Bicomplex Frames

Aiad Elgourari, Allal Ghanmi and Mohammed Souid El Ainin

Sahand Communications in Mathematical Analysis

Print ISSN: 2322-5807
Online ISSN: 2423-3900
Volume: 18
Number: 3
Pages: 69-89

Sahand Commun. Math. Anal.
DOI: 10.22130/scma.2021.140216.875

Sahand Communications in Mathematical Analysis (SCMA) Vol. 18 No. 3 (2021), 69-89
http://scma.maragheh.ac.ir
DOI: 10.22130/scma.2021.140216.875

Bicomplex Frames

Aiad Elgourari[1*], Allal Ghanmi[2] and Mohammed Souid El Ainin[3]

Abstract. We define in a natural way the bicomplex analog of the frames (bc-frames) in the setting of bicomplex infinite Hilbert spaces, and we characterize them in terms of their idempotent components. We also extend some classical results from frames theory to bc-frames and show that some of them do not remain valid for bc-frames in general. The construction of bc-frame operators and Weyl–Heisenberg bc-frames are also discussed.

1. Introduction

Frames are generalizations of orthonormal bases and can be defined as "sets" of vectors (not necessary independent) giving the explicit expansion of any arbitrary vector in the space as a linear combination of the elements in the frame. They were considered by Duffin and Schaeffer, in early fifties, in the framework of nonharmonic Fourier series (see [14]). It is only in 1986 that a landmark development was given by Daubechies, Grossmann and Meyer [11]. Since then, they have been extensively studied in different branches of mathematics and engineering sciences. Frames possess several interesting properties which make them very useful in applications like in signal analysis, especially in connection with sampling theorems [3], image processing, quantum information, filter bank theory, as well as in robust transmission, coding and communications [2, 4, 9, 13, 15, 29]. For further details and tools on frames one can refer to the very nicely written surveys and research tutorials [5, 7, 8, 28].

2010 *Mathematics Subject Classification.* 42C15; 41A58.
Key words and phrases. Bicomplex, bc-frames, bc-frame operator, Weyl-Heisenberg bc-frame.
Received: 21 November 2020, Accepted: 15 May 2021.
* Corresponding author.

This frames theory (for separable Hilbert spaces) had been rapidly generalized to different contexts and has attracted the interest of many authors in the last decades. Notice for instance that a general frame theory for C^*-algebras, Hilbert C^*-modules and countably generated Hilbert C^*-modules was proposed in [22] by Frank and Larson (see also [19–21]). They were able to extend the entire classical frame theory to this generalization undergoing only slight changes but requiring further investigations. See also [6, 22, 23, 27, 30–33, 36, 40] for further generalizations.

In the present paper, we aim to introduce and study bicomplex (bc) frames for infinite bicomplex Hilbert spaces incorporating classical ones for complex Hilbert spaces. The idea of investigating bicomplex frames comes from our recent working on hypercomplex analysis which appears to be consistent and promoter for a number of mathematical problems, especially in signal and image processing. In fact, given an infinite dimensional bicomplex Hilbert space, which was introduced in [24] making use of the idempotent representation, it becomes immediately interesting to extend the notion of frame to this new context and discuss its basic properties. The consideration of the bicomplex setting lies also in the fact that this model can serve to represent color image encoding in image processing [1, 17, 18, 34, 42]. In fact, color image in RGB color is represented using bicomplex numbers by according the red, green and blue components of RGB value as pure bicomplex number

$$f(x,y) = ir(x,y) + jg(x,y) + kb(x,y).$$

The corresponding bicomplex eignaxis represented color and phase, while the modulus is represented by the grayscale image [17, 18]. Another motivation lies in possible interesting extension of almost classical results from signal processing and time-frequency analysis using tools from frame theory to the bicomplex setting. This is reinforced by the existence of many types of modulation and translation operators leading to the consideration and the construction of specific bicomplex analogs of short Fourier (Fourier-Wigner) transform as done in [16]. Example 3.10 and the results we establish in Section 3 will illustrate further the motivation of considering bc-frames. The main feature of bc-frames is Theorem 3.8 concerning the problem of characterizing these bc-frames in terms of the standard ones for appropriate Hilbert spaces. Namely, this result shows that a bicomplex frame is intimately related with a pair of frames arising from idempotent representation. Accordingly, important basic and elementary properties for bc-frames are described forthwith thanks to this special characterization. We establish the mutual relation between bicomplex frames and its components. Several properties from the classical frame theory survive in this generalization. Although this extension is natural, it will be shown that many basic properties for the classical frames fail for bicomplex frames, unless we restrict ourself to complex-valued Hilbert space on bicomplex numbers. Notice for instance that, one of the main surprise results is that an exact bicomplex frame does not follows its components are exact, also, we give a necessary and sufficient condition such that it holds. Special attention will also be given to the problem of defining bicomplex frame operators and bicomplex Weyl–Heisenberg frames.

It should be mentioned here that bicomplex frames for infinite dimensional bicomplex Hilbert spaces are structurally different from frames for Hilbert A-modules, since the positivity is not imposed in the definition of bicomplex scalar product. Moreover, there are many facts making these two generalizations different. One of these facts lies on the classical characterization of standard Riesz bases that are normalized tight frames (i.e., with frame bounds equal to one) as an orthogonormal Hilbert basis, which remains valid for certain Hilbert C^*-modules (see [19, Proposition 2.2] or [22, Corollary 4.2]), fails in the bicomplex extension (see Remark 3.14). However, a rigorous and complete comparison needs further investigations.

This paper is organized as follows. We begin by reviewing the structure of the infinite bicomplex Hilbert space including the Gaussian bicomplex Hilbert space $\mathcal{H}^{2,\sigma}(\mathbb{T})$ (see Section 2). Section 3 is devoted to introducing some basic concepts, the statement and the proof of our main results related to bc-frames. In Section 4, we discuss the associated bicomplex frame operator and Weyl–Heisenberg bc-frames. Final remarks, that can serve as motivations, complete our investigations.

In next sections $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{T}$ will respectively denote the integers, the real, the complex and the bicomplex numbers.

2. Preliminaries: Infinite Bicomplex Hilbert Spaces

This section is a brief review of needed notions and results from the theory of infinite bicomplex Hilbert spaces. For a lot of the material and background we refer the reader to [35, 37–39]. To start, recall first that bicomplex numbers are special generalization of complex numbers. In fact, they are complex numbers $Z = z_1 + jz_2$ with complex coefficients $z_1, z_2 \in \mathbb{C} = \mathbb{C}_i$, where j is a pure imaginary unit independent of i such that $ij = ji$. This defines a commutative (non division) algebra over $\mathbb{C}$, where addition and multiplication operations are defined in a natural way. Conjugates of given $Z = z_1 + jz_2 \in \mathbb{T}$, with respect to i, j and ij, are defined by $\widetilde{Z} = \overline{z_1} + j\overline{z_2}$, $Z^\dagger = z_1 - jz_2$ and $Z^* = \overline{z_1} - j\overline{z_2}$, respectively. It should be mentioned here that the nullity of $ZZ^\dagger = z_1^2 + z_2^2$, which is equivalent to $Z = \lambda(1 \pm ij)$ for certain complex number $\lambda \in \mathbb{C}$, characterizes those that are zero divisors in $\mathbb{T}$, while $ZZ^\dagger \neq 0$ characterizes those that are invertible. Thus by considering the idempotent elements

$$e_+ = \frac{1+ij}{2}, \qquad e_- = \frac{1-ij}{2},$$

we have the identities ${e_+}^2 = e_+$, ${e_-}^2 = e_-$, $e_+ + e_- = 1$, $e_+ - e_- = ij$ and $e_+e_- = 0$. Moreover, for any $Z = z_1 + jz_2 \in \mathbb{T}$ there exist unique complex numbers α, β, such that

$$\begin{aligned} Z &= (z_1 - iz_2)e_+ + (z_1 + iz_2)e_- \\ &= \alpha e_+ + \beta e_-. \end{aligned} \tag{2.1}$$

Here $\alpha = z_1 - iz_2, \beta = z_1 + iz_2 \in \mathbb{C}$. The idempotent representation (2.1) is crucial and simplifies considerably the computation with bicomplex numbers. In particular, the different $Z^\dagger$-, $\widetilde{Z}$- and Z^*-conjugates read simply $Z^\dagger = \beta e_+ + \alpha e_-$, $\widetilde{Z} = \overline{\beta} e_+ + \overline{\alpha} e_-$ and $Z^* = \overline{\alpha} e_+ + \overline{\beta} e_-$.

