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Characteristics of Solutions of Fractional Hybrid Integro-Differential Equations in Banach Algebra

Ahmed El-Sayed[1], Hind Hashem[2] and Shorouk Al-Issa[3*]

Abstract. In this paper, we discuss the existence results for a class of hybrid initial value problems of Riemann-Liouville fractional differential equations. Our investigation is based on the Dhage hybrid fixed point theorem, remarks and some special cases will be discussed. The continuous dependence of the unique solution on one of its functions will be proved.

1. Introduction

Quadratic perturbation of nonlinear differential equations is quite worth studying, as one of the most important types of perturbations, we call it hybrid differential equations. The importance of the investigations of hybrid differential equations lies in the fact that they include several dynamic systems as special cases. This class of hybrid differential equations includes the perturbations of original differential equations in different ways. The consideration of hybrid differential equations is implicit in the works of Krasnoselskii [18] and extensively treated in several papers on hybrid differential equations with different perturbations, see [6–16, 20, 21], and the references therein.

In recent years, a hybrid differential equation (quadrature perturbations of a nonlinear differential equation) has attracted much attention. Dhage and Lakshmikantham [7] started working on hybrid equations. They introduced a new category of nonlinear differential equations called ordinary hybrid differential equations and studied the existence of extremal solutions for this boundary value problem by establishing some

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fundamental differential inequalities. Zhao, Sun, Han, and Li [21], generalized Dhage's work and discussed the fractional hybrid differential equations involving Riemann Liouville differential operators. The existence theorem for fractional hybrid differential equations is proved under mixed Lipschitz and Carathodory conditions. Next, Lu et al. [17] developed the fractional hybrid differential equation involving the Riemann-Liouville differential operators of order $0 < \alpha < 1$, with linear perturbations of the second type. They established the existence and uniqueness results under the ϕ-Lipschitz condition.

Also, M.A. Darwish, and K. Sadarangani et. al [8], studied the existence of the hybrid fractional pantograph equation

$$\begin{cases} D^{\alpha}_{0^+}\left(\frac{x(t)}{f(t,x(t),x(\tau(t))}\right) = g(t,x(t),x(\sigma(t)), & t \in [0,1], \\ x(0) = 0, \end{cases}$$

where $\alpha, \tau, \sigma \in (0,1)$ and $D^{\alpha}_{0^+}$ denotes the Riemann-Liouville fractional derivative. The results obtained by using the technique of measures of non-compactness in the Banach algebras and a fixed point theorem for the product of two operators verifying a Darbo type condition.

In [1], B. Ahmad, S.K. Ntouyas, J. Tariboon, discussed the existence of solutions by using the hybrid fixed point theorems of Dhage [5] for the sum of three operators in a Banach algebra for the following nonlocal boundary value problem of hybrid fractional integro-differential equations

$$\begin{cases} {}^cD^{\alpha}\left(\frac{x(t)-\sum\limits_{j=1}^{m} I^{\beta_j}h_j(t,x(t))}{f(t,x(t))}\right) = g(t,x(t)), & t \in [0,1], \\ x(0) = \mu(x), x(1) = A, \end{cases} \tag{1.1}$$

where ${}^cD^{\alpha}$ denotes the Caputo fractional derivative of order α, $1 < \alpha \leq 2$, I^{β_j} is the Riemann Liouville fractional integral of order $\beta_j, j = 1, 2, \ldots, m$.

Motivated by these works, we focus on a class of initial-value problems of hybrid fractional differential equations (FHDE) involving Riemann Liouville differential operators given by

$$\begin{cases} D^{\alpha}\left(\frac{x(t)-\sum\limits_{i=1}^{m} k_i(t,x(t)).I^{\gamma_i}h_i(t,x(t))}{g(t,x(t))}\right) = f_1(t, I^{\beta}f_2(t,x(t))), & t \in J = [0,T], \\ \left[x(t) - \sum\limits_{i=1}^{m} k_i(t,x(t)).I^{\gamma_i}h_i(t,x(t))\right]_{t=0} = 0, \end{cases} \tag{1.2}$$

where D^{α} denotes the Riemann-Liouville fractional derivative of order α, $0 < \alpha < 1$, I^{γ_i} and I^{β} are the Riemann-Liouville fractional integral of orders $\gamma_i, \beta \in (0,1)$, $g(t,x(t)), \in C(J \times R, R \setminus \{0\})$, $k_i(t,x(t))$ and

