k+\lambda}{1+\lambda}\right)^{n+1} \beta a_k z^{k+\beta-1} + (-1)^{n+1} \sum_{k=1}^{\infty} \left(\frac{k+\lambda}{1+\lambda}\right)^{n+1} \beta \overline{b_k} \overline{z}^{k+\beta-1} \\
&\quad - (1+\alpha)\left[z^{\beta} + \sum_{k=2}^{\infty} \left(\frac{k+\lambda}{1+\lambda}\right)^{n} \beta a_k z^{k+\beta-1} + (-1)^{n} \sum_{k=1}^{\infty} \left(\frac{k+\lambda}{1+\lambda}\right)^{n} \beta \overline{b_k} \overline{z}^{k+\beta-1} \right] \Bigg| \\
&\geq 2(1-\alpha)|z|^{\beta} \left[1 - \sum_{k=2}^{\infty} \left(\frac{k+\lambda}{1+\lambda}\right)^{n} \left(\frac{k-\alpha+\lambda(1-\alpha)}{(1-\alpha)(1+\lambda)}\right) \beta |a_k| \right. \\
&\quad \left. - \sum_{k=1}^{\infty} \left(\frac{k+\lambda}{1+\lambda}\right)^{n} \left(\frac{k+\alpha+\lambda(1+\alpha)}{(1-\alpha)(1+\lambda)}\right) \beta |b_k| \right] \\
&\geq 0, \quad \text{in view of (2.1).}
\end{aligned}
$$

Hence, it completes the proof. $\square$

In the next result we show that the coefficient condition (2.1) is necessary for the functions in the class $\overline{H^{\beta}}(n,\lambda,\alpha)$.

Theorem 2.2. *Let $f_n^{\beta} = h^{\beta} + \overline{g_n^{\beta}}$ be given by (1.5). Then $f_n^{\beta} \in \overline{H^{\beta}}(n,\lambda,\alpha)$ if and only if f_n^{β} satisfies condition (2.1).*

Proof. In view of Theorem 2.1 and the relation $\overline{H^{\beta}}(n,\lambda,\alpha) \subset H^{\beta}(n,\lambda,\alpha)$ we require to prove 'only if' part. Let $f_n^{\beta} = h^{\beta} + \overline{g_n^{\beta}} \in \overline{H^{\beta}}(n,\lambda,\alpha)$, then we observe

$$
\Re\left[\left(\frac{I(n+1,\lambda)f_n(z)^{\beta}}{I(n,\lambda)f_n(z)^{\beta}}\right) - \alpha\right] \geq 0, \quad 0 \leq \alpha < 1.
$$

After substituting $I(n+1,\lambda)f_n(z)^{\beta}$ and $I(n,\lambda)f_n(z)^{\beta}$ in the above inequality we have,

$$
\Re\left[\frac{(1-\alpha)z^{\beta} - \sum_{k=2}^{\infty}\left(\frac{k+\lambda}{1+\lambda}\right)^{n} \frac{k-\alpha+\lambda(1-\alpha)}{(1+\lambda)} \beta a_k z^{k+\beta-1} - (-1)^{2n} \sum_{k=1}^{\infty}\left(\frac{k+\lambda}{1+\lambda}\right)^{n} \frac{k+\alpha+\lambda(1+\alpha)}{(1+\lambda)} \beta \overline{b_k} \overline{z}^{k+\beta-1}}{z^{\beta} - \sum_{k=2}^{\infty}\left(\frac{k+\lambda}{1+\lambda}\right)^{n} \beta a_k z^{k+\beta-1} + (-1)^{2n} \sum_{k=1}^{\infty}\left(\frac{k+\lambda}{1+\lambda}\right)^{n} \beta \overline{b_k} \overline{z}^{k+\beta-1}}\right] \geq 0. \tag{2.2}
$$

The above required condition (2.2) holds true for all values of z, $|z| = r < 1$. Choosing the values of z on positive real axis, where $0 \leq z = r <$

