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Hatim Labrigui and Samir Kabbaj

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Integral K-Operator Frames for $End_{\mathcal{A}}^{*}(\mathcal{H})$

Hatim Labrigui[1*] and Samir Kabbaj[2]

Abstract. In this work, we introduce a new concept of integral K-operator frame for the set of all adjointable operators from a Hilbert C^*-module $\mathcal{H}$ to itself denoted by $End_{\mathcal{A}}^{*}(\mathcal{H})$. We give some properties relating to some constructions of integral K-operator frames and to operators preserving integral K-operator frame and we establish some new results.

1. Introduction and preliminaries

In 1952 Duffin and Schaefer [7] have introduced the concept of frames in the study of nonharmonic Fourier series. Frames possess many nice properties which make them very useful in wavelet analysis, irregular sampling theory, signal processing and many other fields. The theory of frames has been generalized rapidly and various generalizations of frames have emerged in Hilbert spaces and Hilbert C^*-module (see [8, 10, 15–18]).

The concept of continuous frames has been defined by Ali, Antoine and Gazeau [1]. Gabardo and Han in [9] called these frames: frames associated with measurable spaces.

In this paper, we introduce a new concept of integral K-operator frame for the set of all adjointable operators from a Hilbert C^*-module $\mathcal{H}$ to $\mathcal{H}$ denoted by $End_{\mathcal{A}}^{*}(\mathcal{H})$. This concept is a generalization of continuous K-frames for Hilbert C^*-module, and we establish some new results.

In what follows, let $\mathcal{H}$ be a Hilbert C^*-module over a C^*-algebra $\mathcal{A}$, (Ω, μ) a measure space with positive measure μ and $End_{\mathcal{A}}^{*}(\mathcal{H})$ the set

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* Corresponding author.

of all adjointable operators from Hilbert C^*-module $\mathcal{H}$ to $\mathcal{H}$.
Let $K, T \in End^*_{\mathcal{A}}(\mathcal{H})$, if $TK = I$, then T is called the left inverse of K, denoted by K_l^{-1}.

If $KT = I$, then T is called the right inverse of K and we write $K_r^{-1} = T$.

If $KT = TK = I$, then T and K are inverse of each other.

Let,

$$l^2(\Omega, \mathcal{H}) = \left\{ x = (x_\omega)_{\omega \in \Omega} \in \mathcal{H}, \quad \left\| \int_\Omega \langle x_\omega, x_\omega \rangle \, d\mu(\omega) \right\| < \infty \right\}.$$

The inner product on $l^2(\Omega, \mathcal{H})$ is defined for $x = (x_\omega)_{\omega \in \Omega} \in l^2(\Omega, \mathcal{H})$ and $y = (y_\omega)_{\omega \in \Omega} \in l^2(\Omega, \mathcal{H})$ by,

$$\langle x, y \rangle = \int_\Omega \langle x_\omega, y_\omega \rangle \, d\mu(\omega).$$

The norm is defined by $\|x\| = \langle x, x \rangle^{\frac{1}{2}}$.

In this section we briefly recall the definitions and basic properties of C^*-algebra, Hilbert $\mathcal{A}$-module and frame in Hilbert $\mathcal{A}$-module. For information about frames in Hilbert spaces we refer to [4]. Our references for C^*-algebras are [5, 6].

For a C^*-algebra $\mathcal{A}$, if $a \in \mathcal{A}$ is positive we write $a \geq 0$ and $\mathcal{A}^+$ denotes the set of positive elements of $\mathcal{A}$.

Definition 1.1 ([11]). Let $\mathcal{A}$ be a unital C^*-algebra and $\mathcal{H}$ be a left $\mathcal{A}$-module, such that the linear structures of $\mathcal{A}$ and $\mathcal{H}$ are compatible. $\mathcal{H}$ is a pre-Hilbert $\mathcal{A}$-module if $\mathcal{H}$ is equipped with an $\mathcal{A}$-valued inner product $\langle ., . \rangle : \mathcal{H} \times \mathcal{H} \to \mathcal{A}$, which is sesquilinear, positive definite and preserves the module action. In the other words,

(i) $\langle x, x \rangle_{\mathcal{A}} \geq 0$ for all $x \in \mathcal{H}$ and $\langle x, x \rangle_{\mathcal{A}} = 0$ if and only if $x = 0$.
(ii) $\langle ax + y, z \rangle_{\mathcal{A}} = a \langle x, y \rangle_{\mathcal{A}} + \langle y, z \rangle_{\mathcal{A}}$ for all $a \in \mathcal{A}$ and $x, y, z \in \mathcal{H}$.
(iii) $\langle x, y \rangle_{\mathcal{A}} = \langle y, x \rangle^*_{\mathcal{A}}$ for all $x, y \in \mathcal{H}$.

For $x \in \mathcal{H}$, we define $||x|| = || \langle x, x \rangle ||^{\frac{1}{2}}$. If $\mathcal{H}$ is complete with $||.||$, it is called a Hilbert $\mathcal{A}$-module or a Hilbert C^*-module over $\mathcal{A}$. For every a in the C^*-algebra $\mathcal{A}$, we have $|a| = (a^*a)^{\frac{1}{2}}$ and the $\mathcal{A}$-valued norm on $\mathcal{H}$ is defined by $|x| = \langle x, x \rangle^{\frac{1}{2}}$ for $x \in \mathcal{H}$.

Let $\mathcal{H}$ and $\mathcal{K}$ be two Hilbert $\mathcal{A}$-module. A map $T : \mathcal{H} \to \mathcal{K}$ is said to be adjointable if there exists a map $T^* : \mathcal{K} \to \mathcal{H}$ such that $\langle Tx, y \rangle_{\mathcal{A}} = \langle x, T^*y \rangle_{\mathcal{A}}$ for all $x \in \mathcal{H}$ and $y \in \mathcal{K}$.

We also reserve the notation $End^*_{\mathcal{A}}(\mathcal{H}, \mathcal{K})$ for the set of all adjointable operators from $\mathcal{H}$ to $\mathcal{K}$ and $End^*_{\mathcal{A}}(\mathcal{H}, \mathcal{H})$ is abbreviated to $End^*_{\mathcal{A}}(\mathcal{H})$.

Definition 1.2 ([12]). Let $\mathcal{H}$ be a Hilbert $\mathcal{A}$-module over a unital C^*-algebra. A family $\{x_i\}_{i \in I}$ of elements of $\mathcal{H}$ is said to be a frame for $\mathcal{H}$, if there exist two positive constants A, B such that,

$$A \langle x, x\rangle_{\mathcal{A}} \leq \sum_{i\in I} \langle x, x_i\rangle_{\mathcal{A}} \langle x_i, x\rangle_{\mathcal{A}} \leq B \langle x, x\rangle_{\mathcal{A}}, \quad x \in \mathcal{H}. \tag{1.1}$$

The numbers A and B are called lower and upper bounds of the frame, respectively. If $A = B = \lambda$, the frame is called λ-tight. If $A = B = 1$, it is called a normalized tight frame or a Parseval frame. If the sum in the middle of (1.1) is convergent in norm, the frame is called standard. If only upper inequality of (1.1) holds, then $\{x_i\}_{i\in I}$ is called a Bessel sequence for $\mathcal{H}$.