$h_i(t, x(t)) \in C(J \times R, R)$, $h_i(0,0) = 0, i = 1, 2, \ldots, m$, and $f_j(t, x(t)) \in C(J \times R, R), j = 1, 2$, by using a hybrid fixed point theorem for three operators in a Banach algebra due to Dhage [5], and under mixed Lipschitz and Carathodory conditions. Problem (1.2) contains many integral and functional differential equations that appear in applications of nonlinear analysis which seem to be important in the study of dynamics of biological systems [2], some particular cases are presented in Section 3.

Here, we study the existence and uniqueness of solution for the initial value problem of hybrid fractional differential equations (FHDE) (1.2) and prove continuous dependence on one of its functions.

Also, this initial value problem can be studied under another sequence of assumptions as shown in Remark 3.1.

The paper organized as follows: In Section 2, we prove an auxiliary lemma related to the linear variant of the problem (1.2) and state sufficient conditions which guarantee the existence of solutions to the problem (1.2). Where in section 3 particular cases and remarks presented. Section 4, deals with the existence of continuous dependence of unique solutions for (FHDE) (1.2) on function f_1. Our conclusion is presented in Section 5.

2. Fractional Hybrid Differential Equation

Let $X = C(J, R)$ be the space of all real-valued continuous functions on J, we equip the space X with the norm $||x|| = \sup_{t \in J} |x(t)|$. Clearly, $C(J, R)$ is a complete normed algebra with respect to this supremum norm.

Definition 2.1. By a solution of the FHDE (1.2) we mean a function $x \in C(J, R)$ such that

(i) the function $t \to \frac{x(t) - \sum_{i=1}^{m} k_i(t,x(t)).I^{\gamma_i} h_i(t,x(t))}{g(t,x(t))}$ is continuous for each $x \in C(J, R)$, and

(ii) x satisfies equations in (1.2).

In this section, we consider the initial value problem (1.2). The hybrid fixed point theorem for three operators in a Banach algebra, due to Dhage [5] will be used to prove the existence result for the initial value problem (1.2).

Consider the following assumptions:

(A_1) The functions $k_i : J \times R \to R$, and $h_i : J \times R \to R, i = 1, 2, \ldots, m$, are continuous and there exist positive functions $\lambda_i(t)$ and $\psi_i(t)$ with norms $\|\lambda_i\| and \|\psi_i\|$ respectively such that

$$|k_i(t, x) - k_i(t, y)| \leq \lambda_i(t)|x - y|,$$

$$|h_i(t,x) - h_i(t,y)| \leq \psi_i(t)|x-y|.$$

for all $t \in J$ and $x, y \in R$.

(A_2) $g : J \times R \to R\backslash\{0\}$ is continuous with $\|g\| = \sup_{(t,x)\in J\times R} |g(t,x)|$, and there exist a positive function $\omega(t)$ with norm $\|\omega\|$ such that

$$|g(t,x) - g(t,y)| \leq \omega(t)|x-y|.$$

(A_3) $f_j : J \times R \to R, j = 1, 2$, satisfy Carathéodory condition i.e., f_j are measurable functions in t for any $x \in R$ and continuous in x for almost all $t \in J$. Moreover, there exists three functions $t \to a(t), t \to b(t)$ and $t \to m(t)$ such that

$$|f_1(t,x)| \leq a(t) + b(t)|x|, \forall (t,x) \in J \times R,$$
$$|f_2(t,x)| \leq m(t), \forall (t,x) \in J \times R,$$

where $a(\cdot), m(\cdot)$ and $b(\cdot)$ are measurable and bounded on J such that: $|a(\cdot)| \leq M_1$ and $|m(\cdot)| \leq M_2$.