1, we have

$$\frac{(1-\alpha)-\sum\limits_{k=2}^{\infty}\left(\frac{k+\lambda}{1+\lambda}\right)^n\frac{k-\alpha+\lambda(1-\alpha)}{(1+\lambda)}\beta a_k r^{k-1}-\sum\limits_{k=1}^{\infty}\left(\frac{k+\lambda}{1+\lambda}\right)^n\frac{k+\alpha+\lambda(1+\alpha)}{(1+\lambda)}\beta b_k r^{k-1}}{1-\sum\limits_{k=2}^{\infty}\left(\frac{k+\lambda}{1+\lambda}\right)^n\beta a_k r^{k-1}+\sum\limits_{k=1}^{\infty}\left(\frac{k+\lambda}{1+\lambda}\right)^n\beta\overline{b_k}r^{k-1}}$$
$$\geq 0.$$

If condition (2.1) doesn't hold true, then numerator of the above fraction will be negative for successfully large values of r close to 1. Hence, there exists $z_0 = r_0$ in $(0,1)$ for which the quotient in the above inequality will be negative. It contradicts the required condition that $f_n^\beta \in \overline{H^\beta}(n,\lambda,\alpha)$ and so it completes the proof. □

Next we establish that the family $\overline{H^\beta}(n,\lambda,\alpha)$ is non-empty, that is we prove that there exist harmonic functions which are members of the class $\overline{H^\beta}(n,\lambda,\alpha)$ and moreover these functions are the extermums for the family $\overline{H^\beta}(n,\lambda,\alpha)$.

Theorem 2.3. *Let $f_n^\beta = h^\beta + \overline{g_n^\beta}$, then $f_n^\beta \in \overline{H^\beta}(n,\lambda,\alpha)$ if and only if $f_n(z)^\beta = \sum\limits_{k=1}^{\infty}\left(X_k h_k(z)^\beta + Y_k\overline{g_{n_k}(z)^\beta}\right)$, where*

$$h_1(z)^\beta = z^\beta,$$
$$h_k(z)^\beta = z^\beta - \frac{(1-\alpha)(1+\lambda)^{n+1}}{((k-\alpha)+\lambda(1-\alpha))\,(k+\lambda)^n}z^{k+\beta-1}, \quad k = 2,3,4\ldots$$
$$g_{n_k}(z)^\beta = z^\beta + (-1)^n\frac{(1-\alpha)(1+\lambda)^{n+1}}{((k+\alpha)+\lambda(1+\alpha))\,(k+\lambda)^n}z^{k+\beta-1}, \quad k = 1,2,3,\ldots,$$

and $\sum\limits_{k=1}^{\infty}(X_k+Y_k) = 1$, $X_k \geq 0$, $Y_k \geq 0$. In particular, the extreme points of $\overline{H^\beta}(n,\lambda,\alpha)$ are $\{h_k^\beta\}$ and $\{g_{n_k}^\beta\}$.

Proof. For function $f_n^\beta = h^\beta + \overline{g_{n_k}^\beta}$, where h^β and $g_{n_k}^\beta$ are given as above, we have

$$f_n(z)^\beta = \sum_{k=1}^{\infty}\left(X_k h_k(z)^\beta + Y_k\overline{g_{n_k}(z)^\beta}\right),$$

$$(2.3)\quad f_n(z)^\beta = z^\beta - \sum_{k=2}^{\infty}\frac{(1-\alpha)(1+\lambda)^{n+1}}{((k-\alpha)+\lambda(1-\alpha))\,(k+\lambda)^n}X_k z^{k+\beta-1}$$
$$+(-1)^n\sum_{k=1}^{\infty}\frac{(1-\alpha)(1+\lambda)^{n+1}}{((k+\alpha)+\lambda(1+\alpha))\,(k+\lambda)^n}Y_k\overline{z}^{k+\beta-1}.$$

Comparing (2.3) with

$$f_n(z)^\beta = z^\beta - \sum_{k=2}^{\infty} \beta |a_k| z^{k+\beta-1} + (-1)^n \sum_{k=1}^{\infty} \beta |b_k| z^{k+\beta-1},$$

we obtain

$$|a_k| = \frac{(1-\alpha)(1+\lambda)^{n+1}}{((k-\alpha)+\lambda(1-\alpha))\,\beta(k+\lambda)^n} X_k,$$

and

$$|b_k| = (-1)^n \frac{(1-\alpha)(1+\lambda)^{n+1}}{((k+\alpha)+\lambda(1+\alpha))\,\beta(k+\lambda)^n} Y_k.$$