In [10], L. Gavruta introduced K-frames to study atomic systems for operators in Hilbert spaces.

Definition 1.3 ([13])**.** Let $K \in End_{\mathcal{A}}^{*}(\mathcal{H})$. A family $\{x_i\}_{i\in I}$ of elements in a Hilbert $\mathcal{A}$-module $\mathcal{H}$ over a unital C^*-algebra is a K-frame for $\mathcal{H}$, if there exist two positive constants A and B, such that,

$$\begin{aligned} A \langle K^*x, K^*x\rangle_{\mathcal{A}} &\leq \sum_{i\in I} \langle x, x_i\rangle_{\mathcal{A}} \langle x_i, x\rangle_{\mathcal{A}} \\ &\leq B \langle x, x\rangle_{\mathcal{A}}, \quad x \in \mathcal{H}. \end{aligned} \tag{1.2}$$

The numbers A and B are called lower and upper bounds of the K-frame, respectively.

The following lemmas will be used to prove our mains results

Lemma 1.4 ([14])**.** *Let $\mathcal{H}$ be a Hilbert $\mathcal{A}$-module. If $T \in End_{\mathcal{A}}^{*}(\mathcal{H})$, then*

$$\langle Tx, Tx\rangle_{\mathcal{A}} \leq \|T\|^2 \langle x, x\rangle_{\mathcal{A}}, \quad \forall x \in \mathcal{H}.$$

Lemma 1.5 ([20])**.** *Let $\mathcal{H}$ be a Hilbert $\mathcal{A}$-module over a C^*-algebra $\mathcal{A}$ and let T, S two elements of $End_{\mathcal{A}}^{*}(\mathcal{H})$. If $Rang(S)$ is closed, then the following statements are equivalent:*

(i) *$Rang(T) \subseteq Rang(S)$.*
(ii) *$TT^* \leq \lambda SS^*$ for some $\lambda > 0$.*
(iii) *There exists $Q \in End_{\mathcal{A}}^{*}(\mathcal{H})$ such that $T = SQ$.*

Lemma 1.6 ([2])**.** *Let $\mathcal{H}$ and $\mathcal{K}$ be two Hilbert $\mathcal{A}$-module and $T \in End^*(\mathcal{H}, \mathcal{K})$.*

(i) *If T is injective and T has closed range, then the adjointable map T^*T is invertible and*

$$\left\|(T^*T)^{-1}\right\|^{-1} \leq T^*T \leq \|T\|^2.$$

(ii) *If T is surjective, then the adjointable map TT^* is invertible and*

$$\left\|(TT^*)^{-1}\right\|^{-1} \leq TT^* \leq \|T\|^2.$$

Lemma 1.7 ([19]). *Let (Ω, μ) be a measure space, X and Y two Banach spaces, $\lambda : X \longrightarrow Y$ a bounded linear operator and $f : \Omega \longrightarrow X$ measurable function; then,*

$$\lambda\left(\int_{\Omega} f d\mu\right) = \int_{\Omega} (\lambda f) d\mu.$$

Lemma 1.8 ([3]). *Let $\mathcal{H}$ and $\mathcal{K}$ be two Hilbert $\mathcal{A}$-module and $T \in End^*_{\mathcal{A}}(\mathcal{H}, \mathcal{K})$. Then the following statements are equivalent:*

(i) *T is surjective.*
(ii) *T^* is bounded below with respect to norm, i.e., there is $m > 0$ such that $\|T^* x\| \geq m\|x\|$, for all $x \in \mathcal{K}$.*
(iii) *T^* is bounded below with respect to the inner product, i.e., there is $m' > 0$ such that $\langle T^* x, T^* x\rangle_{\mathcal{A}} \geq m' \langle x, x\rangle_{\mathcal{A}}$, for all $x \in \mathcal{K}$.*

2. Integral K-Operator Frames for $End^*_{\mathcal{A}}(\mathcal{H})$

We began this section with the following definition.

Definition 2.1. A family of adjointable operators $\{T_w\}_{w\in\Omega} \subset End^*_{\mathcal{A}}(\mathcal{H})$ on a Hilbert $\mathcal{A}$-module $\mathcal{H}$ over a unital C^*-algebra is said to be an integral operator frame for $End^*_{\mathcal{A}}(\mathcal{H})$, if there exist two positive constants $A, B > 0$ such that

$$A \langle x, x\rangle_{\mathcal{A}} \leq \int_{\Omega} \langle T_\omega x, T_\omega x\rangle_{\mathcal{A}} d\mu(\omega) \leq B \langle x, x\rangle_{\mathcal{A}}, \quad x \in \mathcal{H}. \tag{2.1}$$

Definition 2.2. Let $K \in End^*_{\mathcal{A}}(\mathcal{H})$ and $T = \{T_\omega \in End^*_{\mathcal{A}}(\mathcal{H}), \omega \in \Omega\}$. The family T is said an integral K-operator frame for $End^*_{\mathcal{A}}(\mathcal{H})$, if there exist two positive constants $A, B > 0$ such that

$$A \langle K^* x, K^* x\rangle_{\mathcal{A}} \leq \int_{\Omega} \langle T_\omega x, T_\omega x\rangle_{\mathcal{A}} d\mu(\omega) \leq B \langle x, x\rangle_{\mathcal{A}}, \quad x \in \mathcal{H}. \tag{2.2}$$

The numbers A and B are called respectively lower and upper bounds of the integral K-operator frame.

An integral K-operator frame $\{T_\omega\}_{\omega\in\Omega} \subset End^*_{\mathcal{A}}(\mathcal{H})$ is said to be A-tight if there exists a constant $0 < A$ such that,

$$A \langle K^* x, K^* x\rangle_{\mathcal{A}} = \int_{\Omega} \langle T_\omega x, T_\omega x\rangle_{\mathcal{A}} d\mu(\omega), \quad x \in \mathcal{H}.$$

If $A = 1$, it is called a normalized tight integral K-operator frame or a Parseval integral K-operator frame.

Example 2.3. Let $\mathcal{H}$ be a Hilbert space defined by:
$\mathcal{H} = \left\{\begin{pmatrix} \alpha & 0 & 0 \\ 0 & 0 & \beta \end{pmatrix} \ / \ \alpha, \beta \in \mathbb{C}\right\}$ and $\mathcal{A} = \left\{\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \ / \ a, b \in \mathbb{C}\right\}$.

