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Existence and Uniqueness for a Class of SPDEs Driven by Lévy Noise in Hilbert Spaces

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Abstract. The present paper seeks to prove the existence and uniqueness of solutions to stochastic evolution equations in Hilbert spaces driven by both Poisson random measure and Wiener process with non-Lipschitz drift term. The proof is provided by the theory of measure of noncompactness and condensing operators. Moreover, we give some examples to illustrate the application of our main theorem.

1. Introduction

Stochastic evolution equations are natural development of SDEs and owing to their mathematical and natural science basis, they have attracted much attention. There are several papers and books providing significant applications of the types of equations in various fields of studies (see, for example, [11, 19]).

There are strong results and published papers on the existence and uniqueness of stochastic evolution equations driven by Wiener process, (see, for instance, [11] and references therein). Over the last few years, a number of authors have focused on stochastic evolution equations driven by Poisson random measure or Lévy process with different conditions on coefficients (see, e.g., [2, 7, 9, 10, 14, 15, 19, 20]). In one of these categories of conditions, first, Taniguchi [22] studied the existence and uniqueness of solutions to the following SDE in $\mathbb{R}^n$ by the method of successive approximations

$$dX_t = f(t, X_t)dt + g(t, X_t)dW_t, \quad X_0 = \xi,$$

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where f and g satisfy more general non-Lipschitz conditions including the special result of Yamada [25]. Likewise, some researchers extended the results of Taniguchi [22] to the infinite dimensional case. They considered the stochastic evolution equations with the coefficients satisfying the conditions suggested by Taniguchi [22]. Barbu [4] and then Barbu and Bocan [5, 6] studied the existence and uniqueness of the mild solution of the following equation with the coefficients of Taniguchi type in Hilbert spaces

$$dX_t = AX_t dt + f(t, X_t)dt + g(t, X_t)dW_t, \quad X_0 = \xi.$$

Their proof is based on the method of measure of noncompactness, Picard's method of approximation and successive approximation method, respectively. See also [12, 13, 17, 24] in the case of Wiener noise.

There are some works focused on stochastic evolution equations in Hilbert spaces driven by Poisson random measure with the coefficients of Taniguchi type. For instance, Tanguchi [**?**] investigated the existence and uniqueness of the energy solutions of stochastic functional evolution equation driven by Poisson jumps. He used the Pardoux method [18] to prove his main theorem, (see also [3, 8, 23]).

This paper sets out to prove the existence and pathwise uniqueness of the mild solution of the following SPDE

$$\begin{aligned} dX_t &= AX_t dt + f(t, X_t)dt + g(t, X_t)dW_t \\ &\quad + \int_{E\setminus\{0\}} k\big(t, X_t, y\big)\overline{N}(dt, dy), \quad X_0 = \xi, \end{aligned} \tag{1.1}$$

where W is a Wiener process, $\overline{N}(dt, dy)$ denotes the compensated Poisson random measure, $A : D(A) \subseteq \mathcal{H} \longrightarrow \mathcal{H}$ generates a contraction semigroup $(e^{tA})_{t\geq 0}$ in Hilbert space $\mathcal{H}$, functions f, g and k are measurable and ξ is an $\mathcal{F}_0$-measurable random variable in $\mathcal{H}$. Thanks to Lévy-Itô decomposition theorem, equation (1.1) includes a large class of equations driven by Lévy noise. Our proof is based on the theory of measure of noncompactness, so we extend the result of [4] for stochastic evolution equations driven by Lévy noise.

This paper is structured as follows. The second section introduces some notations and lemmas, used throughout the paper and in the third section, we go through the proof of the main result and corroborate our main claim by the help of an example.

2. Preliminaries

Let $\mathcal{K}$ and $\mathcal{H}$ be two real Hilbert spaces and let $L(\mathcal{K}, \mathcal{H})$ denote the space of linear bounded operators from $\mathcal{K}$ into $\mathcal{H}$. For simplicity, norm and inner product in both $\mathcal{K}$ and $\mathcal{H}$ are denoted by $\|.\|$ and $\langle ., . \rangle$ and we assume that $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ is a filtered probability space. Let $\mathcal{P}$ and $\mathcal{P}_T$ be the predictable σ-fields on $\Omega_\infty = [0,\infty)\times\Omega$ and $\Omega_T = [0,T)\times\Omega$, respectively. Moreover, W is a Wiener process on Hilbert space $\mathcal{H}$ with covariance operator Q. Furthermore, $L_2^0 = L_2(U_0, \mathcal{H})$ stands for the space of all Hilbert-Schmit operators from U_0 into $\mathcal{H}$ where U_0 is the Cameron-Martin space $U_0 = Q^{\frac{1}{2}}(U)$ with the norm $\|u\|_0 = \left\|Q^{\frac{-1}{2}}(u)\right\|$ and L_2^0 is a Hilbert space endowed with the following norm:

$$\|\varphi\|_{L_2^0} = \left\|\varphi Q^{\frac{1}{2}}\right\| t_{L_2(U,\mathcal{H})}.$$

In addition, $N(dt,dx)$ stands for Poisson random measure on $\mathbb{R}^+ \times E$ with intensity measure $\nu(dx)dt$ where $(E,\mathcal{E})$ is a measurable space. We denote by $\overline{N}(dt,dx)$ the compensated Poisson measure defined by $\overline{N}(dt,dx) = N(dt,dx) - \nu(dx)dt$; furthermore, N and W are assumed to be independent.