Now,

$$\begin{aligned}
&\sum_{k=2}^{\infty} \left(\frac{k+\lambda}{1+\lambda}\right)^n \left[\frac{(k-\alpha)+\lambda(1-\alpha)}{(1-\alpha)(1+\lambda)}\right] \beta |a_k| \\
&\qquad + \sum_{k=1}^{\infty} \left(\frac{k+\lambda}{1+\lambda}\right)^n \left[\frac{(k+\alpha)+\lambda(1+\alpha)}{(1-\alpha)(1+\lambda)}\right] \beta |b_k| \\
&\qquad = \sum_{k=2}^{\infty} X_k + \sum_{k=1}^{\infty} Y_k \\
&\qquad = 1 - X_1 \le 1.
\end{aligned}$$

Therefore, $f_n \in \overline{H^\beta}(n, \lambda, \alpha)$. Conversely, let $f_n \in \overline{H^\beta}(n, \lambda, \alpha)$. Set

$$\begin{aligned}
X_k &= \left(\frac{k+\lambda}{1+\lambda}\right)^n \left[\frac{(k-\alpha)+\lambda(1-\alpha)}{(1-\alpha)(1+\lambda)}\right] |a_k|, \quad k = 2, 3, 4, \dots, \\
Y_k &= \left(\frac{k+\lambda}{1+\lambda}\right)^n \left[\frac{(k+\alpha)+\lambda(1+\alpha)}{(1-\alpha)(1+\lambda)}\right] |b_k|, \quad k = 1, 2, 3, \dots,
\end{aligned}$$

and $\sum_{k=1}^{\infty} (X_k + Y_k) = 1$. Since $f_n^\beta \in \overline{H^\beta}(n, \lambda, \alpha)$, therefore

$$f_n(z)^\beta = z^\beta - \sum_{k=2}^{\infty} \beta |a_k| z^{k+\beta-1} + (-1)^n \sum_{k=1}^{\infty} \beta |b_k| \overline{z}^{k+\beta-1},$$

by substituting values of $|a_k|$ and $|b_k|$, we get

$$f_n(z)^\beta = \sum_{k=1}^{\infty} \left(X_k h_k(z)^\beta + Y_k \overline{g_{n_k}(z)} \right),$$

where $h_k(z)^\beta$ and $g_{n_k}(z)^\beta$ are as mentioned above. Hence the proof is complete. $\square$

In the following results we investigate geometric properties of the functions in the class $\overline{H^\beta}(n,\lambda,\alpha)$. We prove that the functions in the class $\overline{H^\beta}(n,\lambda,\alpha)$ are starlike of order α and we obtain the radius of convexity for the functions in this class.

Theorem 2.4. *Let $f_n^\beta \in \overline{H^\beta}(n,\lambda,\alpha)$, then f_n^β maps the unit disk onto a domain which is starlike of order α.*

Proof. A function $f_n^\beta = h^\beta + \overline{g_n^\beta} \in \overline{H^\beta}(n,\lambda,\alpha)$ maps the unit disk $\mathbb{E}$ onto a domain starlike of order α if and only if for $z \in \mathbb{E}$,

$$\Re\left\{\frac{z(h(z)^\beta)' - \overline{z(g_n(z)^\beta)'}}{h(z)^\beta + \overline{g_n(z)^\beta}}\right\} > \alpha. \tag{2.4}$$

We know that $\Re(w) > \alpha$ if and only if $|1-\alpha+w| > |1+\alpha-w|$. Therefore to prove (2.4), it suffices to show that

$$\left|(1-\alpha)\left(h(z)^\beta + \overline{g_n(z)^\beta}\right) + z(h(z)^\beta)' - \overline{z(g_n(z)^\beta)'}\right| \\ - \left|(1+\alpha)\left(h(z)^\beta + \overline{g_n(z)^\beta}\right) - zh'(z)^\beta + \overline{zg_n'(z)^\beta}\right| > 0.$$