(A_4) Let r be a positive root of the equation

$$\begin{aligned}\text{(2.1)}\qquad &\sum_{i=1}^{m} \frac{\|\lambda_i\|\|\psi_i\|T^{\gamma_i}}{\Gamma(\gamma_i+1)} r^2 + \left(1 - \left[\sum_{i=1}^{m} \frac{(\|\lambda_i\|H_i + \|\psi_i\|K_i)T^{\gamma_i}}{\Gamma(\gamma_i+1)}\right]\right) r \\ &\quad + \|g\| \left(\frac{M_1 T^{\alpha}}{\Gamma(\alpha+1)} + \frac{b_1 M_2 T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\right) + \sum_{i=1}^{m} \frac{H_i K_i T^{\gamma_i}}{\Gamma(\gamma_i+1)} \\ &= 0,\end{aligned}$$

where $K_i = \sup_{t\in J} |k_i(t,0)|$, and $H_i = \sup_{t\in J} |h_i(t,0)|$. Let

$$\begin{aligned}&\left(1 - \left[\sum_{i=1}^{m} \frac{(\|\lambda_i\|H_i + \|\psi_i\|K_i)T^{\gamma_i}}{\Gamma(\gamma_i+1)}\right]\right) \\ &> \sqrt{4\left(\sum_{i=1}^{m} \frac{\|\lambda_i\|\|\psi_i\|T^{\gamma_i}}{\Gamma(\gamma_i+1)}\right)\left(\|g\|\left(\frac{M_1 T^{\alpha}}{\Gamma(\alpha+1)} + \frac{b_1 M_2 T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\right) + \sum_{i=1}^{m} \frac{H_i K_i T^{\gamma_i}}{\Gamma(\gamma_i+1)}\right)}.\end{aligned}$$

Then

$$\begin{aligned}&2\left[\sum_{i=1}^{m} \frac{(\|\lambda_i\|H_i + \|\psi_i\|K_i)T^{\gamma_i}}{\Gamma(\gamma_i+1)}\right] - \left[\sum_{i=1}^{m} \frac{(\|\lambda_i\|H_i + \|\psi_i\|K_i)T^{\gamma_i}}{\Gamma(\gamma_i+1)}\right]^2 \\ &\quad + 4\left(\sum_{i=1}^{m} \frac{\|\lambda_i\|\|\psi_i\|T^{\gamma_i}}{\Gamma(\gamma_i+1)}\right)\left[\|g\|\left(\frac{M_1 T^{\alpha}}{\Gamma(\alpha+1)} + \frac{b_1 M_2 T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\right) + \sum_{i=1}^{m} \frac{H_i K_i T^{\gamma_i}}{\Gamma(\gamma_i+1)}\right] \\ &< 1,\end{aligned}$$

i,e.,

$$2\left[\sum_{i=1}^{m}\frac{(\|\lambda_i\|H_i+\|\psi_i\|K_i)T^{\gamma_i}}{\Gamma(\gamma_i+1)}\right]-\left[\sum_{i=1}^{m}\frac{(\|\lambda_i\|H_i+\|\psi_i\|K_i)T^{\gamma_i}}{\Gamma(\gamma_i+1)}\right]^2$$
$$+4\left(\sum_{i=1}^{m}\frac{\|\lambda_i\|\|\psi_i\|T^{\gamma_i}}{\Gamma(\gamma_i+1)}\right)\sum_{i=1}^{m}\frac{H_iK_iT^{\gamma_i}}{\Gamma(\gamma_i+1)}$$
$$+4\|g\|\left(\sum_{i=1}^{m}\frac{\|\lambda_i\|\|\psi_i\|T^{\gamma_i}}{\Gamma(\gamma_i+1)}\right)\left(\frac{M_1T^{\alpha}}{\Gamma(\alpha+1)}+\frac{b_1M_2T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\right)$$
$$<1.$$

To prove our main existence result for continuous solutions of the differential equations of fractional order (1.2), the following useful lemma is immediate and follows from the theory of fractional calculus.