For more details about Poisson random measure and Wiener process in Hilbert spaces, see, [11, 19].

Definition 2.1. A predictable process $X : [0,T]\times\Omega \longrightarrow \mathcal{H}$ is a mild solution of equation (1.1) if for every $t\in[0,T]$ we have

$$P\left(\int_0^t \left\|e^{(t-s)A} f(s,X_s)\right\| ds + \int_0^t \left\|e^{(t-s)A} g(s,X_s)\right\|_{L_2^0}^2 ds \right.$$
$$\left. + \int_0^t \int_{E\setminus\{0\}} \left\|e^{(t-s)A} k\big(s,X_s,y\big)\right\|^2 \nu(dy)ds < \infty\right) = 1,$$

and it satisfies the following equation P-almost surely

$$X_t = e^{tA}\xi + \int_0^t e^{(t-s)A} f(s,X_s)ds + \int_0^t e^{(t-s)A} g(s,X_s)dW_s$$
$$+ \int_0^t \int_{E\setminus\{0\}} e^{(t-s)A} k\big(s,X_s,y\big)\bar{N}(ds,dy).$$

Let us state some important lemmas which will play a fundamental role in the proof of our results.

Lemma 2.2 ([19])**.** *Suppose that A generates a contraction semigroup and $\Phi(t)$ is an L_2^0-valued predictable process satisfying*

$$E\int_0^T \|\Phi(s)\|_{L_2^0}^2 ds < \infty,$$

then for a constant C_T, the following inequality holds

$$E\left(\sup_{t\in[0,T]} \left\|\int_0^t e^{(t-s)A}\,\Phi(s)dW_s\right\|^2\right) \leq C_T\, E\int_0^T \|\Phi(s)\|_{L_2^0}^2 ds, \quad t\in[0,T].$$

Lemma 2.3 ([16])**.** *Assume* $\Phi : \mathbb{R}^+ \times \Omega \times E\backslash\{0\} \longrightarrow \mathcal{H}$ *is a* $\mathcal{P} \times \mathcal{E}$*-measurable function such that*

$$\int_0^t \int_{E\backslash\{0\}} \|\Phi(s,x)\|^2 \nu(dx)ds < \infty,$$

for all $t \geq 0$*, and let* A *generate a contraction semigroup, then for each* $0 < p \leq 2$*, we have*

$$E\left(\sup_{t\in[0,T]} \|Z(t)\|^p\right) \leq C'_{p,T} E\left[\int_0^T \int_{E\backslash\{0\}} \|\Phi(s,x)\|^2 \nu(dx)ds\right]^{\frac{p}{2}},$$

where $C'_{p,T}$ *is a constant and*

$$Z(t) = \int_0^t \int_{E\backslash\{0\}} e^{(t-s)A}\Phi(s,x)\bar{N}(ds,dy).$$

The notation M_T stands for the Banach space of all càdlàg and predictable processes $X_t(w)$ defined on $[0,T] \times \Omega$ and endowed with the following norm

$$\|X\|_{M_T} = \left\{E\left(\sup_{t\in[0,T]} \|X_t(w)\|^2\right)\right\}^{\frac{1}{2}} < \infty.$$

Besides, we denote by

$$N_r[X_0] = \left\{X \in M : \|X - X_0\|_{M_T} \leq r\right\},$$

the closed ball with center X_0 and radius r in M_T.

Let us introduce the basic notations related to measure of noncompactness (see [1] for more details).

Definition 2.4. Assume that $(\mathcal{A}, \leq)$ is a partially ordered set and E is a Banach space. A function α defined from $P(E)$, the power set of E, into $\mathcal{A}$ is called a measure of noncompactness if $\alpha(\overline{coB}) = \alpha(B)$, for all $B \subseteq E$. Here $\overline{coB}$ denotes the closure of the convex hall of B.

Definition 2.5. The Hausdorf measure of noncompactness of a nonempty subset B of a Banach space E, denoted by $\chi(B)$, is the infimum of all numbers ϵ for which B has a finite ϵ-net in E.

Definition 2.6. Assume that α is a measure of noncompactness in a Banach space E with values in $(\mathcal{A}, \leq)$. An α-condensing operator is a continuous operator $\Psi : D(\Psi) \subseteq E \longrightarrow E$ such that for $B \subseteq D(\Psi)$, $\alpha[\Psi(B)] \geq \alpha(B)$ implies that B is relatively compact.

Theorem 2.7 ([1]). *Assume that α is a measure of noncompactness on a Banach space E such that $\alpha(B \cup \{x\}) = \alpha(B)$ for any $B \subseteq E$ and $x \in E$ (called additively nonsingular property), and let $\Psi : E \longrightarrow E$ be an α-condensing operator on E such that for some nonempty, convex and closed subset K satisfies $\Psi(K) \subseteq K$. Then Ψ has at least one fixed point in K.*

We denote by $\mathcal{S}[0,T]$ the space of all real increasing functions on $[0,T]$. Furthermore, notice that $\mathcal{S}[0,T]$ is partially ordered by the usual ordered $\leq$. Also, we indicate the following measure of noncompactness on M_T by μ;

$$\begin{cases} \mu : P(M_T) \to \mathcal{S}[0,T], \\ \Lambda \to \mu(\Lambda). \end{cases}$$

Here

$$\begin{cases} \mu(\Lambda) : [0,T] \to \mathbb{R}, \\ t \to \chi_t(\Lambda_t), \end{cases}$$

where $\Lambda_t = \{X\,|_{[0,t]} : X \in \Lambda\} \subseteq M_t$ and χ_t is the Hausdorff measure of noncompactness on M_t. For more details about μ, see [1].