Infinite bicomplex Hilbert space is defined by means of a special extension of the notions of inner product and norm to the $\mathbb{T}$-modules. More generally, if M is a $\mathbb{T}$-module, we consider the $\mathbb{C}$-vector spaces $V^+ = Me_+$ and $V^- = Me_-$, so that one can see M as the $\mathbb{C}$-vector space $M' = V^+ \oplus V^-$. In general, V^+ and V^- bear no structural similarities. Accordingly, an inner product on M is a given functional $\langle \cdot, \cdot \rangle : M \times M \longrightarrow \mathbb{T}$ satisfying

- (i) $\langle \phi, \tau\psi + \varphi \rangle = \tau^* \langle \phi, \psi \rangle + \langle \phi, \varphi \rangle$ for every $\tau \in \mathbb{T}$ and; $\phi, \psi, \varphi \in M$, and
- (ii) $\langle \phi, \psi \rangle = \langle \psi, \phi \rangle^*$ as well as;
- (iii) $\langle \phi, \phi \rangle = 0$ if and only if $\phi = 0$.

Therefore, the projection $\langle \cdot, \cdot \rangle_{V^\pm}$ of $\langle \cdot, \cdot \rangle$ to $V^\pm$ is a standard scalar product on $V^\pm$. Indeed,

$$\langle \phi, \varphi \rangle = \left\langle \phi^+, \varphi^+ \right\rangle_{V^+} e_+ + \left\langle \phi^-, \varphi^- \right\rangle_{V^-} e_-,$$

where ϕ, φ belong to M identified to $M' = V^+ \oplus V^-$ and $\varphi^\pm := \varphi e_\pm \in V^\pm$ and $\psi^\pm := \psi e_\pm \in V^\pm$. It should be noted here that any $\mathbb{T}$-scalar product on M is completely determined as described above (see [24, Theorem 2.6]).

A bicomplex norm on a $\mathbb{T}$-module M is a map $\|\cdot\| : M \longrightarrow \mathbb{R}$ satisfying

- (i) $\|\cdot\|$ is a norm on the vector space $V^+ \oplus V^-$, and;
- (ii) $\|\lambda\phi\| \leq \sqrt{2}|\lambda| \, \|\phi\|$ for all $\lambda \in \mathbb{T}$ and all $\phi \in M$.

In this case $(M, \|\cdot\|)$ is called a normed bicomplex-module. As for normed $\mathbb{C}$-vector spaces, a bicomplex norm can always be induced from a $\mathbb{T}$-scalar product by considering

$$\begin{aligned} \|\phi\|^2 &= \frac{1}{2}\left(\langle \phi^+, \phi^+\rangle_{V^+} + \langle \phi^-, \phi^-\rangle_{V^-}\right) \\ &= |\langle \phi, \phi\rangle| , \end{aligned} \tag{2.2}$$

where $\phi = \phi^+ + \phi^-$ thanks to the identification of M to $V^+ \oplus V^-$. The modulus $|\cdot|$ denotes the usual Euclidean norm in $\mathbb{R}^4$. The norm in (2.2) obeys a generalized Schwarz inequality ([24, Theorem 3.7])

$$|\langle \phi, \varphi\rangle| \leq \sqrt{2}\, \|\phi\|\, \|\varphi\| .$$

Accordingly, one defines an infinite bicomplex Hilbert space to be a $\mathbb{T}$-inner product space $(M, \langle\cdot,\cdot\rangle)$. This is complete with respect to the induced $\mathbb{T}$-norm (2.2) which is equivalent to $(V^\pm, \langle\cdot,\cdot\rangle_{V^\pm})$ be $\mathbb{C}$-Hilbert spaces. This characterization is contained in Theorems 3.4, 3.5 and Corollary 3.6 of [24]. As an example of infinite bicomplex Hilbert space, one consider the one associated to the trivial bicomplex inner product

$$\begin{aligned} \langle Z, W\rangle_{bc} &= ZW^* \\ &= \alpha\overline{\alpha'}e_+ + \beta\overline{\beta'}e_-, \end{aligned} \tag{2.3}$$

for $Z = \alpha e_+ + \beta e_-$ and $W = \alpha' e_+ + \beta' e_-$ in $\mathbb{T}$, so that the induced bicomplex norm coincides with the usual Euclidean norm in $\mathbb{R}^4$ given by the modulus

$$\begin{aligned} |Z|^2_{bc} &= |z_1|^2 + |z_2|^2 \\ &= \frac{1}{2}\left(|\alpha|^2 + |\beta|^2\right) \\ &= |\langle Z, Z\rangle_{bc}| \end{aligned} \tag{2.4}$$

for given $Z = z_1 + z_2 j = \alpha e_+ + \beta e_-$; $z_1, z_2, \alpha, \beta \in \mathbb{C} =: \mathbb{C}_i$. We next perform the space $\mathcal{H}^{2,\sigma}(\mathbb{T})$ of all $\mathbb{T}$–valued measurable functions f on $\mathbb{T}$ subject to $\|f\|_{bc,\sigma} < +\infty$. Here $\|f\|_{bc,\sigma}$ is the bicomplex norm associated by means of (2.2) to the bicomplex inner product depending in a given fixed given real number $\sigma \geq 0$ and defined through

$$\langle f, g\rangle_{bc,\sigma} := \int_{\mathbb{T}} \langle f(Z), g(Z)\rangle_{bc}\, e^{-\sigma|Z|^2_{bc}} d\lambda(Z),$$

$d\lambda(Z) = d\lambda(z_1)d\lambda(z_2)$ being the Lebesgue measure on $\mathbb{R}^4$. More explicitly, if $L^{2,\sigma}(\mathbb{C}^2)$ denotes the classical Hilbert space of complex-valued square integrable functions in $\mathbb{C}^2$ with respect to the gaussian density $e^{-\sigma(|\xi|^2+|\zeta|^2)}d\lambda(\xi)d\lambda(\zeta)$, then we have

$$\langle f, g\rangle_{bc,\sigma} = \langle f_1, g_1\rangle_{L^{2,\sigma}(\mathbb{C}^2)}\, e_+ + \langle f_2, g_2\rangle_{L^{2,\sigma}(\mathbb{C}^2)}\, e_-, \tag{2.5}$$

for every $f = f_1 e_+ + f_2 e_-, g = g_1 e_+ + g_2 e_-$, where the functions f_k and g_k, for $k = 1, 2$, are seen as $\mathbb{C}$–valued functions on $\mathbb{C}^2$ in the variables

(z_1, z_2); $Z = z_1 + jz_2$. Indeed, this follows using $|Z|_{bc}^2 = |z_1|^2 + |z_2|^2$ in (2.4) as well as

$$\langle f(Z), g(Z)\rangle_{bc} = f_1(z_1, z_2)\overline{g_1(z_1, z_2)}e_+ + f_2(z_1, z_2)\overline{g_2(z_1, z_2)}e_-,$$

which is immediate from the definition of $\langle , \rangle_{bc}$. Thus, the following decomposition

$$\mathcal{H}^{2,\sigma}(\mathbb{T}) = L^{2,\sigma}(\mathbb{C}^2)e_+ + L^{2,\sigma}(\mathbb{C}^2)e_-,$$

holds true thanks to

$$\|f\|_{bc,\sigma}^2 = \frac{1}{2}\left(\|f_1\|_{L^{2,\sigma}(\mathbb{C}^2)}^2 + \|f_2\|_{L^{2,\sigma}(\mathbb{C}^2)}^2\right). \tag{2.6}$$

Another interesting decomposition of the infinite bicomplex Hilbert space $\mathcal{H}^{2,\sigma}(\mathbb{T})$ with respect to the idempotent representation states that ([25])

$$\mathcal{H}^{2,\sigma}(\mathbb{T}) = L^{2,\frac{\sigma}{2}}(\mathbb{C}^2)e_+ + L^{2,\frac{\sigma}{2}}(\mathbb{C}^2)e_-.$$

Succinctly, for every $f \in \mathcal{H}^{2,\sigma}(\mathbb{T})$, there exist $\phi^{\pm} \in L^{2,\frac{\sigma}{2}}(\mathbb{C}^2)$ such that

$$f(\alpha e_+ + \beta e_-) = \phi^+(\alpha, \beta)e_+ + \phi^-(\alpha, \beta)e_-,$$

and (2.6) implies

$$\|f\|_{bc,\sigma}^2 = \frac{1}{2}\left(\left\|\phi^+\right\|_{L^{2,\frac{\sigma}{2}}(\mathbb{C}^2)}^2 + \left\|\phi^-\right\|_{L^{2,\frac{\sigma}{2}}(\mathbb{C}^2)}^2\right).$$

In the sequel, we will denote $\mathcal{H}^{2,\sigma}(\mathbb{T})$ simply by $L^2(\mathbb{T})$ when $\sigma = 0$.