Lemma 2.2. *Assume that hypotheses* $(A_1)-(A_4)$ *hold,* $\alpha,\beta,and\gamma_i\in(0,1),i=1,2,\ldots,m$. *If function* $x\in C(J,R)$ *is a solution of the FHDE* (1.2)*, then it satisfies the following quadratic fractional integral equation*

$$x(t)=\sum_{i=1}^{m}k_i(t,x(t))I^{\gamma_i}h_i(t,x(t))+g(t,x(t))I^{\alpha}f_1(t,I^{\beta}f_2(t,x(t))). \tag{2.2}$$

Proof. Assume that x is a solution of the FHDE (1.2). Applying Riemann-Liouville fractional integral of order α on both sides of (1.2), we obtain

$$I^{\alpha}D^{\alpha}\left(\frac{x(t)-\sum_{i=1}^{m}k_i(t,x(t))I^{\gamma_i}h_i(t,x(t))}{g(t,x(t))}\right)=I^{\alpha}f_1(t,I^{\beta}f_2(t,x(t))).$$

So, we conclude that

$$\frac{1}{g(t,x(t))}\left(x(t)-\sum_{i=1}^{m}k_i(t,x(t))I^{\gamma_i}h_i(t,x(t))\right)$$
$$-\frac{1}{\Gamma(\alpha)}\left(I^{1-\alpha}\left[\frac{x(t)-\sum_{i=1}^{m}k_i(t,x(t))I^{\gamma_i}h_i(t,x(t))}{g(t,x(t))}\right]_{t=0}t^{\alpha-1}\right)$$
$$=I^{\alpha}f_1(t,I^{\beta}f_2(t,x(t))),\quad t\in J.$$

Since

$$\left[\frac{1}{g(t,x(t))}\left(x(t)-\sum_{i=1}^{m}k_i(t,x(t))I^{\gamma_i}h_i(t,x(t))\right)\right]_{t=0}$$

$$= \frac{1}{g(0, x(0))} \left[x(t) - \sum_{i=1}^{m} k_i(t, x(t)) I^{\gamma_i} h_i(t, x(t)) \Big) \right]_{t=0}$$
$$= \frac{0}{g(0, 0)} = 0,$$

(due to the fact that $g(0, 0) \neq 0$), then we have

$$\frac{x(t) - \sum\limits_{i=1}^{m} k_i(t, x(t)) I^{\gamma_i} h_i(t, x(t))}{g(t, x(t))} = I^{\alpha} f_1(t, I^{\beta} f_2(t, x(t)),$$

i.e.,

$$x(t) = g(t, x(t)) I^{\alpha} f_1(t, I^{\beta} f_2(t, x(t)) + \sum_{i=1}^{m} k_i(t, x(t)) I^{\gamma_i} h_i(t, x(t)), t \in J.$$

Thus, eq.(2.2) holds.

Conversely, assume that x satisfies eq.(2.2). Then dividing by $g(t, x(t))$ and applying D^{α} on both sides of eq.(2.2), so eq.(1.2) is satisfied. Again, substituting $t = 0$ in eq.(2.2) yields

$$\frac{\left(x(t) - \sum\limits_{i=1}^{m} k_i(t, x(t)) I^{\gamma_i} h_i(t, x(t)) \right) \Big|_{t=0}}{g(0, x(0))} = I^{\alpha} f_1(t, I^{\beta} f_2(t, x(t)) \Big|_{t=0},$$

for $i = 1, 2, \ldots, m$. The proof is completed. □

At this stage, our target is to prove the following existence theorem.

Theorem 2.3. *Assume that the hypotheses $(A_1) - (A_4)$ hold. Then the FHDE (1.2) has at least one solution defined on J.*

Proof. By Lemma 2.2, problem (1.2) is equivalent to the quadratic fractional integral equation (2.2).

Define a subset S of X as

$$S := \{x \in X, \|x\| \leq r\},$$

where r satisfies inequality (2.1). Clearly S is closed, convex, and bounded subset of the Banach space X. Consider the operators $A : X \to X, B : S \to X$ and $C : X \to X$ defined by:

$$(Ax)(t) = g(t, x(t)), \quad t \in J, \tag{2.3}$$

$$(Bx)(t) = I^{\alpha} f_1(t, I^{\beta} f_2(t, x(t))), \quad t \in J, \tag{2.4}$$

$$(Cx)(t) = \sum_{i=1}^{m} k_i(t, x(t)) I^{\gamma_i} h_i(t, x(t)) \tag{2.5}$$

$$= \sum_{i=1}^{m} k_i(t, x(t)) \int_0^t \frac{(t-s)^{\gamma_i - 1}}{\Gamma(\gamma_i)} h_i(s, x(s)) ds,$$

$t \in J, (i = 1, 2, \ldots, m)$.