3. Main Results

In this section, we prove our main result, based on Theorem 2.7.

Theorem 3.1. *Considering equation (1.1), suppose that the following assumptions hold:*

(i) *There exists a function $G : \mathbb{R}^+ \times \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ for which*

$$E\,\|f(t,X)\|^2 + E\|g(t,X)\|^2_{L_2^0} + E\int_{E\setminus\{0\}} \|k(t,X,y)\|^2 \nu(dy) \leq G\left(t, E\,\|X\|^2\right),$$

for all $t \in [0,T]$ and all $X \in L^2(\Omega, \mathcal{F}, \mathcal{H})$.

(ii) *$G(t,x)$ is continuous and increasing in x and it is locally integrable in t.*

Then the operator

$$\Psi : (M_\tau, \|.\|_{M_\tau}) \longrightarrow (M_\tau, \|.\|_{M_\tau}),$$

defined by

$$\Psi X_t = e^{tA}\xi + \int_0^t e^{(t-s)A} f(s,X_s)ds + \int_0^t e^{(t-s)A} g(s,X_s)dW_s + \int_0^t \int_{E\setminus\{0\}} e^{(t-s)A} k\big(s,X_s,y\big)\bar{N}(ds,dy); \quad t \in [0,T],$$

is well defined and has the following property for a fixed $\tau \in [0,T]$:

$$\Psi\big(N_r\big[e^{\cdot A}\xi\big]\big) \subseteq N_r[e^{\cdot A}\xi].$$

Proof. From Lemmas 2.2 and 2.3, we conclude that the operator Ψ is well defined. And we get

$$\begin{aligned}
&E\left(\sup_{0\le s\le\tau}\left\|\Psi X_s - e^{sA}\xi\right\|^2\right)\\
&\quad\le \underbrace{3E\left(\sup_{0\le s\le\tau}\left\|\int_0^s e^{(s-r)A}f(r,X_r)dr\right\|^2\right)}_{I_1}\\
&\qquad+\underbrace{3E\left(\sup_{0\le s\le\tau}\left\|\int_0^s e^{(s-r)A}g(r,X_r)dW_r\right\|^2\right)}_{I_2}\\
&\qquad+\underbrace{3E\left(\sup_{0\le s\le\tau}\left\|\int_0^s\int_{E\backslash\{0\}} e^{(s-r)A}k\big(r,X_r,y\big)\bar{N}(dr,dy)\right\|^2\right)}_{I_3}.
\end{aligned}$$

The Hölder's inequality implies that

$$\begin{aligned}
I_1 &\le 3E\sup_{0\le s\le\tau}\left(\int_0^s\|f(r,X_r)\|\,dr\right)^2\\
&\le 3E\left(\int_0^\tau\|f(r,X_r)\|dr\right)^2\\
&\le 3\tau E\int_0^\tau\|f(r,X_r)\|^2dr.
\end{aligned}\tag{3.1}$$

Applying Lemma 2.2, we get

$$I_2\le 3C_\tau E\int_0^\tau\|g(r,X_r)\|^2_{L_2^0}\,dr.\tag{3.2}$$

Also, Lemma 2.3 for $p=2$ implies

$$I_3\le 3C'_\tau E\int_0^\tau\int_{E\backslash\{0\}}\left\|k\big(r,X_r,y\big)\right\|^2\nu(dy)dr.\tag{3.3}$$

By the inequalities (3.1), (3.2), (3.3) and assumption (i), we have

$$E\left(\sup_{0\le s\le\tau}\left\|\Psi X_s-e^{sA}\xi\right\|^2\right)\le L\int_0^\tau G\left(s,E\|X_s\|^2\right)ds,$$

where $L=3(\tau+C_\tau+C'_\tau)$. If $X\in N_r[e^{\cdot A}\xi]\subseteq M_\tau$ then $E\left\|X_s-e^{sA}\xi\right\|^2\le r^2$ for any $s\in[0,\tau]$ and we get

$$\begin{aligned}
E\|X_s\|^2&\le E\left(\left\|X_s-e^{sA}\xi\right\|+\left\|e^{sA}\xi\right\|\right)^2\\
&\le 2E\left\|X_s-e^{sA}\xi\right\|^2+2E\left\|e^{sA}\xi\right\|^2
\end{aligned}$$

$$\begin{aligned}&\leq 2r^2 + 2E\left\|e^{sA}\xi\right\|^2\\ &\leq 2r^2 + 2E\left\|\xi\right\|^2\\ &= M.\end{aligned}$$

According to assumption (ii), the function $G(t,x)$ is increasing in x, and therefore, we get

$$E\left(\sup_{0\leq s\leq\tau}\left\|\Psi X_s - e^{sA}\xi\right\|^2\right) \leq L\int_0^\tau G(s,M)ds,$$

for all $X \in N_r[e^{\cdot A}\xi] \subseteq M_\tau$. Notice that $G(.,M)$ is locally integrable and therefore, there exists τ' such that $L\int_0^{\tau'} G(s,M)ds \leq r^2$ and we get

$$\Psi\left(N_r\left[e^{\cdot A}\xi\right]\right) \subseteq N_r[e^{\cdot A}\xi].$$