It's clear that $\mathcal{H}$ is a Hilbert space and $\mathcal{A}\mathcal{H} \subset \mathcal{H}$.
Furthermore, the $\mathcal{A}$-valued inner product,

$$\mathcal{H} \times \mathcal{H} \longrightarrow \mathcal{A},$$
$$(A, B) \longrightarrow \langle A, B\rangle_{\mathcal{A}} = A^t \bar{B},$$

is sesquilinear and positive.
If $A = \begin{pmatrix} a_1 & 0 & 0 \\ 0 & 0 & a_2 \end{pmatrix}$ and $B = \begin{pmatrix} b_1 & 0 & 0 \\ 0 & 0 & b_2 \end{pmatrix}$,
then,

$$\langle A, B\rangle_{\mathcal{A}} = \begin{pmatrix} a_1\bar{b_1} & 0 \\ 0 & a_2\bar{b_2} \end{pmatrix}.$$

Let $(\Omega = [0,1], d\lambda)$ be a measure space, where $d\lambda$ is the Lebesgue measure restricted to the interval $[0,1]$.
For all $w \in [0,1]$, we consider,

$$F : \Omega \longrightarrow \mathcal{H},$$
$$w \longrightarrow F_\omega = \begin{pmatrix} w & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

F is a measurable map and for all $A \in \mathcal{H}$, we have,

$$\begin{aligned}
\int_\Omega \langle A, F_\omega\rangle_{\mathcal{A}} \langle F_\omega, A\rangle_{\mathcal{A}}\, d\lambda(w) &= \int_\Omega \begin{pmatrix} a_1\bar{w} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} w\bar{a_1} & 0 \\ 0 & 0 \end{pmatrix} d\lambda(w) \\
&= \begin{pmatrix} |a_1|^2 \int_\Omega w^2 d\lambda(w) & 0 \\ 0 & 0 \end{pmatrix} \\
&= \frac{1}{3} \begin{pmatrix} |a_1|^2 & 0 \\ 0 & 0 \end{pmatrix} \\
&\leq \frac{1}{3} \begin{pmatrix} |a_1|^2 & 0 \\ 0 & |a_2|^2 \end{pmatrix} \\
&= \frac{1}{3} \langle A, A\rangle_{\mathcal{A}}.
\end{aligned}$$

Wich show that F is an integral Bessel sequence. But F is not an integral operator frame for the Hilbert $\mathcal{A}$-module. Indeed, just take $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & a_2 \end{pmatrix}$ with $a_2 \neq 0$.
Consider now,

$$K : \mathcal{H} \longrightarrow \mathcal{H},$$
$$\begin{pmatrix} a_1 & 0 & 0 \\ 0 & 0 & a_2 \end{pmatrix} \longrightarrow \begin{pmatrix} a_1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

K is a linear, bounded and selfadjoint operator and we have for all $A \in \mathcal{H}$,

$$\langle K^*A, K^*A\rangle_{\mathcal{A}} = \begin{pmatrix} |a_1|^2 & 0 \\ 0 & 0 \end{pmatrix}.$$

So,

$$\begin{aligned}\frac{1}{4}\langle K^*A, K^*A\rangle_{\mathcal{A}} &\leq \int_{\Omega} \langle A, F_\omega\rangle_{\mathcal{A}} \langle F_\omega, A\rangle_{\mathcal{A}} \, d\lambda(w) \\ &\leq \frac{1}{3}\langle A, A\rangle_{\mathcal{A}}.\end{aligned}$$

Remark 2.4. Every integral operator frame is an integral K-operator frame, for any $K \in End^*_{\mathcal{A}}(\mathcal{H})$, $K \neq 0$. Indeed, if $\{T_\omega\}_{\omega\in\Omega}$ is an integral operator frame for $End^*_{\mathcal{A}}(\mathcal{H})$ with bounds A and B, then

$$A\langle x, x\rangle_{\mathcal{A}} \leq \int_{\Omega} \langle T_w x, T_\omega x\rangle_{\mathcal{A}} \, d\mu(\omega) \leq B\langle x, x\rangle_{\mathcal{A}}, \quad x \in \mathcal{H}.$$

By Lemma 1.4, we have,

$$A\|K\|^{-2}\langle K^*x, K^*x\rangle_{\mathcal{A}} \leq \int_{\Omega} \langle T_\omega x, T_\omega x\rangle_{\mathcal{A}} \, d\mu(\omega) \leq B\langle x, x\rangle_{\mathcal{A}}, \quad x \in \mathcal{H}.$$

Therefore the family $\{T_\omega\}_{\omega\in\Omega}$ is an integral K-operator frame with bounds $A\|K\|^{-2}$ and B.

Proposition 2.5. *Let $K \in End^*_{\mathcal{A}}(\mathcal{H})$ and $\{T_\omega\}_{\omega\in\Omega}$ be an integral K-operator frame for $End^*_{\mathcal{A}}(\mathcal{H})$ with frame bounds A and B. If K is surjective then $\{T_\omega\}_{\omega\in\Omega}$ is an integral operator frame for $End^*_{\mathcal{A}}(\mathcal{H})$.*

Proof. Since K is surjective, by Lemma 1.8 there exists $m > 0$ such that

$$\langle K^*x, K^*x\rangle_{\mathcal{A}} \geq m\langle x, x\rangle_{\mathcal{A}}, \quad x \in \mathcal{H}.$$

Also, since $\{T_\omega\}_{\omega\in\Omega}$ is an integral K-operator frame for $End^*_{\mathcal{A}}(\mathcal{H})$, we have for all $x \in \mathcal{H}$

$$\begin{aligned}Am\langle x, x\rangle_{\mathcal{A}} &\leq A\langle K^*x, K^*x\rangle_{\mathcal{A}} \\ &\leq \int_{\Omega} \langle T_\omega x, T_\omega x\rangle_{\mathcal{A}} \, d\mu(\omega) \\ &\leq B\langle x, x\rangle_{\mathcal{A}}.\end{aligned}$$

Hence $\{T_\omega\}_{\omega\in\Omega}$ is an integral operator frame for $End^*_{\mathcal{A}}(\mathcal{H})$ with frame bounds Am and B. □

Let $\{T_\omega\}_{\omega\in\Omega}$ be an integral K-operator frame for $End^*_{\mathcal{A}}(\mathcal{H})$. We define the operator R by,

$$\begin{aligned}&R : \mathcal{H} \longrightarrow l^2(\Omega, \mathcal{H}), \\ &x \longrightarrow Rx = \{T_\omega x\}_{\omega\in\Omega},\end{aligned}$$

The operator R is called the analysis operator of the integral K-operator frame $\{T_{\omega}\}_{\omega\in\Omega}$.

The adjoint of the analysis operator R is defined by,

$$R^{*}: l^{2}(\Omega,\mathcal{H}) \longrightarrow \mathcal{H},$$
$$x \longrightarrow R^{*}(\{x_{\omega}\}_{\omega\in\Omega}) = \int_{\Omega} T_{\omega}^{*} x_{\omega} d\mu(\omega).$$

The operator R^{*} is called the synthesis operator of the integral K-operator frame $\{T_{\omega}\}_{\omega\in\Omega}$.