3. Bicomplex Frames

In the sequel, $\mathcal{H}_{bc}$ will denote an arbitrary separable bicomplex Hilbert space with bicomplex inner product $\langle \cdot, \rangle_{bc}$, linear in the first entry, and denotes by $\|\cdot\|_{bc}$ the associated bicomplex norm, $\|\cdot\|_{bc}^2 = |\langle \cdot, \cdot\rangle_{bc}|_{bc}$. Let $\{f_n, n = 0, 1, 2, \ldots\}$ be a countable family in $\mathcal{H}_{bc}$. The different notions from the classical frame theory can be extended, in a natural way, to the bicomplex setting.

Definition 3.1. The sequence $(f_n)_n$ is said to be a bicomplex basis for $\mathcal{H}_{bc}$ if for every $f \in \mathcal{H}_{bc}$ there exists a unique sequence of bicomplex numbers $(c_n)_n$ such that

$$f = \sum_{n=0}^{\infty} c_n f_n.$$

It is said to be a bicomplex orthonormal basis if in addition $(f_n)_n$ is an orthonormal set,

$$\langle f_m, f_n\rangle_{bc} = \delta_{m,n},$$

where $\delta_{m,n} = \delta_{m,n}e_+ + \delta_{m,n}e_-$ denotes the Kronecker symbol.

Definition 3.2. A basis $(f_n)_n$ is said to be a bicomplex bounded basis if it satisfies the condition

$$\begin{aligned} 0 &< \inf |\langle f_n, f_n\rangle_{bc}|_{bc} \\ &\leq \sup |\langle f_n, f_n\rangle_{bc}|_{bc} \\ &< +\infty. \end{aligned}$$

It is unconditional if for every $f \in \mathcal{H}_{bc}$, the corresponding series $f = \sum_n c_n f_n$; $c_n \in \mathbb{T}$, converges for every rearrangement of its terms.

Proposition 3.3. *If $(f_n)_n$; $f_n = f_n^+ e_+ + f_n^- e_-$, is a bicomplex bounded unconditional basis for $L^2(\mathbb{T})$, then $(f_n^+)_n$ or $(f_n^-)_n$ is a bounded unconditional basis for $L^2(\mathbb{C}^2)$.*

Proof. Using the idempotent decomposition, the bc-basis property, and the bc-unconditionality of $(f_n)_n$ in $L^2(\mathbb{T})$ are clearly equivalent to that $(f_n^+)_n$ and $(f_n^-)_n$ being unconditional bases for $L^2(\mathbb{C}^2)$. We need only to prove the boundedness property. Indeed, starting from (2.6), we obtain

$$\sup |\langle f_n, f_n\rangle_{bc}|_{bc} \leq \frac{1}{\sqrt{2}}\left(\sup \left\|f_n^+\right\| + \sup \left\|f_n^-\right\|\right).$$

Therefore, $\sup |\langle f_n, f_n\rangle_{bc}|_{bc}$ is finite if and only if $\sup \|f_n^+\|$ and $\sup \|f_n^-\|$ are finite. In a similar way, we can prove the following

$$\frac{1}{\sqrt{2}} \sup \left(\inf \left\|f_n^+\right\|, \inf \left\|f_n^-\right\|\right) \leq \inf |\langle f_n, f_n\rangle_{bc}|_{bc}.$$

This proves $\inf |\langle f_n, f_n\rangle|_{bc}$ is positive if and only if $\inf \|f_n^+\|$ or $\inf \|f_n^-\|$ is positive. This completes the proof. □

According to the previous proof, it is evident to see that the converse of Proposition 3.3 holds true if the idempotent components $(f_n^+)_n$ and $(f_n^-)_n$ are both assumed bounded unconditional bases for $L^2(\mathbb{C}^2)$. Moreover, this shows that from two collections in $L^2(\mathbb{C}^2)$ such that the first one is bounded unconditional basis for $L^2(\mathbb{C}^2)$ and the second one is an unconditional basis which is upper bonded, we can perform a bicomplex bounded unconditional basis for $L^2(T)$. We reformulate this as proposition.

Proposition 3.4. *If $(f_n^+)_n$ is a bounded unconditional basis for $L^2(\mathbb{C}^2)$ and the sequence $(f_n^-)_n$ is an unconditional basis for $L^2(\mathbb{C}^2)$ with upper bounded bc-norm, then $(f_n)_n$ is a bicomplex bounded unconditional basis for $L^2(\mathbb{T})$.*

In analogy with the standard case, we propose the following definitions for bicomplex Riesz (bc-Riesz) bases and bicomplex frames (bc-frames).

Definition 3.5. A sequence $(f_n)_n$ is a bicomplex Riesz basis for $\mathcal{H}_{bc}$ if $\overline{span\{f_n; n\}} = \mathcal{H}_{bc}$ and there exist $A, B > 0$ such that

$$A\sum_{n=0}^{\infty}|c_n|^2_{bc} \leq \left\|\sum_{n=0}^{\infty}c_nf_n\right\|^2_{bc} \leq B\sum_{n=0}^{\infty}|c_n|^2_{bc},$$

for all $f = \sum_{n} c_nf_n \in \mathcal{H}_{bc}$.

Definition 3.6. The sequence $(f_n)_n$ is called a bc-frame for $\mathcal{H}_{bc}$ if there are two constants $0 < A \leq B$ (frame bounds) such that for every $f \in H$, we have

$$A\left\|f\right\|^2_{bc} \leq \sum_{n=0}^{\infty}|\left\langle f, f_n\right\rangle_{bc}|^2_{bc} \leq B\left\|f\right\|^2_{bc}.$$

A bc-frame is said to be tight if in addition $A = B$ and a Parseval bc-frame if $A = B = 1$. It is called exact if it ceases to be a bc-frame whenever any single element is deleted from it.

The following example is specific for the bc-frames and is generated by a given orthonormal basis e_n of $L^2(\mathbb{C}^2)$.

Example 3.7. Let $(c_n)_n$ be a bicomplex sequence such that $c_n = a_ne_+ + b_ne_-$; $a_n, b_n \in \mathbb{C}$, and consider the set of bicomplex-valued functions

$$e_{m,n}(z_1 + jz_2) := a_ne_m(z_1, z_2)e_+ + b_me_n(z_1, z_2)e_-.$$

If $\sum\limits_{n=0}^{\infty}|a_n|^2 = a$ and $\sum\limits_{n=0}^{\infty}|b_n|^2 = b$ converge, then $e_{m,n}$, for varying m and n, is a bc-frame for $L^2(\mathbb{T})$. Indeed, by means of (2.5), we get $\left\langle e_{m,n}, e_{j,k}\right\rangle_{bc} = a_n\overline{a_k}\delta_{m,j}e_+ + b_m\overline{b_j}\delta_{n,k}e_-$. This shows that $e_{m,n}$ is not orthogonal nor necessary normalized. However, direct computation shows that

$$\sum_{m,n=0}^{\infty}|\left\langle f, e_{m,n}\right\rangle_{bc}|^2_{bc} = \frac{1}{2}\left(a\left\|f^+\right\|^2_{L^2(\mathbb{C}^2)} + b\left\|f^-\right\|^2_{L^2(\mathbb{C}^2)}\right).$$

Therefore,

$$\min(a,b)\left\|f\right\|^2_{bc} \leq \sum_{m,n=0}^{\infty}|\left\langle f, e_{m,n}\right\rangle_{bc}|^2_{bc} \leq \max(a,b)\left\|f\right\|^2_{bc}.$$

By specifying $(a_n)_n$ and $(b_n)_n$, we get tight and Parseval bc-frames.