□

Theorem 3.2. *Assume that the functions f, g and k satisfy all the conditions of Theorem 3.1. Furthermore, suppose that:*

(i) *There exists a function $D : \mathbb{R}^+ \times \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ which is increasing and continuous in x and it is locally integrable in t. Furthermore, $D(t,0) \equiv 0$ and*

$$\begin{aligned}E\left\|f(t,X) - f(t,Y)\right\|^2 &+ E\|g(t,X) - g(t,Y)\|^2_{L_2^0}\\ &+ E\int_{E\setminus\{0\}} \|k(t,X,y) - k(t,Y,y)\|^2\nu(dy)\\ &\leq D\big(t, E\left\|X-Y\right\|^2\big),\end{aligned} \tag{3.4}$$

for all $t \in [0,T]$ and $X, Y \in L^2(\Omega, \mathcal{F}, \mathcal{H})$.

(ii) *If there exists a nonnegative function $V(t)$ which satisfies*

$$\begin{cases} V(t) \leq \alpha\int_{t_0}^t D\big(s, V(s)\big)ds, & t\in[0,T],\\ V(0) = 0,\end{cases} \tag{3.5}$$

then $V(t) = 0$ for all $t \in (0, T_1]$, where α is a positive number and $T_1 \in (0,T]$.

Then the operator $\Psi' : \big(M_T, \|.\|_{M_T}\big) \longrightarrow \big(M_T, \|.\|_{M_T}\big)$ defined by

$$\begin{aligned}\Psi' X_t = &\int_0^t e^{(t-s)A} f\big(s, X_s\big)ds + \int_0^t e^{(t-s)A} g\big(s, X_s\big)dW_s\\ &+ \int_0^t\int_{E\setminus\{0\}} e^{(t-s)A} k\big(s, X_s, y\big)\bar{N}(ds,dy), \quad t\in[0,T],\end{aligned}$$

is a condensing operator with respect to μ on any bounded subset of M_T.

Proof. Assume $\mu(B) \leq \mu(\Psi^{'}(B))$ for a fixed bounded set $B \subseteq M_T$. To accomplish our goal, we use this fact that the function $t \longrightarrow [\mu(B)](t)$ is increasing and bounded. Consequently, the number of jumps with values greater than ϵ is finite for any fixed $\epsilon \geq 0$.

Remove disjoint δ_1-neighborhoods of these jumps from the interval $[0, T]$, we divide the remaining intervals into smaller intervals by choosing points $\alpha_i, i = 1, \ldots, n$, on which the oscillation of $\mu(B)$ is smaller than ϵ. Now surround the points α_i by δ_2-neighborhoods and consider $\Lambda = \{h_k : k = 1, \ldots, l\}$ which includes all almost surely continuous functions, constructed as follows; h_k coincides with an arbitrary element of $[(\mu(B)(\alpha_i) + \epsilon]$-net of the set B_{α_i} on $\sigma_i = [\alpha_{i-1} + \delta_2, \alpha_i + \delta_2]$,$i = 1, \ldots, n$, and it is linear on the complementar segments. Now assume $u \in \Psi^{'}(B)$, for some $h \in B$ we have $u = \Psi^{'}(h)$ and

$$\|h - h_r^{\alpha_i}\|^2_{M_{\alpha_i}} \leq [(\mu(B)(\alpha_i) + \varepsilon]^2,$$

here $h_r^{\alpha_i}$ is an element of the $[(\mu(B)(\alpha_i) + \epsilon]$-net of B_{α_i}. Since $h_r^{\alpha_i}|_{\sigma_i} = h_{k|\sigma_i}$ for some element h_k of Λ, it implies that for $s \in \sigma_i$ we get

$$\begin{aligned} E\|h(s) - h_k(s)\|^2 &\leq E\left(\sup_{\alpha_{i-1}+\delta_2 \leq s \leq \alpha_i - \delta_2} \|h(s) - h_k(s)\|^2\right) \\ &\leq \|h - h_r^{\alpha_i}\|^2_{M_{\alpha_i}} \\ &\leq [\big(\mu(B)\big)(\alpha_i) + \varepsilon]^2 \\ &\leq [\big(\mu(B)\big)(s) + 2\varepsilon]^2. \end{aligned}$$

Then

$$\begin{aligned} &E\left(\sup_{0\leq s\leq t}\left\|\Psi^{'}h(s) - \Psi^{'}h_k(s)\right\|^2\right) \\ &\leq 3t\int_0^t E\|f(s,h(s)) - f(s,h_k(s))\|^2 ds \\ &\quad + 3C_t\int_0^t E\|g(s,h(s)) - g(s,h_k(s))\|^2_{L_2^0} ds \\ &\quad + 3\,C_t^{'}\int_0^t\int_{E\setminus\{0\}} E\|k(s,h(s),y) - k(s,h_k(s),y)\|^2\,\nu(dy)ds \\ &\leq M\int_0^t D\big(s, E\|h(s) - h_k(s)\|^2\big)ds \\ &= M\sum_{i=1}^n\int_{\sigma_i} D\big(s, E\|h(s) - h_k(s)\|^2\,\big)ds \end{aligned}$$

$$+ M \int_{[0,t]-\bigcup\limits_{i=1}^{n}\sigma_i} D\big(s, E\|h(s)-h_k(s)\|^2\big)ds.$$