By composing R and R^{*}, the frame operator $S_T : \mathcal{H} \to \mathcal{H}$ for the integral K-operator frame T is given by

$$\begin{aligned} S(x) &= R^{*}Rx \\ &= \int_{\Omega} T_{\omega}^{*} T_{\omega} x d\mu(\omega). \end{aligned}$$

Theorem 2.6. *Let $K \in End_{\mathcal{A}}^{*}(\mathcal{H})$ and $\{T_{\omega}\}_{\omega\in\Omega} \subset End_{\mathcal{A}}^{*}(\mathcal{H})$. The following statements are equivalent:*

(1) *$\{T_{\omega}\}_{\omega\in\Omega}$ is an integral K-operator frame for $End_{\mathcal{A}}^{*}(\mathcal{H})$.*
(2) *There exists $A > 0$ such that $AKK^{*} \leq S$.*
(3) *$K = S^{\frac{1}{2}}Q$, for some $Q \in End_{\mathcal{A}}^{*}(\mathcal{H})$.*

Proof. (1) $\Rightarrow$ (2) Let $\{T_{\omega}\}_{\omega\in\Omega}$ be an integral K-operator frame for $End_{\mathcal{A}}^{*}(\mathcal{H})$ with frame bounds A, B and frame operator S, we have

$$\begin{aligned} A\langle K^{*}x, K^{*}x\rangle_{\mathcal{A}} &\leq \int_{\Omega} \langle T_{\omega}x, T_{\omega}x\rangle_{\mathcal{A}} d\mu(\omega) \\ &\leq B\langle x, x\rangle_{\mathcal{A}}, \quad x \in \mathcal{H}. \end{aligned}$$

Then

$$\begin{aligned} \langle AKK^{*}x, x\rangle_{\mathcal{A}} &\leq \langle Sx, x\rangle_{\mathcal{A}} \\ &\leq \langle Bx, x\rangle_{\mathcal{A}}, \quad x \in \mathcal{H}. \end{aligned}$$

Hence

$$AKK^{*} \leq S. \tag{2.3}$$

(2) $\Rightarrow$ (3) Suppose that there exists $A > 0$ such that $AKK^{*} \leq S$.
Note that

$$AKK^{*} \leq \left(S^{\frac{1}{2}}\right)\left(S^{\frac{1}{2}}\right)^{*}.$$

By Lemma 1.5 we have: $K = S^{\frac{1}{2}}Q$, for some $Q \in End_{\mathcal{A}}^{*}(\mathcal{H})$.
(3) $\Rightarrow$ (1) Suppose that $K = S^{\frac{1}{2}}Q$, for some $Q \in End_{\mathcal{A}}^{*}(\mathcal{H})$. Then by

the Lemma 1.5, there exists $A > 0$ such that $AKK^* \leq \left(S^{\frac{1}{2}}\right)\left(S^{\frac{1}{2}}\right)^*$. So,

$$\langle AKK^*x, x\rangle_{\mathcal{A}} \leq \left\langle \left(S^{\frac{1}{2}}\right)\left(S^{\frac{1}{2}}\right)^* x, x\right\rangle_{\mathcal{A}}, \quad x \in \mathcal{H},$$

which gives,

$$A\left\langle K^*x, K^*x\right\rangle_{\mathcal{A}} \leq \left\langle Sx, x\right\rangle_{\mathcal{A}}, \quad x \in \mathcal{H},$$

moreover, by Lemma 1.4, we have,

$$\left\langle Sx, x\right\rangle_{\mathcal{A}} = \left\langle S^{\frac{1}{2}}x, S^{\frac{1}{2}}x\right\rangle_{\mathcal{A}} \leq \left\|S^{\frac{1}{2}}\right\|^2 \left\langle x, x\right\rangle_{\mathcal{A}}.$$

This shows that $\{T_\omega\}_{\omega\in\Omega}$ is an integral K-operator frame for $End_{\mathcal{A}}^*(\mathcal{H})$ with frame bounds A and $\left\|S^{\frac{1}{2}}\right\|^2$. □

3. Some Constructions of Integral K-Operator frame

Theorem 3.1. *Let $Q \in End_{\mathcal{A}}^*(\mathcal{H})$ and let $\{T_\omega\}_{\omega\in\Omega}$ be an integral K-operator frame for $End_{\mathcal{A}}^*(\mathcal{H})$. Then $\{T_\omega Q\}_{\omega\in\Omega}$ is an integral (Q^*K)-operator frame for $End_{\mathcal{A}}^*(\mathcal{H})$.*

Proof. Let $\{T_\omega\}_{\omega\in\Omega}$ be an integral K-operator frame for $End_{\mathcal{A}}^*(\mathcal{H})$ with frame bounds A and B if and only if,

$$\begin{aligned} A\left\langle K^*x, K^*x\right\rangle_{\mathcal{A}} &\leq \int_\Omega \left\langle T_\omega x, T_\omega x\right\rangle_{\mathcal{A}} d\mu(\omega) \\ &\leq B\left\langle x, x\right\rangle_{\mathcal{A}}, \quad x \in \mathcal{H}. \end{aligned}$$

This give for all $x \in \mathcal{H}$,

$$\begin{aligned} A\left\langle K^*Qx, K^*Qx\right\rangle_{\mathcal{A}} &\leq \int_\Omega \left\langle T_\omega Qx, T_\omega Qx\right\rangle_{\mathcal{A}} d\mu(\omega) \\ &\leq B\left\langle Qx, Qx\right\rangle_{\mathcal{A}}. \end{aligned}$$

So,

$$\begin{aligned} A\left\langle (Q^*K)^*x, (Q^*K)^*x\right\rangle_{\mathcal{A}} &\leq \int_\Omega \left\langle T_\omega Qx, T_\omega Qx\right\rangle_{\mathcal{A}} d\mu(\omega) \\ &\leq B\|Q\|^2 \left\langle x, x\right\rangle_{\mathcal{A}}, \end{aligned}$$

which shows that $\{T_\omega Q\}_{\omega\in\Omega}$ is an integral (Q^*K)-operator frame for $End_{\mathcal{A}}^*(\mathcal{H})$ with bounds A and $B\|Q\|^2$. □

Theorem 3.2. *Let $K \in End_{\mathcal{A}}^*(\mathcal{H})$ and let $\{T_\omega\}_{\omega\in\Omega} \subset End_{\mathcal{A}}^*(\mathcal{H})$ be a tight integral K-operator frame for $End_{\mathcal{A}}^*(\mathcal{H})$ with frame bound A_1. Then $\{T_\omega\}_{\omega\in\Omega}$ is a tight integral operator frame for $End_{\mathcal{A}}^*(\mathcal{H})$ with frame bound A_2 if and only if $K_r^{-1} = \frac{A_1}{A_2}K^*$.*