Since

$$\begin{aligned}
&\left|(1-\alpha)\left(h(z)^\beta + \overline{g_n(z)^\beta}\right) + zh'(z)^\beta - \overline{zg_n'(z)^\beta}\right| \\
&\quad - \left|(1+\alpha)\left(h(z)^\beta + \overline{g_n(z)^\beta}\right) - zh'(z)^\beta + \overline{zg_n'(z)^\beta}\right| \\
&= \Bigg|(1-\alpha)\left(z^\beta - \sum_{k=2}^{\infty}|a_k|z^{k+\beta-1} + (-1)^n\sum_{k=1}^{\infty}|b_k|\overline{z}^{k+\beta-1}\right) \\
&\quad + \left(\beta z^\beta - \sum_{k=2}^{\infty}(k+\beta-1)|a_k|z^{k+\beta-1}\right) - (-1)^n\sum_{k=1}^{\infty}(k+\beta-1)|b_k|\overline{z}^{k+\beta-1}\Bigg| \\
&\quad - \Bigg|(1+\alpha)\left(z^\beta - \sum_{k=2}^{\infty}|a_k|z^{k+\beta-1} + (-1)^n\sum_{k=1}^{\infty}|b_k|\overline{z}^{k+\beta-1}\right) \\
&\quad - \left(\beta z^\beta - \sum_{k=2}^{\infty}(k+\beta-1)|a_k|z^{k+\beta}\right) + \sum_{k=1}^{\infty}(k+\beta-1)|b_k|\overline{z}^{k+\beta}\Bigg| \\
&= \left|(1-\alpha+\beta)z^\beta - \sum_{k=2}^{\infty}(\beta+k-\alpha)|a_k|z^{k+\beta-1} + (-1)^n\sum_{k=1}^{\infty}(2-\alpha-\beta-k)|b_k|\overline{z}^{k+\beta-1}\right| \\
&\quad - \left|(1-\alpha-\beta)z^\beta\sum_{k=2}^{\infty}(2-\alpha-\beta-k)|a_k|z^{k+\beta-1} + (-1)^n\sum_{k=1}^{\infty}(\beta+k-\alpha)|b_k|\overline{z}^{k+\beta-1}\right| \\
&\geq 2(1-\alpha)|z|^\beta\left[1 - \sum_{k=2}^{\infty}|a_k||z|^{k-1} - \sum_{k=1}^{\infty}|b_k||z|^{k-1}\right] \\
&\geq 2(1-\alpha)|z|^\beta\left[1 - \sum_{k=2}^{\infty}|a_k| - \sum_{k=1}^{\infty}|b_k|\right]
\end{aligned}$$

$$
\begin{aligned}
&> 2(1-\alpha)|z|^{\beta}\left[1-\sum_{k=2}^{\infty}\left(\frac{k+\lambda}{1+\lambda}\right)^{n}\frac{k-\alpha+\lambda(1-\alpha)}{(1-\alpha)(1+\lambda)}\beta|a_k|\right.\\
&\quad\left.-\sum_{k=1}^{\infty}\left(\frac{k+\lambda}{1+\lambda}\right)^{n}\frac{k+\alpha+\lambda(1+\alpha)}{(1-\alpha)(1+\lambda)}\beta|b_k|\right]\\
&> 0,
\end{aligned}
$$

so the theorem follows. □

Theorem 2.5. *If $f_n^{\beta} \in \overline{H^{\beta}}(n,\lambda,\alpha)$ then f_n^{β} is convex of order α in the disk*

$$|z| \le \min_{q}\left(\frac{(1-\alpha)(1-|b_1|)}{q\left((1-\alpha)|-(1+\alpha)|b_1|\right)}\right)^{1/(q+\beta-1)}.$$

Proof. Let $f_n^{\beta} \in \overline{H^{\beta}}(n,\lambda,\alpha)$ and let r, $0<r<1$ be fixed. If f_n^{β} is convex of order α in the disk $|z|\le r$, then r will satisfy

$$
\begin{aligned}
&\sum_{k=2}^{\infty}\left(\frac{k(k-\alpha)}{(1-\alpha)}|a_k|+\frac{k(k+\alpha)}{(1-\alpha)}|b_k|\right)r^{k+\beta-1}\\
&\quad\le \sum_{k=2}^{\infty}\left(\frac{k+\lambda}{1+\lambda}\right)^{n}\left(\left(\frac{k-\alpha+\lambda(1-\alpha)}{(1-\alpha)(1+\lambda)}\right)|a_k|+\left(\frac{k+\alpha+\lambda(1+\alpha)}{(1-\alpha)(1+\lambda)}\right)|b_k|\right)kr^{k+\beta-1}\\
&\quad\le 1-|b_1|,
\end{aligned}
$$