The set Λ is finite and B is bounded and therefore, there exists $b \geq 0$ for which we have $E(\|h(s)-h_k(s)\|^2) < b$ for all $h \in B$, $h_k \in \Lambda$ and $s \in [0,T]$. Using Theorem 3.2 (i), we find $\delta_1, \delta_2 \geq 0$ sufficiently small such that

$$\begin{aligned}[(\mu(B))(t)]^2 &\leq \Big[(\mu(\Psi' B))(t)\Big]^2 \\ &\leq \varepsilon + M\int_0^t D(s,[(\mu(B))(s)+2\varepsilon]^2)ds.\end{aligned}$$

The function $D(t,x)$ is continuous in x and ϵ is arbitrary and as a result, we have:

$$[(\mu(B))(t)]^2 \leq M\int_0^t D(s,[(\mu(B))(s)]^2)ds. \tag{3.6}$$

Finally, by inequality (3.6), Lemma 2.2 in [4] and Theorem 3.2 (ii), we conclude that $\mu(B)=0$. Hence, B is totally bounded in M_T and therefore it is relatively compact.

To prove the continuity of the operator Ψ', assume $\{X^n\}_{n\geq 1}$ to be a sequence converging to X in M_T, thus

$$\begin{aligned}&\Big\|\Psi' X - \Psi' X^n\Big\|_{M_T}^2 \\ &= E\left(\sup_{0\leq t\leq T}\Big\|\Psi' X_t - \Psi' X_t^n\Big\|^2\right) \\ &\leq 3T\int_0^T E\|f(s,X_s)-f(s,X_s^n)\|^2 ds \\ &\quad + 3C_T\int_0^T E\|g(s,X_s)-g(s,X_s^n)\|_{L_2^0}^2 ds \\ &\quad + 3\,C_T'\int_0^T\int_{E\setminus\{0\}} E\|k(s,X_s,y)-k(s,X_s^n,y)\|^2\,\nu(dy)ds \\ &\leq M\int_0^T D\left(s,\|X-X^n\|_{M_T}^2\right)ds,\end{aligned}$$

which implies $\Big\|\Psi' X - \Psi' X^n\Big\|_{M_T}^2 \longrightarrow 0$ as $\|X-X^n\|_{M_T}\longrightarrow 0$. □

Remark 3.3. (i) Notice that, under the assumptions of Theorem 3.2, the operator $\Psi : M_T \longrightarrow M_T$ defined by

$$\Psi X_t = e^{tA}\xi + \Psi' X_t, \quad t\in[0,T], \tag{3.7}$$

is also μ-condensing.

(ii) Notice that if $D(t,x)$ is concave in x and it satisfies the following inequality

$$\begin{aligned}\|f(t,x)-f(t,y)\|^2 &+ \|g(t,x)-g(t,y)\|^2_{L_2^0}\\ &+\int_{E\setminus\{0\}}\|k(t,x,z)-k(t,y,z)\|^2\nu(dz)\\ &\le D\big(t,\|x-y\|^2\big),\end{aligned}$$

for all $y,x\in\mathcal{H}$ and any $t\ge 0$, then the inequality (3.4) is followed immediately by Jensens inequality.

(iii) Let $D(t,x)=\lambda(t)\varphi(x)$, where $\lambda(t)$ is a locally integrable, nonnegative function and $\varphi:\mathbb{R}^+\longrightarrow\mathbb{R}^+$ is a continuous and increasing function such that $\varphi(0)=0$ and $\int_{0^+}\frac{1}{\varphi(x)}dx=\infty$, then the function $D(t,x)$ satisfies inequality (3.5) of Theorem 3.2, see [21].

(vi) Let $D(t,x)=\lambda(t)\varphi^2(x^{\frac{1}{2}})$, where $\lambda(t)$ is a locally integrable, nonnegative function and $\varphi:\mathbb{R}^+\longrightarrow\mathbb{R}^+$ is a concave increasing function on $\mathbb{R}^+$ such that $\varphi(0)=0$ and $\int_{0^+}\frac{x}{\varphi^2(x)}dx=\infty$, then the function $D(t,x)$ satisfies inequality (3.5) of Theorem 3.2, see [8].

We are now in a position to prove our main theorem.

Theorem 3.4. *Under the assumptions of Theorem 3.2, equation (1.1) has a unique solution for some $T^{'}\in(0,T]$ in $M_{T'}$.*

Proof. According to Theorem 3.1 for some $T^{'}\in(0,T]$, the operator Ψ has the following property:

$$\begin{aligned}\Psi\left(N_r\left[e^{\cdot A}\xi\right]\right)&\subseteq N_r[e^{\cdot A}\xi]\\ &\subseteq M_{T'}.\end{aligned}$$

Notice that $N_r[e^{\cdot A}\xi]$ is a convex, closed and nonempty subset of $M_{T'}$, and Ψ is a μ-condensing operator on $N_r[e^{\cdot A}\xi]$, then by Theorem 2.7, Ψ has at least one fixed point in $N_r[e^{\cdot A}\xi]\subseteq M_{T'}$. Let us prove the uniqueness of solution. Assume $X,Y\in M_{T'}$ to be two fixed points for Ψ. Then

$$\begin{aligned}&E\left(\sup_{0\le s\le t}\|X_s-Y_s\|^2\right)\\ &\le 3t\int_0^t E\|f(s,X_s)-f(s,Y_s)\|^2ds\\ &\quad+3C_t\int_0^t E\|g(s,X_s)-g(s,Y_s)\|^2_{L_2^0}ds\end{aligned}$$

$$
\begin{aligned}
&+ 3\,C_t^{'} \int_0^t \int_{E\setminus\{0\}} E\left\|k(s,X_s,y) - k(s,Y_s,y)\right\|^2 \nu(dy)ds \\
&\leq M \int_0^t D\big(s, E\left\|X_s - Y_s\right\|^2\big)ds.
\end{aligned}
$$