The next result is a fundamental tool in our expository and characterizes the bc-frames in terms of the classical ones via its idempotent components.

Theorem 3.8. *The sequence* $(f_n)_n$ *is a bc-frame for* $L^2(\mathbb{T})$ *with best frame bounds* A *and* B *if and only if their components* $(f_n^{\pm})_n$*;* $f_n = f_n^+ e_+ + f_n^- e_-$*, are frames for* $L^2(\mathbb{C}^2)$ *with best frame bounds* a^+, b^+ *and* a^-, b^-*. Moreover, we have* $A = \min\{a^+, a^-\}$ *and* $B = \max\{b^+, b^-\}$*.*

Proof. Let $f_n, f \in L^2(\mathbb{T})$ and assume that $(f_n)_n$ is a bc-frame. By writing them as $f_n = f_n^+ e_+ + f_n^- e_-$ and $f = f^+ e_+ + f^- e_- \in L^2(\mathbb{T})$ with $f^+, f^-, f_n^+, f_n^- \in L^2(\mathbb{C}^2)$, and making use of (2.6) and (2.3), the condition

$$A \|f\|_{bc}^2 \leq \sum_{n=0}^{\infty} |\langle f, f_n \rangle_{bc}|_{bc}^2 \leq B \|f\|_{bc}^2 ,$$

becomes equivalent to

$$\begin{aligned} A &\left(\|f^+\|_{L^2(\mathbb{C}^2)}^2 + \|f^-\|_{L^2(\mathbb{C}^2)}^2 \right) \\ &\leq \sum_{n=0}^{\infty} \left(\left| \langle f^+, f_n^+ \rangle_{L^2(\mathbb{C}^2)} \right|^2 + \left| \langle f^-, f_n^- \rangle_{L^2(\mathbb{C}^2)} \right|^2 \right) \\ &\leq B \left(\|f^+\|_{L^2(\mathbb{C}^2)}^2 + \|f^-\|_{L^2(\mathbb{C}^2)}^2 \right), \end{aligned} \tag{3.1}$$

for every $f^+, f^- \in L^2(\mathbb{C}^2)$. Accordingly, if $(f_n)_n$ is a bc-frame for $L^2(\mathbb{T})$ with best frame bounds A and B then the component sequence $(f_n^+)_n$ (resp. $(f_n^-)_n$) is a frame for $L^2(\mathbb{C}^2)$ by taking $f^- = 0$ (resp. $f^+ = 0$) in (3.1). Their best frame bounds a^+, b^+ (resp. a^-, b^-) satisfy $A \leq \min\{a^+, a^-\}$ and $B \geq \max\{b^+, b^-\}$. Conversely, if $(f_n^+)_n$ and $(f_n^-)_n$ are frames for $L^2(\mathbb{C}^2)$ with best frame bounds a^+, b^+ and a^-, b^-, respectively, then $(f_n)_n$ is a bc-frame for $L^2(\mathbb{T})$ with best frame bounds A and B satisfying $A \geq \min\{a^+, a^-\}$ and $B \leq \max\{b^+, b^-\}$. This completes the proof of that $(f_n)_n$ is a bc-frame for $L^2(\mathbb{T})$ if and only $(f_n^+)_n$ and $(f_n^-)_n$ are frames for $L^2(\mathbb{C}^2)$ with $A = \min\{a^+, a^-\}$ and $B = \max\{b^+, b^-\}$. □

Remark 3.9. The assertion of Proposition 3.3 and Theorem 3.8 remain valid for bc-frames $(f_n)_n$ for general bicomplex Hilbert space $\mathcal{H}_{bc} = \mathcal{H}^+ e_+ + \mathcal{H}^- e_-$, with $f = f^+ e_+ + f^- e_-$ and $f^{\pm} \in \mathcal{H}^{\pm}$.

Example 3.10. Let $(c_n)_n$ be a bicomplex sequence as in Example 3.7, and $(f_n)_n$, with $f_n = f_n^+ e_+ + f_n^- e_-$, be a bc-frame for $L^2(\mathbb{T})$. Then, the set of bicomplex-valued functions

$$f_{m,n}(z_1 + j z_2) := a_n f_m^+(z_1, z_2) e_+ + b_m f_n^-(z_1, z_2) e_-,$$

defines a new bc-frame for $L^2(\mathbb{T})$. This immediately follows from Theorem 3.8 and generalizes Example 3.7.

Remark 3.11. Theorem 3.8 and Example 3.10 show why bc-frames may be of interest and that they contain the classical ones as particular subclasses. In fact, $L^2(\mathbb{C}^2)$ can be embedded in a natural way in $L^2(\mathbb{T})$, since any frame $(h_n)_n$ for $L^2(\mathbb{C}^2)$ can be seen as a bc-frame for $L^2(\mathbb{T})$ by considering $h_n e_+ + h_n e_- = h_n$.

According to Theorem 3.8, a number of important properties for bc-frames are described forthwith thanks to the previous characterization. For example, we have the following characterization of completeness of bc-frame $(f_n)_n$.

Proposition 3.12. *The bc-frame $(f_n)_n$ is complete in $L^2(\mathbb{T})$ if and only if $(f_n^+)_n$ and $(f_n^-)_n$ are both complete in $L^2(\mathbb{C}^2)$. The same observation holds true for the Parseval bc-frames.*

Moreover, it is well known that classical Riesz basis can be characterized as the data of a sequence $(f_n)_n$ in $\mathcal{H}_{bc}$ which is the image of an orthonormal basis under a bounded invertible linear operator. We claim that this remains valid for bicomplex Riesz basis. This readily follows from the following fact (whose proof is similar to the one provided to Theorem 3.8).

Proposition 3.13. *The sequence $(f_n)_n$ is a bc-Riesz basis if and only the idempotent components $(f_n^{\pm})_n$ are Riesz bases.*

However, one has to be careful as shown in the sequel.

Remark 3.14. bc-Riesz bases with frame bounds equal to one are not necessary orthonormal bases. Indeed, given a bc-Riesz basis $(f_n)_n$ for $L^2(\mathbb{T})$ with frame bounds $A = B = 1$ is equivalent the idempotent components $(f_n^+)_n$ and $(f_n^-)_n$ be Riesz bases for $L^2(\mathbb{C}^2)$ with frame bounds a^+, b^+ and a^-, b^-, respectively, and such that $A = \min(a^+, a^-) = 1$ and $B = \max(b^+, b^-) = 1$, according to the proof of Theorem 3.8 adapted for Riesz bases. This is clearly not equivalent to $(f_n^+)_n$ and $(f_n^-)_n$ being Riesz bases with frame bounds $a^+ = b^+ = 1$ and $a^- = b^- = 1$, respectively, which characterizes orthonormal bases in $L^2(\mathbb{T})$.

Proposition 3.15. *If $(f_n)_n$ is a tight bc-frame for $L^2(\mathbb{T})$, then $(f_n^+)_n$ and $(f_n^-)_n$ are tight frames for $L^2(\mathbb{C}^2)$.*

Proof. The proof is straightforward using the direct implication in Theorem 3.8. □

Remark 3.16. The converse is not in general true unless $(f_n^+)_n$ and $(f_n^-)_n$ are tight frames with the same best frame bounds.

Proposition 3.17. *Let $(f_n)_n$ be a bc-frame for $L^2(\mathbb{T})$ and assume that $(f_n^+)_n$ (or $(f_n^-)_n$) is an exact frame for $L^2(\mathbb{C}^2)$. Then $(f_n)_n$ is an exact bc-frame for $L^2(\mathbb{T})$.*

Proof. The non-exactness of $(f_n)_n$ is equivalent to the existence of some u such that $(f_n)_{n\neq u}$ is still a bc-frame. By Theorem 3.8, this is equivalent to the sets $(f_n^+)_{n\neq u}$ and $(f_n^-)_{n\neq u}$ be frames for $L^2(\mathbb{C}^2)$. This means that $(f_n^+)_n$ and $(f_n^-)_n$ are both not exact (at least at u). This completes the proof. □

From this, there is no reason to have the converse of Proposition 3.17 "$(f_n)_n$ is exact for $L^2(\mathbb{T})$ implies $(f_n^+)_n$ or $(f_n^-)_n$ is an exact frame for $L^2(\mathbb{C}^2)$". This is not true in general as shown by the following counterexample.