Then the integral equation (2.2) is transformed into the operator equation as:

$$x(t) = Ax(t) \cdot Bx(t) + Cx(t), \quad t \in J. \tag{2.6}$$

We shall show that A, B and C satisfy all the conditions of theorem for three operators in a Banach algebra, due to Dhage [5]. This will be achieved in the following series of steps.

Step 1. We first show that A and C are Lipschitzian on X. To see this, let $x, y \in X$. So

$$\begin{aligned} |Ax(t) - Ay(t)| &= |g(t, x(t)) - g(t, y(t))| \\ &\leq \omega(t)|x(t) - y(t)| \leq |\omega(t)||x(t) - y(t)|. \end{aligned}$$

Taking the supremum over $t \in J$, we get

$$\|Ax - Ay\| \leq \|\omega\| \|x - y\|, \quad \forall x, y \in X.$$

Therefore, A is Lipschitzian on X with Lipschitz constant $\|\omega\|$.

Analogously, for any $x, y \in X$, we have

$$\begin{aligned} &|Cx(t) - Cy(t)| \\ &\quad = \left| \sum_{i=1}^{m} k_i(t, x(t)) I^{\gamma_i} h_i(t, x(t)) - \sum_{i=1}^{m} k_i(t, y(t)) I^{\gamma_i} h_i(t, y(t)) \right| \\ &\quad \leq \left| \sum_{i=1}^{m} k_i(t, x(t)) I^{\gamma_i} h_i(t, x(t)) - \sum_{i=1}^{m} k_i(t, x(t)) I^{\gamma_i} h_i(t, y(t)) \right| \\ &\qquad + \left| \sum_{i=1}^{m} k_i(t, x(t)) I^{\gamma_i} h_i(t, y(t)) - \sum_{i=1}^{m} k_i(t, y(t)) I^{\gamma_i} h_i(t, y(t)) \right| \\ &\quad \leq \sum_{i=1}^{m} |k_i(t, x(t))| |I^{\gamma_i} h_i(t, x(t)) - I^{\gamma_i} h_i(t, y(t))| \\ &\qquad + \sum_{i=1}^{m} |k_i(t, x(t)) - k_i(t, y(t))| |I^{\gamma_i} h_i(t, y(t))| \\ &\quad \leq \sum_{i=1}^{m} [|k_i(t, x(t)) - k_i(t, 0)| + |k_i(t, 0)|] \int_0^t \frac{(t-s)^{\gamma_i - 1}}{\Gamma(\gamma_i)} \psi_i(t) |x(s) - y(s)| ds \\ &\qquad + \sum_{i=1}^{m} \lambda_i(t) |x(t) - y(t)| \int_0^t \frac{(t-s)^{\gamma_i - 1}}{\Gamma(\gamma_i)} [|h_i(s, y(s)) - h_i(s, 0)| + h_i(s, 0)|] ds \\ &\quad \leq \sum_{i=1}^{m} [|\lambda_i(t)||x(t)| + K_i] \int_0^t \frac{(t-s)^{\gamma_i - 1}}{\Gamma(\gamma_i)} |\psi_i(s)||x(s) - y(s)| ds \end{aligned}$$

$$
\begin{aligned}
&+\sum_{i=1}^{m}|\lambda_i(t)||x(t)-y(t)|\int_0^t\frac{(t-s)^{\gamma_i-1}}{\Gamma(\gamma_i)}[|\psi_i(s)||y(s)|+H_i]ds\\
&\le\|x-y\|\sum_{i=1}^{m}[\|\lambda_i\|\|x\|+K_i]\frac{\|\psi_i\|T^{\gamma_i}}{\Gamma(\gamma_i+1)}\\
&+\sum_{i=1}^{m}\|\lambda_i\|[\|\psi_i\|\|y\|+H_i]\|x-y\|\frac{T^{\gamma_i}}{\Gamma(\gamma_i+1)}\\
&\le\|x-y\|\sum_{i=1}^{m}\frac{[\|\lambda_i\|r+K_i]\|\psi_i\|+\|\lambda_i\|[\|\psi_i\|r+H_i]}{\Gamma(\gamma_i+1)}T^{\gamma_i}.
\end{aligned}
$$