Proof. Let $\{T_\omega\}_{\omega\in\Omega} \subset End_{\mathcal{A}}^{*}(\mathcal{H})$ be a tight integral K-operator frame for $End_{\mathcal{A}}^{*}(\mathcal{H})$ with frame bound A_1, then

$$\int_\Omega \langle T_\omega x, T_\omega x\rangle_{\mathcal{A}}\, d\mu(\omega) = A_1 \langle K^*x, K^*x\rangle_{\mathcal{A}}, \quad x \in \mathcal{H}.$$

Since $\{T_\omega\}_{\omega\in\Omega}$ is a tight integral operator frame for $End_{\mathcal{A}}^{*}(\mathcal{H})$ with frame bound A_2, then,

$$\int_\Omega \langle T_\omega x, T_\omega x\rangle_{\mathcal{A}}\, d\mu(\omega) = A_2 \langle x, x\rangle_{\mathcal{A}}, \quad x \in \mathcal{H}.$$

We deduce that, for each $x \in \mathcal{H}$, we have

$$A_1 \langle K^*x, K^*x\rangle_{\mathcal{A}} = A_2 \langle x, x\rangle_{\mathcal{A}}.$$

So,

$$\langle KK^*x, x\rangle_{\mathcal{A}} = \left\langle \frac{A_2}{A_1}x, x\right\rangle_{\mathcal{A}}, \quad x \in \mathcal{H}.$$

Then $KK^* = \frac{A_2}{A_1}I$, Hence $K_r^{-1} = \frac{A_1}{A_2}K^*$.

Conversely, suppose that $K_r^{-1} = \frac{A_1}{A_2}K^*$. Then $KK^* = \frac{A_2}{A_1}I$. Thus

$$\langle KK^*x, x\rangle_{\mathcal{A}} = \left\langle \frac{A_2}{A_1}x, x\right\rangle_{\mathcal{A}}, \quad x \in \mathcal{H}.$$

Since $\{T_\omega\}_{\omega\in\Omega}$ is a tight K-operator frame for $End_{\mathcal{A}}^{*}(\mathcal{H})$, we have

$$\int_\Omega \langle T_\omega x, T_\omega x\rangle_{\mathcal{A}}\, d\mu(\omega) = A_2 \langle x, x\rangle_{\mathcal{A}}, \quad x \in \mathcal{H},$$

which completes the proof. □

Theorem 3.3. *Let $\{T_\omega\}_{\omega\in\Omega}$ be an integral K-operator frame for $End_{\mathcal{A}}^{*}(\mathcal{H})$ with best frame bounds A and B. If $Q \in End_{\mathcal{A}}^{*}(\mathcal{H})$ be an adjointable and invertible operator such that $Q^{-1}K^* = K^*Q^{-1}$, then $\{T_\omega Q\}_{\omega\in\Omega}$ is an integral K-operator frame for $End_{\mathcal{A}}^{*}(\mathcal{H})$ with best frame bounds C and D satisfying the inequalities*

$$A\|Q^{-1}\|^{-2} \leq C \leq A\|Q\|^2, \qquad B\|Q^{-1}\|^{-2} \leq D \leq B\|Q\|^2. \tag{3.1}$$

Proof. Let $\{T_\omega\}_{\omega\in\Omega}$ be an integral K-operator frame for $End_{\mathcal{A}}^{*}(\mathcal{H})$ with best frame bounds A and B.

In the one hand, we have for all $x \in \mathcal{H}$,

$$\begin{aligned}\int_\Omega \langle T_\omega Qx, T_\omega Qx\rangle_{\mathcal{A}}\, d\mu(\omega) &\leq B \langle Qx, Qx\rangle_{\mathcal{A}} \\ &\leq B\|Q\|^2 \langle x, x\rangle_{\mathcal{A}}.\end{aligned}$$

On the other hand, we have for all $x \in \mathcal{H}$,

$$A \langle K^*x, K^*x\rangle_{\mathcal{A}} = A \left\langle K^*Q^{-1}Qx, K^*Q^{-1}Qx\right\rangle_{\mathcal{A}}$$

$$= A \left\langle Q^{-1}K^*Qx, Q^{-1}K^*Qx \right\rangle_{\mathcal{A}}$$
$$\leq \|Q^{-1}\|^2 \int_\Omega \langle T_\omega Qx, T_\omega Qx\rangle_{\mathcal{A}} \, d\mu(\omega).$$

So, we conclude,

$$A\|Q^{-1}\|^{-2} \langle K^*x, K^*x\rangle_{\mathcal{A}} \leq \int_\Omega \langle T_\omega Qx, T_\omega Qx\rangle_{\mathcal{A}} \, d\mu(\omega)$$
$$\leq B\|Q\|^2 \langle x, x\rangle_{\mathcal{A}},$$

which shows that $\{T_\omega Q\}_{\omega\in\Omega}$ is an integral K-operator frame for $End^*_{\mathcal{A}}(\mathcal{H})$ with bounds $A\|Q^{-1}\|^{-2}$ and $B\|Q\|^2$. Now, let C and D be the best bounds of the integral K-operator frame $\{T_\omega Q\}_{\omega\in\Omega}$. Then

$$A\|Q^{-1}\|^{-2} \leq C, \qquad D \leq B\|Q\|^2, \tag{3.2}$$

Since $\{T_\omega Q\}_{\omega\in\Omega}$ is an integral K-operator frame for $End^*_{\mathcal{A}}(\mathcal{H})$ with frame bounds C and D and

$$\langle K^*x, K^*x\rangle_{\mathcal{A}} = \left\langle QQ^{-1}K^*x, QQ^{-1}K^*x \right\rangle_{\mathcal{A}}$$
$$\leq \|Q\|^2 \left\langle K^*Q^{-1}x, K^*Q^{-1}x \right\rangle_{\mathcal{A}},$$

hence

$$C\|Q\|^{-2} \langle K^*x, K^*x\rangle_{\mathcal{A}} \leq C \left\langle K^*Q^{-1}x, K^*Q^{-1}x \right\rangle_{\mathcal{A}}$$
$$\leq \int_\Omega \left\langle T_\omega QQ^{-1}x, T_\omega QQ^{-1}x \right\rangle_{\mathcal{A}} d\mu(\omega)$$
$$= \int_\Omega \langle T_\omega x, T_\omega x\rangle_{\mathcal{A}} \, d\mu(\omega)$$
$$\leq D \left\|Q^{-1}\right\|^2 \langle x, x\rangle_{\mathcal{A}}.$$