provided

$$kr^{k+\beta-1} \le \frac{1-|b_1|}{1-\frac{1+\alpha}{1-\alpha}\beta|b_1|},$$

or

$$r \le \min_{q}\left(\frac{(1-\alpha)(1-|b_1|)}{q\left((1-\alpha)-(1+\alpha|b|)\beta|b_1|\right)}\right)^{1/(q+\beta-1)}.$$

□

3. Inclusion Properties

In this section we discuss that the class $\overline{H^{\beta}}(n,\lambda,\alpha)$ is closed under the harmonic convolution and convex combinations. Further we prove that the class is also invariant under Bernandi-Libera-Livingston integral operator for harmonic functions.

For harmonic functions

$$f_n(z)^{\beta} = z^{\beta}-\sum_{k=2}^{\infty}|a_k|z^{k+\beta-1}+(-1)^{n}\sum_{k=1}^{\infty}|b_k|\overline{z}^{k+\beta-1},$$

and

$$F_n(z)^{\beta} = z^{\beta}-\sum_{k=2}^{\infty}|A_k|z^{k+\beta-1}+(-1)^{n}\sum_{k=1}^{\infty}|B_k|\overline{z}^{k+\beta-1},$$

the convolution of f_n^β and F_n^β is given by

$$\begin{aligned}(f_n^\beta * F_n^\beta)(z) &= f_n(z)^\beta * F_n(z)^\beta \\ &= z^\beta - \sum_{k=2}^{\infty} |a_k A_k| z^{k+\beta-1} + (-1)^n \sum_{k=1}^{\infty} |b_k B_k| \overline{z}^{k+\beta-1}.\end{aligned}$$

Theorem 3.1. *Let $f_n^\beta \in \overline{H^\beta}(n,\lambda,\alpha_1)$ and $F_n^\beta \in \overline{H^\beta}(n,\lambda,\alpha_2)$, where $0 \leq \alpha_1 \leq \alpha_2 < 1$. Then $f_n^\beta * F_n^\beta \in \overline{H^\beta}(n,\lambda,\alpha_2) \subset \overline{H^\beta}(n,\lambda,\alpha_1)$.*

Proof. We wish to show that $\left(f_n^\beta * F_n^\beta\right)$ satisfies the coefficient condition (2.1). For $F_n^\beta \in \overline{H^\beta}(n,\lambda,\alpha_2)$, we note that $|A_k| \leq 1$ and $|B_k| \leq 1$. Now, for the coefficients of convolution function $\left(f_n^\beta * F_n^\beta\right)$, we have

$$\begin{aligned}&\sum_{k=2}^{\infty} \left(\frac{k+\lambda}{1+\lambda}\right)^n \left[\frac{(k-\alpha_2)+\lambda(1-\alpha_2)}{(1-\alpha_2)(1+\lambda)}\right] \beta |a_k A_k| \\ &\qquad + \sum_{k=1}^{\infty} \left(\frac{k+\lambda}{1+\lambda}\right)^n \left[\frac{(k+\alpha_2)+\lambda(1+\alpha_2)}{(1-\alpha_2)(1+\lambda)}\right] \beta |b_k B_k| \\ &\leq \sum_{k=2}^{\infty} \left(\frac{k+\lambda}{1+\lambda}\right)^n \frac{(k-\alpha_2)+\lambda(1-\alpha_2)}{(1-\alpha_2)(1+\lambda)} \beta |a_k| \\ &\quad + \sum_{k=1}^{\infty} \left(\frac{k+\lambda}{1+\lambda}\right)^n \frac{(k+\alpha_2)+\lambda(1+\alpha_2)}{(1-\alpha_2)(1+\lambda)} \beta |b_k| \\ &\leq \sum_{k=2}^{\infty} \left(\frac{k+\lambda}{1+\lambda}\right)^n \frac{(k-\alpha_1)+\lambda(1-\alpha_1)}{(1-\alpha_1)(1+\lambda)} \beta |a_k| \\ &\quad + \sum_{k=1}^{\infty} \left(\frac{k+\lambda}{1+\lambda}\right)^n \frac{(k+\alpha_1)+\lambda(1+\alpha_1)}{(1-\alpha_1)(1+\lambda)} \beta |b_k| \\ &\leq 1.\end{aligned}$$