Taking the supremum over $t\in J$, we get

$$
\|Cx-Cy\|\le\sum_{i=1}^{m}\frac{[\|\lambda_i\|r+K_i]\|\psi_i\|+\|\lambda_i\|[\|\psi_i\|r+H_i]}{\Gamma(\gamma_i+1)}T^{\gamma_i}\|x-y\|.
$$

This shows that C is a Lipschitz mapping on X with the Lipschitz constant

$$
\sum_{i=1}^{m}\frac{[\|\lambda_i\|r+K_i]\|\psi_i\|+\|\lambda_i\|[\|\psi_i\|r+H_i]}{\Gamma(\gamma_i+1)}T^{\gamma_i}.
$$

Step 2. We show that B is a compact and continuous operator on S into X.

First we show that B is continuous on X. Let $\{x_n\}$ be a sequence in S converging to a point $x\in S$, let us assume that $t\in J$ and since $f_2(t,x(t))$ is continuous in X, then $f_2(t,x_n(t))$ converges to $f_2(t,x(t))$, (see assumption (A_3)). Applying Lebesgue Dominated Convergence Theorem, we get

$$
\lim_{n\to\infty}I^{\beta}f_2(t,x_n(t))=I^{\beta}f_2(t,x(t)).
$$

Also, since $f_1(t,x(t))$ is continuous in x, then using the properties of the fractional-order integral and applying Lebesgue Dominated Convergence Theorem, we get

$$
\begin{aligned}
\lim_{n\to\infty}Bx_n(t)&=\lim_{n\to\infty}I^{\alpha}f_1\left(t,I^{\beta}f_2(t,x_n(t))\right)\\
&=I^{\alpha}f_1\left(t,I^{\beta}f_2(t,x(t))\right)\\
&=Bx(t).
\end{aligned}
$$

Thus, $Bx_n\to Bx$ as $n\to\infty$ uniformly on R, and hence B is a continuous operator on S into S.

Now, we show that B is a compact operator on S. It is enough to show that $B(S)$ is a uniformly bounded and equicontinuous set in X. On the one hand, let $x\in S$ be arbitrary. Then by hypothesis (A_2),

$$
|Bx(t)|\le\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}|f_1(s,I^{\beta}f_2(s,x_n(s))|ds
$$

$$
\begin{aligned}
&\leq \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}[a(s)+b(s)I^{\beta}|f_2(s,x(s)|]ds\\
&\leq \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}a(s)ds+\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}|b(s)|I^{\beta}|f_2(s,x(s))|ds\\
&\leq I^{\alpha}a(t)+\|b\|\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}I^{\beta}|m(s)|ds\\
&\leq I^{\alpha}|a(t)|+\|b\|I^{\alpha+\beta}|m(t)|\\
&\leq M_1\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}ds+\|b\|M_2\int_0^t \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}ds\\
&\leq M_1\frac{T^{\alpha}}{\Gamma(\alpha+1)}+\|b\|M_2\frac{T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\\
&=\Lambda.
\end{aligned}
$$

Taking supremum over $t \in J$, we have

$$\|Bx(t)\| \leq \Lambda,$$

for all $x \in S$. This shows that B is uniformly bounded on S.

Now, we proceed to show that $B(S)$ is also equi-continuous set in X. Let $t_1, t_2 \in J$, and $x \in S$. (without loss of generality assume that $t_1 < t_2$), then we have

$$
\begin{aligned}
&(Bx)(t_2)-(Bx)(t_1)\\
&\quad\leq \int_0^{t_2} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}f_1(s,I^{\beta}f_2(s,x(s)))ds\\
&\qquad-\int_0^{t_1} \frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)}f_1(s,I^{\beta}f_2(s,x(s)))ds\\
&\quad\leq \int_0^{t_1} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}f_1(s,I^{\beta}f_2(s,x(s)))ds\\
&\qquad+\int_{t_1}^{t_2} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}f_1(s,I^{\beta}f_2(s,x(s)))ds\\
&\qquad-\int_0^{t_1} \frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)}f_1(s,I^{\beta}f_2(s,x(s))ds\\
&\quad\leq \int_0^{t_1} \frac{(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}}{\Gamma(\alpha)}f_1(s,I^{\beta}f_2(s,x(s)))ds\\
&\qquad+\int_{t_1}^{t_2} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}f_1(s,I^{\beta}f_2(s,x(s)))ds,
\end{aligned}
$$