Example 3.18 (Counterexample)**.** Let $(\varphi_n)_n$ and $(\psi_n)_n$ be two frames for $L^2(\mathbb{C}^2)$ such that $(\varphi_n)_{n\neq u}$ and $(\psi_n)_{n\neq v}$ are exact with $u \neq v$. We assume that u (resp. v) is the only value satisfying this property. Example of such frame exists and one can consider $(\varphi_n)_n = \{e_0\} \cup (e_n)_{n\geq 0}$, where $(e_n)_n$ is an orthonormal basis. We next perform $f_n = \varphi_n e_+ + \psi_n e_-$ which clearly is a bc-frame for $L^2(\mathbb{T})$ (by Theorem 3.8). Moreover, it is exact, i.e., $(f_n)_{n\neq m}$ ceases to be a bc-frame for any arbitrary m. To this end, notice that we necessary have $m \neq u$ or $m \neq v$. For the first case for example ($m \neq u$), the first component $(f_n^+)_{n\neq m} = (\varphi_n)_{n\neq m}$ is not a frame for $L^2(\mathbb{C}^2)$ by assumption ($(\varphi_n)_n$ is exact at u only), and therefore one concludes for $(f_n)_{n\neq m}$ by making again use of Theorem 3.8.

If we denote by $N_{Exact}((f_n)_n)$ the set of indices k for which $(f_n)_{n\neq k}$ remains a frame,

$$N_{Exact}((f_n)_n) = \{k;\, (f_n)_{n\neq k} \text{ is a frame}\},$$

then the exactness of $(f_n)_n$ becomes equivalent to $N_{Exact}((f_n)_n) = \emptyset$. Using this equivalent definition of exactness, Proposition 3.17 appears as a particular case of the following result.

Theorem 3.19. *A bc-frame $(f_n)_n$ for $L^2(\mathbb{T})$ is exact if and only if*

$$N_{Exact}\left((f_n^+)_n\right) \cap N_{Exact}\left((f_n^-)_n\right) = \emptyset.$$

Proof. By means of Theorem 3.8 and the definition of $N_{Exact}((f_n)_n)$, it is clear that

$$N_{Exact}((f_n)_n) = N_{Exact}((f_n^+)_n) \cap N_{Exact}((f_n^-)_n), \tag{3.2}$$

holds true, where $f_n = f_n^+ e_+ + f_n^- e_-$. Therefore, one concludes for Theorem 3.19, since exactness of a given collection in $L^2(\mathbb{T})$ is equivalent to the triviality of the set N_{Exact}. □

Remark 3.20. The assertion in the counterexample 3.18 can be reproved easily making use of $N_{Exact}((f_n)_n)$. Indeed, since $N_{Exact}((\varphi_n)_n) = \{u\}$ and $N_{Exact}((\psi_n)_n) = \{v\}$, we get $N_{Exact}((\varphi_n e_+ + \psi_n e_-)_n) = \emptyset$ by means of (3.2), and therefore $\varphi_n e_+ + \psi_n e_-$ is an exact bc-frame for $L^2(\mathbb{T})$.

Accordingly, one proves that the well-known fact that "a frame for $L^2_{\mathbb{C}}(\mathbb{R}^d)$ is exact if and only if it is a Riesz basis (see e.g. [8, Theorem 7.1.1, p. 166])" is no longer valid for bc-frames. However, we assert the following

Proposition 3.21. *If $(f_n)_n$ is a bc-Riesz basis for $L^2(\mathbb{T})$, then $(f_n)_n$ is an exact bc-frame for $L^2(\mathbb{T})$.*

Proof. This is immediate making use of Proposition 3.13 asserting that $(f_n)_n$ is a Riesz basis for $L^2(\mathbb{T})$ if and only if the component sequences $(f_n^{\pm})_n$ are Riesz bases for $L^2(\mathbb{C}^2)$ combined with that the exactness of ordinary frames for $L^2(\mathbb{R}^d)$ is equivalent to being Riesz bases [8, Theorem 7.1.1, p. 166] and Theorem 3.8. □

For a complex Hilbert space $\mathcal{H}$, it is a known fact that bounded unconditional bases $(\psi_n)_n$ can be characterized as orthonormal bases $(e_n)_n$ via a bounded invertible operator $U : \mathcal{H} \longrightarrow \mathcal{H}$ such that $\psi_n = Ue_n$ for each n. This result remains valid for bc-frames thanks to Proposition 3.3. The close relation to exact frames is also provided.

Theorem 3.22 ([28, 41])**.** *A frame $(f_n)_n$ for a complex Hilbert space $\mathcal{H}$ is exact if and only if it is a bounded unconditional basis.*

For bc-frames, we assert the following

Proposition 3.23. *If $(f_n)_n$ is a bicomplex bounded unconditional basis for $\mathcal{H}_{bc}$, then $(f_n)_n$ is exact for $\mathcal{H}_{bc}$.*

Proof. This follows from Theorem 3.22 combined with Propositions 3.3 and 3.17. □

4. bc-frame Operator and Weyl-Heisenberg bc-frames

4.1. **bc-frame Operator.** Let $(\mathcal{H}_{bc}, \langle , \rangle_{bc})$ be a functional bc-Hilbert space on $\mathbb{T}$ endowed with the bc-scalar product

$$\langle f, g\rangle_{bc} = \int_{\mathbb{T}} f(Z)[g(Z)]^* d\lambda(Z).$$

Thus, we decompose $\mathcal{H}_{bc}$ idempotentically as $\mathcal{H}_{bc} = \mathcal{H}^+ e_+ + \mathcal{H}^- e_-$ where $\mathcal{H}^{\pm}$ are $\mathbb{C}$-Hilbert spaces on $\mathbb{C}$. For given bc-bicomplex frame $(f_n)_n$ for $\mathcal{H}_{bc}$, the components $(f_n^{\pm})_n$ are ordinary frames for $\mathcal{H}^{\pm}$. Therefore, one may define the analysis operators $T_{\pm} : \mathcal{H}^{\pm} \longrightarrow \ell^2_{\mathbb{C}}$ and their adjoint $T^{adj}_{\pm} : \ell^2_{\mathbb{C}} \longrightarrow \mathcal{H}^{\pm}$ by

$$T_{\pm} f^{\pm} = \left(\langle f^{\pm}, f_n^{\pm}\rangle\right)_n, \qquad T^{adj}_{\pm}((c_n)_n) = \sum_{n=0}^{\infty} c_n f_n^{\pm}.$$

We can define the bc-analysis operator $T : \mathcal{H}_{bc} \longrightarrow \ell^2_{\mathbb{T}}$ for the bicomplex Hilbert space $\mathcal{H}_{bc}$ by

$$\begin{aligned} Tf &:= (T_+ e_+ + T_- e_-) f \\ &= T_+ f^+ e_+ + T_- f^- e_- \\ &= (\langle f, f_n \rangle_{bc})_n, \end{aligned}$$

with $f = f^+ e_+ + f^- e_- \in \mathcal{H}_{bc}$. Its adjoint $T^{adj}_{bc} : \ell_{\mathbb{T}} \longrightarrow \mathcal{H}_{bc}$ with respect to bicomplex hilbertian structure is shown to be given by

$$T^{adj}_{bc}((c_n)_n) = T^{adj}_{+}((c^+_n)_n) e_+ + T^{adj}_{-}((c^-_n)_n) e_-,$$

for given $c_n = c^+_n e_+ + c^-_n e_- \in \mathbb{T}$. Therefore, we define the bc-frame operator $S : \mathcal{H}_{bc} \longrightarrow \mathcal{H}_{bc}$ to be

$$\begin{aligned} Sf &= (T_+ e_+ + T_- e_-)^{adj}_{bc} (T_+ e_+ + T_- e_-) \\ &= S_+ f^+ e_+ + S_- f^- e_-, \end{aligned}$$

where $S_\pm = T^{adj}_{\pm} T_\pm$ are the classical frame operators associated to $(f^\pm_n)_n$ for $\mathcal{H}^\pm$. By construction, the operator S inherits from $S_\pm$ their basic properties. Notice for instance that we have following

$$\begin{aligned} Sf &= \sum_{n=0}^{\infty} \langle f, f_n \rangle_{bc} f_n, \\ \langle Sf, f \rangle_{bc} &= \sum_{n=0}^{\infty} \langle \langle f, f_n \rangle_{bc}, \langle f_n, f \rangle_{bc} \rangle_{bc}, \\ f &= \sum_{n=0}^{\infty} \left\langle f, S^{-1} f_n \right\rangle_{bc} f_n. \end{aligned}$$

Moreover, S is clearly invertible, self-adjoint $\langle Sf, g \rangle_{bc} = \langle f, Sg \rangle_{bc}$ and bounded operator. Its norm satisfies the following estimation

$$\begin{aligned} \|S\|^2_{op} &\leq \max \left(\|S_+\|^2_{op}, \|S_-\|^2_{op} \right) \\ &= \max \left({b^+}^2_{opt}, {b^-}_{opt} \right). \end{aligned}$$

Moreover, S is hyperbolicpositive in the sense that $\langle Sf, f \rangle_{bc} \in \mathbb{D}^+$ for every $f \in \mathcal{H}_{bc}$, where $\mathbb{D}^+ = \mathbb{R}^+ e_+ + \mathbb{R}^+ e_-$ denotes the set of positive hyperbolic numbers $\mathbb{D}^+$.