Since $0 \leq \alpha_1 \leq \alpha_2$ and $f_n^\beta \in \overline{H^\beta}(n,\lambda,\alpha_2)$, therefore,

$$(f_n^\beta * F_n^\beta) \in \overline{H^\beta}(n,\lambda,\alpha_2) \subset \overline{H^\beta}(n,\lambda,\alpha_1).$$

□

For $i = 1, 2, 3, \ldots$ let the function $f_{n,i}(z)^\beta$ be defined as

$$f_{n,i}(z)^\beta = z^\beta - \sum_{k=2}^{\infty} |a_{k,i}| z^{k+\beta-1} + (-1)^n \sum_{k=1}^{\infty} |b_{k,i}| \overline{z}^{k+\beta-1}. \tag{3.1}$$

Theorem 3.2. *Let $f_{n,i}(z)^\beta$ defined by (3.1) be in the class $\overline{H^\beta}(n,\lambda,\alpha)$ for every $i = 1, 2, 3, \ldots, m$. Then $c_i(z)$ defined by*

$$c_i(z) = \sum_{i=1}^{m} t_i f_{n,i}(z)^\beta, \quad (0 \le t_i \le 1), \tag{3.2}$$

is also in the class $\overline{H^\beta}(n,\lambda,\alpha)$, where $\sum\limits_{i=1}^{m} t_i = 1$.

Proof. For $i = 1, 2, 3, \cdots m$, let $f_{n,i}^\beta \in \overline{H^\beta}(n,\lambda,\alpha)$, where $f_{n,i}^\beta$ is given by (3.1). Since each $f_{n,i}(z)^\beta$ is in $\overline{H^\beta}(n,\lambda,\alpha)$ therefore by Theorem 2.2, we obtain

(3.3)
$$\sum_{k=1}^{\infty} \left(\frac{k+\lambda}{1+\lambda}\right)^n \left[\left(\frac{(k-\alpha)+\lambda(1-\alpha)}{(1+\lambda)}\right)\beta|a_{k,i}| + \left(\frac{(k+\alpha)+\lambda(1+\alpha)}{(1+\lambda)}\right)\beta|b_{k,i}|\right] \le (1+\beta)(1-\alpha).$$

For $\sum\limits_{i=1}^{m} t_i = 1$, $0 \le t_i \le 1$, the convex combinations of $f_{n,i}^\beta$ may be written as

$$\begin{aligned} c_i(z) &= \sum_{i=1}^{m} t_i f_{n,i}(z)^\beta \\ &= z^\beta - \sum_{k=2}^{\infty} \left(\sum_{i=1}^{m} t_i |a_{k,i}|\right) z^{k+\beta-1} + (-1)^n \sum_{k=1}^{\infty} \left(\sum_{i=1}^{m} t_i |b_{k,i}|\right) \overline{z}^{k+\beta-1}. \end{aligned}$$

Now for $c_i(z)$, consider

$$\begin{aligned} &\sum_{k=1}^{\infty} \left(\frac{k+\lambda}{1+\lambda}\right)^n \left[\left(\frac{(k-\alpha)+\lambda(1-\alpha)}{(1+\lambda)}\right)\beta\left|\sum_{i=1}^{m} t_i a_{k,i}\right| \right. \\ &\qquad \left. + \left(\frac{(k+\alpha)+\lambda(1+\alpha)}{(1+\lambda)}\right)|\sum_{i=1}^{m} t_i b_{k,i}|\right] \\ &= \sum_{i=1}^{m} t_i \left[\sum_{k=1}^{\infty} \left(\frac{k+\lambda}{1+\lambda}\right)^n \left[\left(\frac{(k-\alpha)+\lambda(1-\alpha)}{(1+\lambda)}\right)|a_{k,i}| \right.\right. \\ &\qquad \left.\left. + \left(\frac{(k+\alpha)+\lambda(1+\alpha)}{(1+\lambda)}\right)|b_{k,i}|\right]\beta\right] \\ &\le (1+\beta)(1-\alpha)\sum_{i=1}^{m} t_i \\ &= (1+\beta)(1-\alpha). \end{aligned}$$