and

$|(Bx)(t_2)-(Bx)(t_1)|$

$$
\begin{aligned}
&\leq \int_0^{t_1} \frac{(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}}{\Gamma(\alpha)} |f_1(s, I^\beta f_2(s, x(s)))| ds \\
&\quad + \int_{t_1}^{t_2} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)} |f_1(s, I^\beta f_2(s, x(s)))| ds \\
&\leq \int_0^{t_1} \frac{(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}}{\Gamma(\alpha)} \Big[|a(s)| + |b(s)| I^\beta |f_2(s, x(s))|\Big] ds \\
&\quad + \int_{t_1}^{t_2} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)} \Big[|a(s)| + |b(s)| I^\beta |f_2(s, x(s))|\Big] ds \\
&\leq \|a\| \left[\int_0^{t_1} \frac{(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}}{\Gamma(\alpha)} ds + \int_{t_1}^{t_2} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)} ds\right] \\
&\quad + \|b\| \Bigg[\int_0^{t_1} \frac{(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}}{\Gamma(\alpha)} I^\beta |f_2(s, x(s))| ds \\
&\quad + \int_{t_1}^{t_2} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)} I^\beta |f_2(s, x(s))| ds\Bigg] \\
&\leq \|a\| \left(\frac{|t_2^\alpha - t_1^\alpha - 2(t_2-t_1)^\alpha|}{\Gamma(\alpha+1)}\right) \\
&\quad + \|b\| \Bigg[\int_0^{t_1} \frac{(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}}{\Gamma(\alpha)} I^\beta m(s) ds \\
&\quad + \int_{t_1}^{t_2} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)} I^\beta m(s) ds\Bigg] \\
&\leq \|a\| \left(\frac{|t_2^\alpha - t_1^\alpha - 2(t_2-t_1)^\alpha|}{\Gamma(\alpha+1)}\right) \\
&\quad + \|b\| M_2 \Bigg[\int_0^{t_1} \frac{(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}}{\Gamma(\alpha)} \int_0^s \frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)} d\tau ds \\
&\quad + \int_{t_1}^{t_2} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)} \int_0^s \frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)} d\tau ds\Bigg] \\
&\leq \|a\| \left(\frac{|t_2^\alpha - t_1^\alpha - 2(t_2-t_1)^\alpha|}{\Gamma(\alpha+1)}\right) \\
&\quad + \|b\| M_2 \Bigg[\int_0^{t_1} \frac{(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}}{\Gamma(\alpha)} \frac{(s)^\beta}{\Gamma(\beta+1)} ds \\
&\quad + \int_{t_1}^{t_2} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)} \frac{(s)^\beta}{\Gamma(\beta+1)} ds\Bigg] \\
&\leq \|a\| \left(\frac{|t_2^\alpha - t_1^\alpha - 2(t_2-t_1)^\alpha|}{\Gamma(\alpha+1)}\right)
\end{aligned}
$$

$$
\begin{aligned}
&+\|b\|M_2\frac{T^\beta}{\Gamma(\beta+1)}\Big[\int_0^{t_1}\frac{(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}}{\Gamma(\alpha)}ds\\
&+\int_{t_1}^{t_2}\frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}ds\Big]\\
\le{}&\|a\|\left(\frac{|t_2^\alpha-t_1^\alpha-2(t_2-t_1)^\alpha|}{\Gamma(\alpha+1)}\right)+\|b\|M_2\left(\frac{|t_2^\alpha-t_1^\alpha-2(t_2-t_1)^\alpha|T^\beta}{\Gamma(\alpha+1)\Gamma(\beta+1)}\right),
\end{aligned}
$$

i.e.,

$$
\begin{aligned}
&|(Bx)(t_2)-(Bx)(t_1)|\\
&\le\|a\|\left(\frac{|t_2^\alpha-t_1^\alpha-2(t_2-t_1)^\alpha|}{\Gamma(\alpha+1)}\right)+\|b\|M_2\left(\frac{|t_2^\alpha-t_1^\alpha-2(t_2-t_1)^\alpha|T^\beta}{\Gamma(\alpha+1)\Gamma(\beta+1)}\right),
\end{aligned}
$$