Hence,

$$
\|X - Y\|_{M_t}^2 \leq M \int_0^t D(s, \|X - Y\|_{M_s}^2)ds, \tag{3.8}
$$

finally, consider $V(t) = \|X - Y\|_{M_t}$ in inequality (3.8), therefore $V(t) = \|X - Y\|_{M_t}$ satisfies inequality (3.5) in Theorem 3.2 (ii) and it follows that $V(t) = \|X - Y\|_{M_t} = 0$ and we get $X = Y$. □

Eventually, in an attempt to prove equation (1.1) has a global solution, we assume that A generates a compact infinitesimal C_0-semigroup.

Theorem 3.5. *For equation* (1.1)*, suppose that the functions* $g(t,w,x)$*,* $f(t,w,x)$ *and* $k(t,w,x,y)$ *satisfy all the conditions of Theorem 3.2 with* $T = \infty$*. Furthermore, let equation*

$$
\frac{dv(t)}{dt} = \alpha G\big(t, v(t)\big), \tag{3.9}
$$

have a global solution on (t_0,∞) *for all* $T > 0, \alpha > 0$ *and for each initial value* (t_0,v_0)*,* $t_0 > 0, v_0 \geq 0$*. Then equation* (1.1) *has a global solution on* $[0,\infty)$*.*

Proof. Let the set $\mathcal{G}$ include all s such that equation (1.1) has a mild solution on $[0,s]$ and put $s_1 = \sup_{s\in\mathcal{G}} s$. According to Theorem 3.4, we get $s_1 > 0$. Assume $s_1 < \infty$ and let $s_1 < T < \infty$. We shall prove that there exists a continuous extension on $[0,s_1]$ for equation (1.1) defined on $[0,s_1)$. As a result, by considering Theorem 3.4, equation (1.1) has an extension to the right of s_1 and this is in contradiction to the definition of s_1.

Assume X_t is the mild solution of equation (1.1). Then we get

$$
\begin{aligned}
E\left\|X_t\right\|^2 \leq 4E\left\|\xi\right\|^2 &+ 4t\int_0^t E\left\|f(s,X_s)\right\|^2 ds \\
&+ 4C_t \int_0^t E\left\|g(s,X_s)\right\|_{L_2^0}^2 ds \\
&+ 4C_t^{'} \int_0^t \int_{E\setminus\{0\}} E\left\|k\big(s,X_s,y\big)\right\|^2 \nu(dy)ds,
\end{aligned}
$$

for a fixed $t \in [0, s_1)$, and from Theorem 3.1, we have

$$E\|X_t\|^2 \leq 4E\|\xi\|^2 + 4(t + C_t + C_t^{'}) \int_0^t G\big(s, E\|X_s\|^2\big) ds.$$

Let $v_0 > 4E\big(\|\xi\|\big)^2$, $\alpha = 4(t + C_t + C_t^{'})$ and assume $v(t)$ is the global solution of the following equation

$$\begin{cases} \frac{dv(t)}{dt} = \alpha G\big(t, v(t)\big), \\ v(0) = v_0. \end{cases}$$

Then we have

$$E\|X_t\|^2 - \alpha \int_0^t G\big(s, E\|X_s\|^2\big) ds < v_0$$
$$= v(t) - \alpha \int_0^t G\big(s, v(s)\big) ds,$$

for all $t \in [0, s_1)$.

Then by Lemma 4 in [22], we get $E\|X_t\|^2 < v(t) \leq v(T)$ for all $t \in [0, s_1)$.

We have:

$$\begin{aligned}
&E\|X_t - X_s\|^2 \\
&\quad = E\Bigg(\bigg\|\big(e^{tA} - e^{sA}\big)\xi + \int_0^s \big(e^{(t-r)A} - e^{(s-r)A}\big) f(r, X_r) dr \\
&\qquad + \int_s^t e^{(t-s)A} f(r, X_r) dr + \int_0^s \big(e^{(t-r)A} - e^{(s-r)A}\big) g(r, X_r) dW_r \\
&\qquad + \int_s^t e^{(t-s)A} g(r, X_r) dW_r \\
&\qquad + \int_0^s \int_{E\setminus\{0\}} \big(e^{(t-r)A} - e^{(s-r)A}\big) k\big(r, X_r, y\big) \overline{N}(dr, dy) \\
&\qquad + \int_s^t \int_{E\setminus\{0\}} e^{(t-s)A} k\big(r, X_r, y\big) \overline{N}(dr, dy) \bigg\|\Bigg)^2 \\
&\quad \leq 6\left(e^{tA} - e^{sA}\right)^2 E\|\xi\|^2 \\
&\qquad + 6TE \int_0^s \Big\|\big(e^{(t-r)A} - e^{(s-r)A}\big)\Big\|^2 \|f(r, X_r)\|^2 dr \\
&\qquad + 6T \int_s^t E\|f(r, X_r)\|^2 dr \\
&\qquad + 6C_T E \int_0^s \Big\|\big(e^{(t-r)A} - e^{(s-r)A}\big)\Big\|^2 \|g(r, X_r)\|^2 dr
\end{aligned}$$

$$+ 6C_T \int_s^t E \left\| g(r, X_r) \right\|^2 dr$$
$$+ 6C'_T E \int_0^s \int_{E\setminus\{0\}} \left\| \left(e^{(t-r)A} - e^{(s-r)A}\right) \right\|^2 \left\| k(r, X_r, y) \right\|^2 \nu(dy)dr$$
$$+ 6C'_T \int_s^t \int_{E\setminus\{0\}} E \left\| k(r, X_r, y) \right\|^2 \nu(dy)dr.$$