Although, the frame operator is naturally extended to the bicomplex context, we will be careful when examining their properties. This is closely connected to materials discussed in the previous section. In fact, from Proposition 3.15, we know that tightness of a bc-frame $(f_n)_n$ for $\mathcal{H}_{bc}$ implies that the frames $(f^\pm_n)_n$ are tight for $\mathcal{H}^\pm$. Thus, by means of

[26, Proposition 5.1.1., p. 86], we have $S_\pm = a^\pm_{opt} Id_{\mathcal{H}^\pm}$; $a^\pm_{opt} = b^\pm_{opt} \in \mathbb{R}^+$, and therefore

$$\begin{aligned} S &= a^+_{opt} Id_{H^+} e_+ + a^-_{opt} Id_{H^-} e_+ \\ &= \left(a^+_{opt} e_+ + a^-_{opt} e_+\right) Id_{\mathcal{H}_{bc}}. \end{aligned}$$

This proves the following assertion

Proposition 4.1. *If $(f_n)_n$ is a tight bc-frame for $\mathcal{H}_{bc}$, then $S = hId_{\mathcal{H}_{bc}}$ for certain positive hyperbolic number $h \in \mathbb{D}^+$.*

Remark 4.2. The converse in not valid in general unless we assume that $h \in \mathbb{R}^+$ (i.e., $a^+_{opt} = a^-_{opt}$). Thus for complex-valued bc-frame, we recover the classical result characterizing tight frames as the operator $hId_{\mathcal{H}_{bc}}$.

4.2. **Weyl-Heisenberg bc-frames.** The so-called Weyl–Heisenberg (or Gabor) frames are the famous frames for $L^2(\mathbb{R})$ that can be generated from a single element (called mother wavelet). More exactly, they are frames of functions

$$\mathcal{G}(a,b,g) := \left\{ W_{na,mb}(g)(t) = e^{imbt} g(t-na);\ m,n \in \mathbb{Z} \right\},$$

where $a, b > 0$ and $g \in L^2(\mathbb{R})$ are fixed, and $W_{a,b}$ denotes the Weyl operator

$$\begin{aligned} W_{a,b}(g)(t) &:= e^{ibt} g(t-a) \\ &= M_b T_a g(t), \end{aligned}$$

where M_b and T_a denote the classical modulation and translation operators defined by $M_b g(t) := e^{ibt} g(t)$ and $T_a g(t) := g(t-a)$, respectively. The frameness of Weyl–Heisenberg systems $\mathcal{G}(a,b,g)$ in $L^2_{\mathbb{C}}(\mathbb{R})$ has been extensively discussed in several papers. See for instance [8, 10, 12, 26, 28]. The next result is an example of assertions providing us with sufficient conditions on g and the lattice parameters a and b to $\mathcal{G}(a,b,g)$ be a frame. Namely,

Theorem 4.3 ([28])**.** *Let g be a compactly supported function with support contained in some interval I of length $1/b$ and such that*

$$\alpha \leq \sum_{n=0}^{\infty} |T_{na}(g)(t)|^2 \leq \beta,$$

almost everywhere on $\mathbb{R}$ for some constants $\alpha, \beta > 0$. Then, the Weyl–Heisenberg system $\mathcal{G}(a,b,g)$ is a frame for $L^2_{\mathbb{C}}(\mathbb{R})$ with frame bounds α/b and β/b.

The introduction of bicomplex analog of Weyl–Heisenberg (W-H) systems can be accomplished in different ways. In the sequel, we confine our attention to the natural ones. By considering two classical W-H systems $\mathcal{G}(a,b,g)$ and $\mathcal{G}(c,d,h)$ for $L^2_{\mathbb{C}}(\mathbb{R})$, we perform the following

$$\mathcal{G}^{\nu,\mu}(\Gamma_{hyp}(A,B),f) := \left\{ W^{\nu,\mu}_{nA,mB} f; m,n \in \mathbb{Z} \right\}, \tag{4.1}$$

associated to $f = ge_+ + he_-$, the hyperbolic lattice $\Gamma_{hyp}(A,B) = \mathbb{Z}A + \mathbb{Z}B$; $A = ae_+ + ce_-, B = be_+ + de_- \in \mathbb{D}^+$ and the modified Weyl operator [16]

$$W^{\nu,\mu}_{nA,mB} f(t) := \left[W^{\nu}_{na,mb} g \right](t) e_+ + \left[W^{\mu}_{nc,md} h \right](t) e_-,$$

defined as projective representation of two copies of $W^{\nu}_{a,b} g(t) = e^{\nu bt} g(t-a)$. Here $\nu^2 = \mu^2 = -1$. Clearly $W^{\nu,\mu}_{nA,mB}$ belongs to the Hilbert space

$$L^2_{\mathbb{T}}(\mathbb{R}) := \left\{ \varphi : \mathbb{R} \longrightarrow \mathbb{T}; \|\varphi\|^2_{bc} := \left| \int_{\mathbb{R}} |\varphi(t)(\varphi(t))^* dt \right|_{bc} < +\infty \right\},$$

of all bicomplex-valued functions on $\mathbb{R}$ with finite norm, where the norm is the one induced from the bicomplex inner product

$$\langle \varphi, \phi \rangle_{bc} := \int_{\mathbb{R}} \langle \varphi(t), \phi(t) \rangle_{bc} \, dt, \tag{4.2}$$

through (2.2). By taking $a = b$, $c = d$ and $g = h$ we recover the classical notion of W-H system.

Definition 4.4. The system $\mathcal{G}^{\nu,\mu}(\Gamma_{hyp}(A,B),f)$ in (4.1) is called bicomplex W-H system for the Hilbert space $L^2_{\mathbb{T}}(\mathbb{R})$.

Therefore, by means of Theorems 3.8 and 4.3, we deduce easily the following result for W-H bc-systems in the bicomplex Hilbert space $L^2_{\mathbb{T}}(\mathbb{R})$.

Proposition 4.5. *$\mathcal{G}^{\nu,\mu}(\Gamma_{hyp}(A,B),f)$ generates a bc-frame for $L^2_{\mathbb{T}}(\mathbb{R})$ under the assumption that f are compactly supported with support contained in some interval I of length* $\min(1/b, 1/d)$ *and*

$$\alpha' \leq \sum_{n=0}^{\infty} |(T_{na}e_+ + T_{nc}e_-)(f)(t)|^2_{bc} \leq \beta',$$

almost everywhere on $\mathbb{R}$ for certain constants α' and β'.

Thus, the existence of W-H bc-frames for $L^2_{\mathbb{T}}(\mathbb{R})$ requires in particular $ab \leq 1$ or $cd \leq 1$ by Theorem 3.8 and [26, Corollary 7.5.1., p. 138]. Moreover, the most properties valid for complex W-H frames can be established easily for the W-H bc-frames. Notice then even the critical case $ab = 1$ characterizes the exact W-H frame for $L^2(\mathbb{R})$, the exactness of $\mathcal{G}^{\nu,\mu}(\Gamma_{hyp}(A,B),f)$ is not characterized by $AB = 1$ in $\mathbb{D}^+$, i.e., $ab = 1$

and $cd = 1$. Namely, the following assertion readily follows using [26, Corollary 7.5.2. p. 139] and Proposition 3.17

Proposition 4.6. *If $\mathcal{G}^{\nu,\mu}(\Gamma_{hyp}(A,B), f)$ is a Weyl–Heisenberg bc-frame for $L^2_{\mathbb{T}}(\mathbb{R})$ and $ab = 1$ or $cd = 1$, then it is exact.*

Remark 4.7. The converse of Proposition 4.6 is not true in general.