Since A and B are the best bounds of integral K-operator frame $\{T_\omega\}_{\omega\in\Omega}$, we have

$$C\|Q\|^{-2} \leq A, \qquad B \leq D\|Q^{-1}\|^2, \tag{3.3}$$

which completes the proof. □

4. Operators Preserving Integral K-Operator Frames

Proposition 4.1. *Let $K, L \in End^*_{\mathcal{A}}(\mathcal{H})$ such that $R(L) \subset R(K)$ and $R(K)$ is closed. Let $\{T_\omega\}_{\omega\in\Omega}$ be an integral K-operator frame for $End^*_{\mathcal{A}}(\mathcal{H})$. Then $\{T_\omega\}_{\omega\in\Omega}$ is an integral L-operator frame for $End^*_{\mathcal{A}}(\mathcal{H})$.*

Proof. Suppose that $\{T_\omega\}_{\omega\in\Omega}$ is an integral K-operator frame for $End^*_{\mathcal{A}}(\mathcal{H})$. Then there exist two positive constants A and B such that

$$A \langle K^*x, K^*x\rangle_{\mathcal{A}} \leq \int_\Omega \langle T_\omega x, T_\omega x\rangle_{\mathcal{A}} \, d\mu(\omega) \leq B \langle x, x\rangle_{\mathcal{A}}, \quad x \in \mathcal{H}. \tag{4.1}$$

From Lemma 1.5, there exists $0<\lambda$ such that

$$LL^{*}\leq \lambda KK^{*},$$

which gives for all $x\in\mathcal{H}$,

$$\begin{aligned}\frac{A}{\lambda}\left\langle L^{*}x,L^{*}x\right\rangle_{\mathcal{A}} &\leq A\left\langle K^{*}x,K^{*}x\right\rangle_{\mathcal{A}}\\ &\leq \int_{\Omega}\left\langle T_{\omega}x,T_{\omega}x\right\rangle_{\mathcal{A}}d\mu(\omega)\\ &\leq B\left\langle x,x\right\rangle_{\mathcal{A}}.\end{aligned}$$

Hence $\{T_{\omega}\}_{\omega\in\Omega}$ is an integral L-operator frame for $End_{\mathcal{A}}^{*}(\mathcal{H})$. □

Theorem 4.2. *Let $K\in End_{\mathcal{A}}^{*}(\mathcal{H})$ with a dense range. Let $\{T_{\omega}\}_{\omega\in\Omega}$ be an integral K-operator frame for $End_{\mathcal{A}}^{*}(\mathcal{H})$ and $L\in End_{\mathcal{A}}^{*}(\mathcal{H})$ have a closed range and commutes with T_{ω} for each $\omega\in\Omega$. If $\{LT_{\omega}\}_{\omega\in\Omega}$ is an integral K-operator frame for $End_{\mathcal{A}}^{*}(\mathcal{H})$ then L is surjective.*

Proof. Assume that the family $\{LT_{\omega}\}_{\omega\in\Omega}$ is an integral K-operator frame for $End_{\mathcal{A}}^{*}(\mathcal{H})$ with bounds A and B, then

$$\begin{aligned}A\left\langle K^{*}x,K^{*}x\right\rangle_{\mathcal{A}} &\leq \int_{\Omega}\left\langle LT_{\omega}x,LT_{\omega}x\right\rangle_{\mathcal{A}}d\mu(\omega)\\ &\leq B\left\langle x,x\right\rangle_{\mathcal{A}},\quad x\in\mathcal{H}.\end{aligned}\tag{4.2}$$

Since K has a dense range, then K^{*} is injective.

By (4.2), L^{*} is injective since, $N(L^{*})\subset N(K^{*})$. Moreover, $R(L)=N(L^{*})^{\perp}=\mathcal{H}$, which shows that L is surjective. □

Theorem 4.3. *Let $K,L\in End_{\mathcal{A}}^{*}(\mathcal{H})$ and let $\{T_{\omega}\}_{\omega\in\Omega}$ be an integral K-operator frame for $End_{\mathcal{A}}^{*}(\mathcal{H})$. If L has a closed range and commutes with K^{*} and T_{ω} for each $\omega\in\Omega$, then $\{LT_{w}\}_{\omega\in\Omega}$ is an integral K-operator frame for $End_{\mathcal{A}}^{*}(R(L))$.*

Proof. let $\{T_{\omega}\}_{\omega\in\Omega}$ be an integral K-operator frame with bounds A,B. If L has closed range, it has the pseudo-inverse $L^{\dagger}$ such that $L^{\dagger}L=I$. So, we have for all $x\in R(L)$,

$$\begin{aligned}\left\langle K^{*}x,K^{*}x\right\rangle_{\mathcal{A}} &= \left\langle L^{\dagger}LK^{*}x,L^{\dagger}LK^{*}x\right\rangle_{\mathcal{A}}\\ &\leq \|L^{\dagger}\|^{2}\left\langle LK^{*}x,LK^{*}x\right\rangle_{\mathcal{A}}.\end{aligned}$$

So,

$$\|L^{\dagger}\|^{-2}\left\langle K^{*}x,K^{*}x\right\rangle_{\mathcal{A}}\leq\left\langle LK^{*}x,LK^{*}x\right\rangle_{\mathcal{A}}.$$

In the one hand, for each $x\in R(T)$, we have,

$$\begin{aligned}\int_{\Omega}\left\langle LT_{\omega}x,LT_{\omega}x\right\rangle_{\mathcal{A}}d\mu(\omega) &= \int_{\Omega}\left\langle T_{\omega}Lx,T_{\omega}Lx\right\rangle_{\mathcal{A}}d\mu(\omega)\\ &\geq A\left\langle K^{*}Lx,K^{*}Lx\right\rangle_{\mathcal{A}}\end{aligned}$$

$$= A \langle LK^*x, LK^*x \rangle_{\mathcal{A}}$$
$$\geq A\|L^\dagger\|^{-2} \langle K^*x, K^*x \rangle_{\mathcal{A}} .$$

On the other hand, we have,

$$\begin{aligned}\int_\Omega \langle LT_\omega x, LT_\omega x \rangle_{\mathcal{A}} d\mu(\omega) &= \int_\Omega \langle T_\omega Lx, T_\omega Lx \rangle_{\mathcal{A}} d\mu(\omega) \\ &\leq B \langle Lx, Lx \rangle_{\mathcal{A}} \\ &= B\|L\|^2 \langle x, x \rangle_{\mathcal{A}}\end{aligned}$$

which shows that the family $\{LT_\omega\}_{\omega\in\Omega}$ is an integral K-operator frame for $End^*_{\mathcal{A}}(R(L))$ with bounds $A\|L^\dagger\|^{-2}$ and $B\|L\|^2$. □

5. Perturbation of Integral K-Operator Frames

In this section we consider perturbation of an integral K-operator frame by non-zero operators.