By Theorem 3.1, we get

$$E \left\| X_t - X_s \right\|^2 \le 6 \left\| \left(e^{tA} - e^{sA}\right) \right\|^2 E \left\| \xi \right\|^2$$
$$+ 6(T + C_T + C'_T) \int_0^s \left\| \left(e^{(t-r)A} - e^{(s-r)A}\right) \right\|^2 G\big(r, v(r)\big) dr$$
$$+ 6(T + C_T + C'_T) \int_s^t G\big(r, v(r)\big) dr.$$

Now, owing to the fact that the function $r \longrightarrow G\big(r, v(r)\big)$ is integrable on $[0, T]$ and the function $t \longrightarrow e^{tA}$ is continuous in operator norm, by using Lebesgue convergence theorem we get

$$\lim_{s,t\uparrow s_1} E\|X_t - X_s\|^2 = 0.$$

Therefore, it follows that there exists $\lim_{t\uparrow s_1} X_t \overset{def}{=} X_{s_1}$ and $E \left\| X_{s_1} \right\|^2 < \infty$. □

4. Illustrated Examples

In this section, motivated by [8], to illustrate the application of our main theorem, we provide some examples of equation (1.1).

Example 4.1. Consider equation (1.1). Let $N(dt, dx)$ be a Poisson random measure on $\mathbb{R}^+ \times E$ with intensity measure $\nu(dx)dt$, where E is a Banach space and $\nu(dx)$ satisfies $\int_{E\setminus\{0\}} \|x\|^2 \nu(dx) < \infty$. Also, set $k(t, X, y) = \sqrt{\lambda_t}\|y\|_E \sum_{m=0}^{\infty} \beta_m(t, X)e_m$, $f(t, X) = g(t, X) = 0$, where $X \in L^2(\Omega, \mathcal{F}, \mathcal{H})$, e_m is an orthonormal basis for Hilbert space $\mathcal{H}$, $\lambda(t)$ is a nonnegative, locally integrable and increasing function, $\beta_{2m+1}(t, X) = \alpha_{2m+1} sin(m^q\|X\|)$, $\beta_{2m}(t, X) = \alpha_{2m} cos(m^q\|X\|)$, which $q > 0$, $\alpha_{2m} = O\left(m^{-(q+\frac{1}{2})}\right)$ and $\alpha_{2m+1} = O\left(m^{-(q+\frac{1}{2})}\right)$. Then we have

$$\int_{E\setminus\{0\}} \|k(t, X, y) - k(t, Y, y)\|^2_{\mathcal{H}} \nu(dy)$$
$$= \int_{E\setminus\{0\}} \left\| \sqrt{\lambda_t}\|y\|_E \sum_{m=0}^{\infty} (\beta_m(t, X) - \beta_m(t, Y))e_m \right\|^2_{\mathcal{H}} \nu(dy)$$

$$= \lambda_t \int_{E\setminus\{0\}} \|y\|_E^2 \nu(dy) \sum_{m=0}^{\infty} \left|\beta_m(t,X) - \beta_m(t,Y)\right|^2.$$

By [8] we have

(4.1)
$$\sum_{m=0}^{\infty} \left|\beta_m(t,X) - \beta_m(t,Y)\right|^2 \leq c \sum_{m=1}^{\infty} m^{-(2q+1)} \sin^2\left(\frac{m^q\|X-Y\|}{2}\right) \leq c\varphi^2(\|X-Y\|),$$

for all sufficiently small $\|X - X'\|$, where

$$\varphi(x) = \begin{cases} 0, & x = 0, \\ cx(-\ln x)^{\frac{1}{2}}, & 0 < x \leq \delta, \\ c\delta(-\ln\delta)^{\frac{1}{2}}, & x > \delta, \end{cases}$$

and $\delta \in (0,1)$ is sufficiently small. Therefore, by (4.5), (4.1) we conclude

$$\int_{E\setminus\{0\}} \|k(t,X,y) - k(t,Y,y)\|_{\mathcal{H}}^2 \nu(dy) \leq c\lambda_t\varphi^2(\|X-Y\|) \int_{E\setminus\{0\}} \|y\|_E^2 \nu(dy) = D(t, \|X-Y\|^2),$$

where $D(t,x) = c\lambda(t)\varphi^2\left(x^{\frac{1}{2}}\right) \int_{E\setminus\{0\}} \|y\|_E^2\nu(dy)$. Notice that $g(x) = \varphi^2\left(x^{\frac{1}{2}}\right)$ is an increasing, continuous and concave function on $\mathbb{R}^+$ for which $\int_{0^+} \frac{1}{g(x)}dx = \infty$, hence, by Remark 3.3, equation (1.1) has a unique solution.

In the next example, we consider two special cases of Example 4.1 to make it more clear.