The previous formalism for constructing W-H bc-frames for $L^2_{\mathbb{T}}(\mathbb{R})$ can be extended (in a similar way), thanks to Theorem 3.8, to the Hilbert space $L^2_{\mathbb{T}}(\mathbb{D})$ on hyperbolic numbers by considering the family of functions

$$\left[W^{\nu}_{na,mb}\varphi\right](xe_+ + ye_-)e_+ + \left[W^{\mu}_{pc,qd}\psi\right](xe_+ + ye_-),$$

for varying 4-uplet $M = (m,n,p,q) \in \mathbb{Z}^4$. Here $a,b,c,d > 0$ are fixed reals and φ, ψ are fixed complex-valued functions in $L^2_{\mathbb{C}}(\mathbb{R})$. The following result shows that it is not true in general that from a given W-H frames for $L^2_{\mathbb{C}}(\mathbb{R})$, one can generate W-H bc-frames for $L^2_{\mathbb{T}}(\mathbb{D})$ by means of Theorem 3.8. In fact, for fixed $g, h \in L^2_{\mathbb{C}}(\mathbb{R})$, we define

(4.3)
$$\psi^{\nu,\mu}_M(xe_+ + ye_-) := e^{-y^2/2}\left[W^{\nu}_{na,mb}g\right](x)e_+ + e^{-x^2/2}\left[W^{\mu}_{nc,md}h\right](y).$$

Thus, we can prove the following

Proposition 4.8. *Assume $\mathcal{G}(a,b,g)$ and $\mathcal{G}(c,d,h)$ are frames for $L^2_{\mathbb{C}}(\mathbb{R})$. Then, the bicomplex W-H system $\psi^{\nu,\mu}_M$ in (4.3) is not a bc-frame for $L^2_{\mathbb{T}}(\mathbb{D})$.*

Proof. By means of Theorem 3.8 the frameness of $\psi^{\nu,\mu}_M$, in $L^2_{\mathbb{T}}(\mathbb{D})$, is equivalent to the frameness of their components $e^{-y^2/2}[W^{\nu}_{na,mb}g](x)$ and $e^{-x^2/2}[W^{\mu}_{nc,md}h](y)$ in $L^2_{\mathbb{C}}(\mathbb{D})$. It should be noticed here that the tensor product $h_u \oplus h_v(x,y) = h_u(x)h_v(y)$ of Hermite functions is an orthogonal basis of

$$L^2_{\mathbb{C}}(\mathbb{D}) = \left\{\psi : \mathbb{D} \longrightarrow \mathbb{C};\ \int_{\mathbb{D}} \psi(\xi)\overline{\psi(\xi)}d\lambda(\xi) < +\infty\right\},$$

and that direct computation using Fubini's theorem shows that we have

$$\sum_{m,n\in\mathbb{Z}} \left|\left\langle h_u \otimes h_v, W^{\nu}_{na,mb}g \otimes h_0\right\rangle_{L^2_{\mathbb{C}}(\mathbb{D})}\right|^2 = \pi \sum_{m,n\in\mathbb{Z}} \left|\left\langle h_u, W^{\nu}_{na,mb}g\right\rangle_{L^2_{\mathbb{C}}(\mathbb{R})}\right|^2 \delta_{v,0}.$$

This shows in particular that the considered sequence $W^{\nu}_{na,mb}g \otimes h_0$ is not a frame in $L^2_{\mathbb{C}}(\mathbb{D})$. This completes the proof. $\square$

However, we have

Proposition 4.9. *If (a,b,g) and (c,d,h) generate Bessel sequence of W-H type in $L^2_{\mathbb{C}}(\mathbb{R})$, then the family $\psi^{\nu,\mu}_M$ in (4.3), for varying $M\in\mathbb{Z}^4$, is a Bessel sequence in $L^2_{\mathbb{T}}(\mathbb{D})$.*

Proof. By similar arguments as in the proof of Theorem 3.8, we need only to prove the components $e^{-y^2/2}\left[W^{\nu}_{na,mb}g\right](x)$ and $e^{-x^2/2}\left[W^{\mu}_{nc,md}h\right](y)$ are Bessel sequences in $L^2_{\mathbb{C}}(\mathbb{D})$. Thus, for arbitrary $\Phi\in L^2_{\mathbb{C}}(\mathbb{D})$, the partial function $x\longmapsto\Phi_y(x):=\Phi(xe_+ + ye_-)$ is clearly in $L^2_{\mathbb{C}}(\mathbb{R})$. Moreover,

$$\begin{aligned}\left\langle\Phi, e^{-y^2/2}W^{\nu}_{na,mb}g\right\rangle_{L^2_{\mathbb{C}}(\mathbb{D})} &= \int_{\mathbb{D}}\Phi(xe_+ + ye_-)\left(e^{-y^2/2}W^{\nu}_{na,mb}g(x)\right)^* dxdy\\ &= \int_{\mathbb{R}} e^{-y^2/2}\left\langle\Phi_y, W^{\nu}_{na,mb}g\right\rangle_{L^2_{\mathbb{C}}(\mathbb{R})} dy.\end{aligned}$$

Now, since g generates a Bessel sequence in $L^2_{\mathbb{C}}(\mathbb{R})$, we see that

$$\begin{aligned}\sum_{m,n\in\mathbb{Z}}\left|\left\langle\Phi, e^{-y^2/2}W^{\nu}_{na,mb}g\right\rangle_{L^2_{\mathbb{C}}(\mathbb{D})}\right|^2 &\leq \int_{\mathbb{R}} e^{-y^2}\sum_{m,n\in\mathbb{Z}}\left|\left\langle\Phi_y, W^{\nu}_{na,mb}g\right\rangle_{L^2_{\mathbb{C}}(\mathbb{R})}\right|^2 dy\\ &\leq \int_{\mathbb{R}} e^{-y^2}\left\|\Phi_y\right\|^2_{L^2_{\mathbb{C}}(\mathbb{R})} dy\\ &\leq c\left\|\Phi_y\right\|^2_{L^2_{\mathbb{C}}(\mathbb{D})},\end{aligned}$$

for some constant c. This shows that $e^{-y^2/2}W^{\nu}_{na,mb}g$ is a Bessel sequence in $L^2_{\mathbb{C}}(\mathbb{D})$. □

5. Concluding Remarks

In this note, we have presented a natural extension of some notions from classical frame theory including frame operator and Weyl-Heisenberg frames to the bicomplex setting. We have briefly discussed their similarities and the differences to classical ones. The mean feature of bc-frame is Theorem 3.8. The complete description of bc-frames needs further investigations. However, the bc-hilbertian structure allows the consideration of non-trivial extensions for the existence of three complex conjugates and the divisibility by zero. Notice for instance that we define a †-bc-frame to be a bicomplex sequence $(f_n)_n$ in a given infinite bc-Hilbert space $\mathcal{H}_{bc}$ if there exist $A,B>0$ such that

$$A\left\|f\right\|^2_{bc}\leq\sum_{n=0}^{\infty}\left|\langle f,f_n\rangle^{\dagger}_{bc}\langle f,f_n\rangle^{*}_{bc}\right|\leq B\left\|f\right\|^2_{bc},$$

holds true for every $f\in\mathcal{H}_{bc}$, so that one recovers the classical definition if we restrict ourself to complex valued functional Hilbert spaces. We claim that this class possesses several interesting and surprising results that deserve special study. We hope to return back to this in detail in a forthcoming paper.

We conclude by noticing that other constructions of W-H bc-frames and bicomplex Wilson bases for bicomplex Bargmann space in [16] can be considered by benefiting from the rich structure of bicomplex Hilbert space, including the one the hyperbolic numbers and those arising from discretization of bicomplex Fourier–Wigner transform in [16] and associated to the bicomplex projective representations considered there.

Acknowledgment. The authors are grateful to the anonymous referees for their valuable remarks, comments and suggestions. The authors are thankful to S. Kabbaj for his valuable comments on the first draft of this work.