Thus, $c_i(z) \in \overline{H^\beta}(n,\lambda,\alpha)$. $\square$

Theorem 3.3. *Let $\mathfrak{I}_c[f(z)^\beta]$ denote the Bernandi-Libera-Livingston integral operator for harmonic functions $f^\beta = h^\beta + \overline{g^\beta}$, where,*

$$\mathfrak{I}_c[f(z)^\beta] = \frac{c+\beta}{z^c}\int_0^z \xi^{c-1} h(\xi)^\beta d\xi + \overline{\frac{c+\beta}{z^c}\int_0^z \xi^{c-1} g(\xi)^\beta d\xi}, \quad (c>0).$$

If $f_n^\beta \in \overline{H^\beta}(n,\lambda,\alpha)$, then $\mathfrak{I}_c[f_n(z)^\beta] \in \overline{H^\beta}(n,\lambda,\alpha)$.

Proof. Let $f_n(z)^\beta = h(z)^\beta + \overline{g_n(z)^\beta} \in \overline{H^\beta}(n,\lambda,\alpha)$. Then,

$$\begin{aligned}
\mathfrak{I}_c[f_n(z)^\beta] &= \frac{c+\beta}{z^c}\int_0^z \xi^{c-1} h(\xi)^\beta d\xi + \overline{\frac{c+\beta}{z^c}\int_0^z \xi^{c-1} g_n(\xi)^\beta d\xi} \\
&= \frac{c+\beta}{z^c}\int_0^z \xi^{c-1}\left(\xi^\beta - \sum_{k=2}^{\infty}|a_k|\xi^{k+\beta-1}\right) d\xi \\
&\quad + \overline{\frac{c+\beta}{z^c}\int_0^z \xi^{c-1}\left((-1)^n\sum_{k=1}^{\infty}|b_k|\xi^{k+\beta-1}\right) d\xi} \\
&= z^\beta - \sum_{k=2}^{\infty}|A_k| z^{k+\beta-1} + (-1)^n \sum_{k=1}^{\infty}|B_k| z^{k+\beta-1},
\end{aligned}$$

where, $|A_k| = \left(\dfrac{c+\beta}{c+k+\beta-1}\right)|a_k|$ and $|B_k| = \left(\dfrac{c+\beta}{c+k+\beta-1}\right)|b_k|$.
Thus

$$\begin{aligned}
&\sum_{k=1}^{\infty}\left(\frac{k+\lambda}{1+\lambda}\right)^n\left[\left(\frac{(k-\alpha)+\lambda(1-\alpha)}{(1+\lambda)}\right)\beta|A_k| + \left(\frac{(k+\alpha)+\lambda(1+\alpha)}{(1+\lambda)}\right)\beta|B_k|\right] \\
&\quad = \sum_{k=1}^{\infty}\left[\left(\frac{k+\lambda}{1+\lambda}\right)^n\left(\frac{(k-\alpha)+\lambda(1-\alpha)}{(1+\lambda)}\right)\beta\left(\frac{c+\beta}{c+k+\beta-1}\right)|a_k|\right. \\
&\quad\quad \left. + \left(\frac{(k+\alpha)+\lambda(1+\alpha)}{(1+\lambda)}\right)\beta\left(\frac{c+\beta}{c+k+\beta-1}\right)|b_k|\right] \\
&\quad \le \sum_{k=1}^{\infty}\left[\left(\frac{k+\lambda}{1+\lambda}\right)^n\left(\frac{(k-\alpha)+\lambda(1-\alpha)}{(1+\lambda)}\right)\beta|a_k| + \left(\frac{(k+\alpha)+\lambda(1+\alpha)}{(1+\lambda)}\right)\beta|b_n|\right] \\
&\quad \le (1+\beta)(1-\alpha).
\end{aligned}$$

Hence, $\mathfrak{I}_c[f_n(z)^\beta] \in \overline{H^\beta}(n,\lambda,\alpha)$. □

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