Example 4.2. Let us consider $\mathcal{H} = \ell^2$ in Example 4.1 where ℓ^2 is the Hilbert space of square summable sequences and β_i's are defined as in Example 4.1, also let $A : D(A) \subseteq \ell^2 \longrightarrow \ell^2$ be an operator such that $Ae_n = \lambda_n e_n$ where $\{e_n\}_{n\geq 1}$ denotes the standard basis of ℓ^2 and λ_n is negative for each $n \in \mathbb{N}$. Also let $(e^{tA})_{t\geq 0}$ be the C_0-semigroup $(e^{tA})_{t\geq 0}$ generated by A.

For each $x = (x_1, x_2, \ldots) \in \ell^2$, we have

$$\left\|e^{tA}x\right\|_2^2 = \sum_{n=1}^{\infty} e^{2t\lambda_n} x_n^2 \leq \sum_{n=1}^{\infty} x_n^2 = \|x\|_2^2,$$

therefore, $(e^{tA})_{t\geq 0}$ is a contraction semigroup on ℓ^2.

In this case, Example 4.1 is equivalent to the following infinite system of stochastic differential equations and it has a unique mild solution $X(t) = \big(X_1(t), X_2(t), \dots\big) \in \ell^2$

$$dX_i(t) = \lambda_i X_i(t)dt + \int_{E\setminus\{0\}} \sqrt{\lambda(t)}\, \|y\| \beta_i\big(t, X(t)\big)\bar{N}(dt, dy), \tag{4.2}$$

for each $i \in \mathbb{N}$.

As another special case of Example 4.1, consider $\mathcal{H} = (\mathbb{R}^m, \|.\|_2)$ and the C_0-semigroup $(e^{tA})_{t\geq 0}$ on $\mathbb{R}^m$, where $A = diag(a_1, \dots, a_m)$ is a diagonal matrix with negative entries $a_1, \dots, a_m$ and $e^{tA} = diag(e^{a_1 t}, \dots, e^{a_m t})$. For each $x = (x_1, \dots, x_m) \in \mathbb{R}^m$, we have

$$\big\|e^{tA}x\big\|_2 = \sqrt{e^{2a_1 t}x_1^2 + \cdots + e^{2a_m t}x_m^2} \leq e^{\alpha t}\|x\|_2 \leq \|x\|_2, \tag{4.3}$$

where $\alpha = max\{a_1, \dots, a_m\}$.

Therefore, $(e^{tA})_{t\geq 0}$ is a contraction semigroup with respect to the Euclidean norm $\|.\|_2$. In this case, Example 4.1 is equivalent to the following system of stochastic differential equations

$$dX_i(t) = a_i X_i(t)dt + \int_{E\setminus\{0\}} \sqrt{\lambda(t)}\, \|y\| \beta_i\big(t, X(t)\big)\bar{N}(dt, dy), \tag{4.4}$$

for each $i = 1, 2, 3, \dots, m$.

Now let us give an example of equation (1.1) without jumps.

Example 4.3. Consider equation (1.1). Let $f(t, X) = k(t, X, y) = 0$ for each $t \geq 0$, $y \in E$, $X \in L^2(\Omega, \mathcal{F}, \mathcal{H})$ and $g(t, X) = \sqrt{\lambda_t}\sum_{m=0}^{\infty} \beta_m(t, X)e_m$ where $\{e_m\}_{m\geq 1}$ is an orthonormal basis for $L_2(U, \mathcal{H})$, $\lambda(t)$ is a non-negative, locally integrable and increasing function and $\beta_m(t, X) = m^{-p}\varphi(m^q\|X\|)$ where φ^2 is an increasing and concave function on $\mathbb{R}^+$, $\varphi(0) = 0$, $O\big(\varphi(x)\big) = x^\alpha$, $\alpha < 1$ and $2q - 2p < -1$.

Notice that φ is a concave function and $\varphi(0) = 0$, therefore it satisfies $\varphi(x) - \varphi(y) \leq \varphi(x - y)$ and we get

$$\begin{aligned} &\Big\|g(t, X) - g(t, X^{'})\Big\|_{L_2(U,\mathcal{H})} \\ &= \left(\lambda(t)\sum_{m=1}^{\infty}\Big(\beta_m(t, X) - \beta_m(t, X^{'})\Big)^2\right)^{\frac{1}{2}} \\ &= \sqrt{\lambda(t)}\left(\sum_{m=1}^{\infty} m^{-2p}\Big(\varphi(m^q\|X\|) - \varphi(m^q\|X^{'}\|)\Big)^2\right)^{\frac{1}{2}} \end{aligned} \tag{4.5}$$

$$\leq \sqrt{\lambda(t)}\left(\sum_{m=1}^{\infty} m^{-2p}\varphi^2\left(m^q\|X-X^{'}\|\right)\right)^{\frac{1}{2}}$$
$$= \sqrt{\lambda(t)}\rho\left(\left\|X-X^{'}\right\|\right)$$
$$= D\left(t, \left\|X-X^{'}\right\|^2\right)^{\frac{1}{2}},$$

therefore $D(t,x) = \lambda(t)\rho^2\left(x^{\frac{1}{2}}\right)$ where $\rho(x) = \left(\sum_{m=1}^{\infty} m^{-2p}\varphi^2(m^q x)\right)^{\frac{1}{2}}$. Notice that ρ is an increasing, continuous and concave function on $\mathbb{R}^+$ for which $\int_{0^+}\frac{x}{\rho^2(x)}dx = \infty$. Hence, by Remark 3.3, equation (1.1) has a unique mild solution in $\mathcal{H}$.

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