References

1. D.S. Alexiadis and G.D. Sergiadis, *Estimation of motions in color image sequences using hypercomplex Fourier transforms*, IEEE Trans. Image Process., 18 (1) (2009), pp. 168-187.
2. P. Balazs, N. Holighaus, T. Necciari and D. Stoeva, *Frame theory for signal processing in psychoacoustics*, in: Excursions in harmonic analysis. vol.5, Appl. Numer. Harmon. Anal., Birkhäuser, Springer, Cham, 2017, pp. 225-268.
3. J.J. Benedetto, *Irregular sampling and frames in Wavelets*, in: A Tutorial in Theory and Applications, Chui. C.K. ed. Cambridge, MA. Academic Press, 1992, pp. 445-507.
4. H. Bolcskei, F. Hlawatsch and H.G. Feichtinger, *Frame-theoretic analysis of oversampled filter banks*, IEEE Trans. Signal Process., 46 (12) (1998), pp. 3256-3268.
5. P.G. Casazza, *The art of frame theory*, Taiwanese J. Math., 4 (2) (2000), pp. 129-202.
6. P.G. Casazza and G. Kutyniok, *Frames of subspaces*, Contemp. Math., 345 (2004), pp. 87-113.
7. P.G. Casazza and R.G. Lynch, *A brief introduction to Hilbert space frame theory and its applications, Finite frame theory*, Proc. Sympos. Appl. Math., 73 (2016), pp. 1-51.
8. O. Christensen, *An Introduction to Frames and Riesz Bases*, Birkhäuser, 2016.
9. I. Daubechies, *Ten Lectures on Wavelets*, Philadelphia. SIAM, 1992.
10. I. Daubechies, *The wavelet transformation, time-frequency localization and signal analysis*, IEEE Trans. Inform. Theory., 36 (1990), pp. 961-1005.

11. I. Daubechies, A. Grassmann and Y. Meyer, *Painless nonorthogonal expansions*, J. Math. Physics., 27 (1986), pp. 1271-1283.
12. B. Deng, W. Schempp, C. Xiao and Z. Wu, *On the existence of Weyl-Heisenberg and affine frames*, Preprint, 1997.
13. D.L. Donoho and M. Elad, *Optimally sparse representation in general (nonorthogonal) dictionaries via l^1 minimization*, in: Proc. Natl. Acad. Sci. USA, 100 (5) (2003), pp. 2197-2202.
14. R. Duffin and A. Schaeffer, *A class of nonharmonic Fourier series*, Trans. Amer. Math. Soc., 72 (1952), pp. 341-366.
15. Y.C. Eldar and G.D. Jr Forney, *Optimal tight frames and quantum measurement*, IEEE Trans. Inf. Theory., 48 (3) (2002), pp. 599-610.
16. A. El Gourari, A. Ghanmi and K. Zine, *On bicomplex Fourier-Wigner transforms*, Int. J. Wavelets Multiresolut. Inf. Process., 18 (3) (2020), 2050008, 16 pages.
17. M.T. El-Melegy and A.T. Kamal, *Color Image Processing Using Reduced Biquaternions With Application to Face Recognition in a PCA Framework*, Proc. IEEE Inte. Conf. Computer Vision (ICCV)., (2017), pp. 3039-3046.
18. T.A. Ell and S.J. Sangwine, *Hypercomplex Fourier transforms of color images*, IEEE Trans. Image Process., 16 (1) (2007), pp. 22-35.
19. M. Frank and D.R. Larson, *Frames in Hilbert C^*–modules and C^*-algebras*, preprint, University of Houston and Texas A&M University, Texas, U.S.A., 1998.
20. M. Frank and D.R. Larson, *A module frame concept for Hilbert C^*-modules*, in: the Functional and Harmonic Analysis of Wavelets and frames, Contemp. Math., 247, (1999), pp. 207-233.
21. M. Frank and D.R. Larson, *Modular frames for Hilbert C^*-modules and symmetric approximation of frames*, Proc. SPIE., 4119 (2000), pp. 325-336.
22. M. Frank and D.R. Larson, *Frames in Hilbert C^*-modules and C^*-algebras*, J. Operator Theory., 2 (48) (2002), pp. 273-314.
23. P. Gävruta, *On some identities and inequalities for frames in Hilbert spaces*, J. Math. Anal. Appl., 321 (2006), pp. 469-478.
24. R. Gervais Lavoie, L. Marchildon and D. Rochon, *Infinite dimensional bicomplex Hilbert spaces*, Ann. Funct. Anal., 2 (2010), pp. 75-91.
25. A. Ghanmi and K. Zine, *Bicomplex analogs of Segal-Bargmann and fractional Fourier transforms*, Adv. Appl. Clifford Algebr., 29 (4) (2019), 20 pages.
26. K. Gröchenig, *Foundations of Time-Frequency Analysis*, Birkhäuser, Boston 2001.

27. E. Guariglia and S. Silvestrov, *Fractional-wavelet analysis of positive definite distributions and wavelets on $D'(\mathbb{C})$*, in: Engineering Mathematics II, Springer Proc. Math. Stat., 179, Springer, Cham, 2016, pp. 337-353.
28. C. Heil and D. Walnut, *Continuous and discrete wavelet transforms*, SIAM Rev., 31 (1989), pp. 628-666 .
29. C. Heil, *A Basis Theory Primer*, Birkhäuser, Boston, 2010.
30. M. Joita, *Tensor products of Hilbert modules over locally C^*-algebras*, Czechoslovak Math. J., 54 (129) (2004), pp. 727-737.
31. A. Khosravi and B. Khosravi, *Frames and bases in tensor products of Hilbert spaces and Hilbert C^*-modules*, Proc. Indian Acad. Sci. Math. Sci., 117 (1) (2007), pp. 1-12.
32. A. Khosravi and B. Khosravi, *Fusion frames and g-frames in Hilbert C^*-modules*, Int. J. Wavelet. Multiresolut. Inf. Process., 6 (3) (2008), pp. 433-446.
33. J. Koväcević and A. Chebira, *An introduction to frames*, Found. Trends Signal Process., 2 (1) (2008), pp. 1-94.
34. C.E. Moxey, S.J. Sangwine and T. Ell, *A. Hypercomplex correlation techniques for vector images*, IEEE Trans. Signal Process., 51 (7) (2003), pp. 1941-1953.
35. G.B. Price, *An Introduction to Multicomplex Spaces and Functions*, Monographs and Textbooks, Pure and Appl Math., 140, Marcel Dekker Inc., New York, 1991.
36. I. Raeburn and S.J. Thompson, *Countably generated Hilbert modules, the Kasparov stabilisation theorem, and frames with Hilbert modules*, Proc. Amer. Math. Soc., 131 (5) (2003), pp. 1557-1564.
37. D. Rochon and M. Shapiro, *On algebraic properties of bicomplex and hyperbolic numbers*, An Univ. Oradea Fasc. Mat., 11 (2004), pp. 71-110.
38. D. Rochon and S. Tremmblay, *Bicomplex quantum mechanics,I: The generalized Schrödinger equation*, Adv. Appl. Clifford Algebr., 14 (2004), pp. 231-248.
39. D. Rochon and S. Tremmblay, *Bicomplex quantum mechanics, II: The Hilbert space*, Adv. Appl. Clifford Algebr., 16 (2006), pp. 135-157.
40. J. Wu, *Frames in Hilbert C^*-modules*, Electronic Theses and Dissertations, 899, pp. 2004-2019.
41. R. Young, *An Introduction to Nonharmonic Fourier Series*, Academic Press, New York, 1980.
42. M. Zaied, O. Jemai and C. Ben Amar, *Training of the Beta wavelet networks by the frames theory: Application to face recognition*, in:

first Workshops on Image Processing Theory, Tools and Applications, 2008, pp. 1-6.

---

[1] Lab. P.D.E., Algebra and Spectral Geometry, Department of mathematics, Faculty of sciences, P.O.Box 133. Ibn Tofaïl University in Kenitra; Morocco.

E-mail address: aiadelgourari@gmail.com

[2] Analysis, P.D.G & Spectral Geometry. Lab. M.I.A.-S.I., CeReMAR, Department of Mathematics, P.O. Box 1014, Faculty of Sciences, Mohammed V University in Rabat, Morocco.

E-mail address: allal.ghanmi@um5.ac.ma

[3] Faculty of Law, Economics and Social Sciences, Ibn Zohr University, Agadir, Morocco.

E-mail address: msouidelainin@yahoo.fr