Theorem 5.1. *Let $K \in End^*_{\mathcal{A}}(\mathcal{H})$ and $\{T_\omega\}_{\omega\in\Omega}$ be an integral K-operator frame for $End^*_{\mathcal{A}}(\mathcal{H})$ with frame bounds A and B. Let $L \in End^*_{\mathcal{A}}(\mathcal{H}), (L \neq 0)$, and $\{a_\omega\}_{\omega\in\Omega}$ be any family of scalars. Then the perturbed family of operators $\{T_\omega + a_\omega LK^*\}_{\omega\in\Omega}$ is an integral K-operator frame for $End^*_{\mathcal{A}}(\mathcal{H})$ if $\int_\Omega |a_\omega|^2 d\mu(\omega) < \frac{A}{\|L\|^2}$.*

Proof. Let $\Gamma_\omega = T_\omega + a_\omega LK^*$, where $\omega \in \Omega$. Then for all $x \in \mathcal{H}$, we have,

$$\begin{aligned}\int_\Omega \langle T_\omega x - \Gamma_\omega x, T_\omega x - \Gamma_\omega x \rangle_{\mathcal{A}} d\mu(\omega) &= \int_\Omega \langle a_\omega LK^*x, a_\omega LK^*x \rangle_{\mathcal{A}} d\mu(\omega), \\ &\leq \int_\Omega |a_\omega|^2\|L\|^2\|K^*\|^2 \langle x, x \rangle_{\mathcal{A}} d\mu(\omega) \\ &= \int_\Omega |a_\omega|^2\|L\|^2\|K^*\|^2 \langle x, x \rangle_{\mathcal{A}} d\mu(\omega), \\ &\leq R\|K^*\|^2 \langle x, x \rangle_{\mathcal{A}} ,\end{aligned}$$

where $R = \int_\Omega |a_\omega|^2\|L\|^2 d\mu(\omega)$.

In the one hand, for all $x \in H$,we have,

$$\begin{aligned}&\left(\int_\Omega \langle (T_\omega + a_\omega LK^*)x, (T_\omega + a_\omega LK^*)x \rangle_{\mathcal{A}} d\mu(\omega)\right)^{\frac{1}{2}} \\ &\quad = \|(T_\omega + a_\omega LK^*)x\|_{l^2(\Omega,H)} \\ &\quad \leq \|T_\omega x\|_{l^2(\Omega,H)} + \|a_\omega LK^*x\|_{l^2(\Omega,H)} \\ &\quad = (\int_\Omega \langle T_\omega x, T_\omega x \rangle_{\mathcal{A}} d\mu(\omega))^{\frac{1}{2}} + \left(\int_\Omega \langle (a_\omega LK^*)x, (a_\omega LK^*)x \rangle_{\mathcal{A}} d\mu(\omega)\right)^{\frac{1}{2}}\end{aligned}$$

$$\leq \sqrt{B}\,\langle x,x\rangle_{\mathcal{A}}^{\frac{1}{2}} + \sqrt{R}\|K^*\|\,\langle x,x\rangle_{\mathcal{A}}^{\frac{1}{2}}$$
$$\leq \left(\sqrt{B} + \sqrt{R}\|K^*\|\right)\langle x,x\rangle_{\mathcal{A}}^{\frac{1}{2}}.$$

Then

(5.1)
$$\int_{\Omega} \langle (T_\omega + a_\omega LK^*)x, (T_\omega + a_\omega LK^*)x\rangle_{\mathcal{A}}\, d\mu(\omega) \leq \left(\sqrt{B} + \sqrt{R}\|K^*\|\right)^2 \langle x,x\rangle_{\mathcal{A}}.$$

On other hand, for all $x \in H$, we have,

$$\left(\int_{\Omega} \langle (T_\omega + a_\omega LK^*)x, (T_\omega + a_\omega LK^*)x\rangle_{\mathcal{A}}\, d\mu(\omega)\right)^{\frac{1}{2}}$$
$$= \|(T_\omega + a_\omega LK^*)x\|_{l^2(\Omega,\mathcal{H})}$$
$$\geq \|T_\omega x\|_{l^2(\Omega,\mathcal{H})} - \|a_\omega LK^* x\|_{l^2(\Omega,\mathcal{H})}$$
$$\geq \left(\int_{\Omega} \langle T_\omega x, T_\omega x\rangle_{\mathcal{A}}\, d\mu(\omega)\right)^{\frac{1}{2}} - \left(\int_{\Omega} \langle a_\omega LK^* x, a_\omega LK^* x\rangle_{\mathcal{A}}\, d\mu(\omega)\right)^{\frac{1}{2}}$$
$$\geq \sqrt{A}\,\langle K^*x, K^*x\rangle_{\mathcal{A}}^{\frac{1}{2}} - \sqrt{R}\,\langle K^*x, K^*x\rangle_{\mathcal{A}}^{\frac{1}{2}}$$
$$\geq (\sqrt{A} - \sqrt{R})\,\langle K^*x, K^*x\rangle_{\mathcal{A}}^{\frac{1}{2}}$$

So, if $R < A$, we have,

(5.2)
$$\int_{\Omega} \langle (T_\omega + a_\omega LK^*)x, (T_\omega + a_\omega LK^*)x\rangle_{\mathcal{A}}\, d\mu(\omega) \geq \left(\sqrt{A} - \sqrt{R}\right)^2 \langle K^*x, K^*x\rangle_{\mathcal{A}}.$$

From (5.1) and (5.2) we conclude that $\{T_\omega + a_\omega LK^*\}_{\omega\in\Omega}$ is an integral K-operator frame for $End_{\mathcal{A}}^*(\mathcal{H})$ if $R < A$, that is , if :

$$\int_{\Omega} |a_\omega|^2 d\mu(\omega) < \frac{A}{\|L\|^2}.$$

□

Theorem 5.2. *Let $K \in End_{\mathcal{A}}^*(\mathcal{H})$ and $\{T_\omega\}_{\omega\in\Omega}$ be an integral K-operator frame for $End_{\mathcal{A}}^*(\mathcal{H})$. Let $\{\Gamma_\omega\}_{\omega\in\Omega}$ be any family in $End_{\mathcal{A}}^*(\mathcal{H})$, and $\{a_\omega\}_{\omega\in\Omega}$, $\{b_\omega\}_{\omega\in\Omega} \subset \mathbb{R}$ be two positively confined sequences. If there exist constants α, β with $0 \leq \alpha, \beta < \frac{1}{2}$ such that,*

(5.3)
$$\int_{\Omega} \langle a_\omega T_\omega x - b_\omega \Gamma_\omega x, a_\omega T_\omega x - b_\omega \Gamma_\omega x\rangle_{\mathcal{A}}\, d\mu(\omega)$$
$$\leq \alpha \int_{\Omega} \langle a_\omega T_\omega x, a_\omega T_\omega x\rangle_{\mathcal{A}}\, d\mu(\omega) + \beta \int_{\Omega} \langle b_\omega \Gamma_\omega x, b_\omega \Gamma_\omega x\rangle_{\mathcal{A}}\, d\mu(\omega),$$

then $\{\Gamma_\omega\}_{\omega\in\Omega}$ is an integral K-operator frame for $End_{\mathcal{A}}^*(\mathcal{H})$.

Proof. Suppose (5.3) holds for the assumptions given in Theorem 5.2. Then for all $x \in H$ we have,

$$\begin{aligned}
&\int_{\Omega} \langle b_\omega \Gamma_\omega x, b_\omega \Gamma_\omega x \rangle_{\mathcal{A}} \, d\mu(\omega) \\
&\quad \leq 2 \Bigg(\int_{\Omega} \langle a_\omega T_\omega x, a_\omega T_\omega x \rangle_{\mathcal{A}} \, d\mu(\omega) \\
&\qquad + \int_{\Omega} \langle a_\omega T_\omega x - b_\omega \Gamma_\omega x, a_\omega T_\omega x - b_\omega \Gamma_\omega x \rangle_{\mathcal{A}} \, d\mu(\omega) \Bigg) \\
&\quad \leq 2 \Bigg(\int_{\Omega} \langle a_\omega T_\omega x, a_\omega T_\omega x \rangle_{\mathcal{A}} \, d\mu(\omega) + \alpha \int_{\Omega} \langle a_\omega T_\omega x, a_\omega T_\omega x \rangle_{\mathcal{A}} \, d\mu(\omega) \\
&\qquad + \beta \int_{\Omega} \langle b_\omega \Gamma_\omega x, b_\omega \Gamma_\omega x \rangle_{\mathcal{A}} \, d\mu(\omega) \Bigg).
\end{aligned}$$

Therefore,

$$\begin{aligned}
(1 - 2\beta) &\int_{\Omega} \langle b_\omega \Gamma_\omega x, b_\omega \Gamma_\omega x \rangle_{\mathcal{A}} \, d\mu(\omega)) \\
&\leq 2(1 + \alpha) \int_{\Omega} \langle a_\omega T_\omega x, a_\omega T_\omega x \rangle_{\mathcal{A}} \, d\mu(\omega).
\end{aligned} \tag{5.4}$$

This gives,

$$\begin{aligned}
(1 - 2\beta) &\left[\inf_{\omega \in \Omega} (b_\omega) \right]^2 \int_{\Omega} \langle \Gamma_\omega x, \Gamma_\omega x \rangle_{\mathcal{A}} \, d\mu(\omega) \\
&\leq 2(1 + \alpha) \left[\sup_{\omega \in \Omega} (a_\omega) \right]^2 \int_{\Omega} \langle T_\omega x, T_\omega x \rangle_{\mathcal{A}} \, d\mu(\omega).
\end{aligned}$$

Thus,

$$\int_{\Omega} \langle \Gamma_\omega x, \Gamma_\omega x \rangle_{\mathcal{A}} \, d\mu(\omega) \leq \frac{2(1 + \alpha) \left[\sup_{\omega \in \Omega} (a_\omega) \right]^2}{(1 - 2\beta) \left[\inf_{\omega \in \Omega} (b_\omega) \right]^2} \int_{\Omega} \langle T_\omega x, T_\omega x \rangle_{\mathcal{A}} \, d\mu(\omega). \tag{5.5}$$

Also, for all $x \in H$, we have,

$$\begin{aligned}
&\int_{\Omega} \langle a_\omega T_\omega x, a_\omega T_\omega x \rangle_{\mathcal{A}} \, d\mu(\omega) \\
&\quad \leq 2 \Bigg(\int_{\Omega} \langle a_\omega T_\omega x - b_\omega \Gamma_\omega x, a_\omega T_\omega x - b_\omega \Gamma_\omega x \rangle_{\mathcal{A}} \, d\mu(\omega)
\end{aligned}$$

$$+ \int_\Omega \langle b_\omega \Gamma_\omega x, b_\omega \Gamma_\omega x\rangle_{\mathcal{A}}\, d\mu(\omega)\Bigg)$$

$$\leq 2\Bigg(\alpha \int_\Omega \langle a_\omega T_\omega x, a_\omega T_\omega x\rangle_{\mathcal{A}}\, d\mu(\omega) + \beta \int_\Omega \langle b_\omega \Gamma_\omega x, b_\omega \Gamma_\omega x\rangle_{\mathcal{A}}\, d\mu(\omega)$$

$$+ \int_\Omega \langle b_\omega \Gamma_\omega x, b_\omega \Gamma_\omega x\rangle_{\mathcal{A}}\, d\mu(\omega)\Bigg).$$

Therefore,

$$(1-2\alpha)\left[\inf_{\omega\in\Omega}(a_\omega)\right]^2 \int_\Omega \langle T_\omega x, T_\omega x\rangle_{\mathcal{A}}\, d\mu(\omega)$$

$$\leq 2(1+\beta)\left[\sup_{\omega\in\Omega}(b_\omega)\right]^2 \int_\Omega \langle \Gamma_\omega x, \Gamma_\omega x\rangle_{\mathcal{A}}\, d\mu(\omega).$$

This gives:
(5.6)

$$\frac{(1-2\alpha)\left[\inf_{\omega\in\Omega}(a_\omega)\right]^2}{2(1+\beta)\left[\sup_{\omega\in\Omega}(b_\omega)\right]^2} \int_\Omega \langle T_\omega x, T_\omega x\rangle_{\mathcal{A}}\, d\mu(\omega) \leq \int_\Omega \langle \Gamma_\omega x, \Gamma_\omega x\rangle_{\mathcal{A}}\, d\mu(\omega)$$

From (5.5) and (5.6) we conclude,

$$\frac{(1-2\alpha)\left[\inf_{\omega\in\Omega}(a_\omega)\right]^2}{2(1+\beta)\left[\sup_{\omega\in\Omega}(b_\omega)\right]^2} \int_\Omega \langle T_\omega x, T_\omega x\rangle_{\mathcal{A}}\, d\mu(\omega)$$

$$\leq \int_\Omega \langle \Gamma_\omega x, \Gamma_\omega x\rangle_{\mathcal{A}}\, d\mu(\omega)$$

$$\leq \frac{2(1+\alpha)\left[\sup_{\omega\in\Omega}(a_\omega)\right]^2}{(1-2\beta)\left[\inf_{\omega\in\Omega}(b_\omega)\right]^2} \int_\Omega \langle T_\omega x, T_\omega x\rangle_{\mathcal{A}}\, d\mu(\omega).$$

Hence, $\{\Gamma_\omega\}_{\omega\in\Omega}$ is an integral K-operator frame for $End^*_{\mathcal{A}}(\mathcal{H})$. □

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[1] Department of Mathematics, Faculty of Science, University of Ibn Tofail, B.P. 133, Kenitra, Morocco.
E-mail address: hlabrigui75@gmail.com

[2] Department of Mathematics, Faculty of Science, University of Ibn Tofail, B.P. 133, Kenitra, Morocco.
E-mail address: samkabbaj@yahoo.fr