which is independent of $x\in S$. Hence, for $\epsilon>0$, there exists a $\delta>0$ such that

$$
|t_2-t_1|<\delta \quad\Rightarrow\quad |(Bx)(t_2)-(Bx)(t_1)|<\epsilon,
$$

for all $t_1,t_2\in J$ and for all $x\in S$. This shows that $B(S)$ is an equicontinuous set in X. Now, the set $B(S)$ is a uniformly bounded and equicontinuous set in X, so it is compact by the Arzela-Ascoli theorem. As a result, B is a complete continuous operator on S.

Step 3. Let $x\in X$ and $y\in S$ be arbitrary elements such that $x=AxBy+Cx$. Then we have

$$
\begin{aligned}
&|x(t)|\\
&\le|Ax(t)||By(t)|+|Cx(t)|\\
&\le|g(t,x(t))|\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}|f_1(s,I^\beta f_2(s,y(s)))|ds\\
&\quad+\sum_{i=1}^m k_i(t,x(t))I^{\gamma_i}|h_i(t,x(t))|\\
&\le|g(t,x(t))|\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}|f_1(s,I^\beta f_2(s,y(s))|ds\\
&\quad+\sum_{i=1}^m(|k_i(t,x(t))-k_i(t,0)|+|k_i(t,0)|)\int_0^t\frac{(t-s)^{\gamma_i-1}}{\Gamma(\gamma_i)}(|h_i(s,x(s))\\
&\quad-h_i(s,0)|+|h_i(s,0)|)ds\\
&\le|g(t,x(t))|\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}(|a(s)|+|b(s)|I^\beta|f_2(s,y(s)|)ds\\
&\quad+\sum_{i=1}^m(\lambda_i(t)|x(t)|+K_i)\int_0^t\frac{(t-s)^{\gamma_i-1}}{\Gamma(\gamma_i)}(\psi_i(s)|x(s)|+H_i)ds
\end{aligned}
$$

$$
\begin{aligned}
&\leq |g(t,x(t))| \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}[|a(s)| + |b(s)| I^\beta m(s)]ds \\
&\quad + \sum_{i=1}^m (|\lambda_i(t)||x(t)| + K_i) \int_0^t \frac{(t-s)^{\gamma_i-1}}{\Gamma(\gamma_i)}(|\psi_i(s)||x(s)| + H_i)ds \\
&\leq \|g\|[I^\alpha a(t) + \|b\| I^{\alpha+\beta} m(t)] \\
&\quad + \sum_{i=1}^m (\|\lambda_i\|\|x\| + K_i)(\|\psi_i\|\|x\| + H_i) \int_0^t \frac{(t-s)^{\gamma_i-1}}{\Gamma(\gamma_i)} ds \\
&\leq \|g\| \left[M_1 \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} ds + \|b\| M_2 \int_0^t \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} ds \right] \\
&\quad + \sum_{i=1}^m \frac{(\|\lambda_i\| r + K_i)(\|\psi_i\| r + H_i) T^{\gamma_i}}{\Gamma(\gamma_i+1)} \\
&\leq \|g\| \left(M_1 \frac{T^\alpha}{\Gamma(\alpha+1)} + \|b\| M_2 \frac{T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} \right) \\
&\quad + \sum_{i=1}^m \frac{(\|\lambda_i\| r + K_i)(\|\psi_i\| r + H_i) T^{\gamma_i}}{\Gamma(\gamma_i+1)}.
\end{aligned}
$$

Taking supremum over $t \in J$, we have

$$
\begin{aligned}
\|x(t)\| &\leq \sum_{i=1}^m \frac{(\|\lambda_i\| r + K_i)(\|\psi_i\| r + H_i) T^{\gamma_i}}{\Gamma(\gamma_i+1)} \\
&\quad + \|g\| \left(\frac{M_1 T^\alpha}{\Gamma(\alpha+1)} + \frac{\|b\| M_2 T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} \right) \\
&\leq r.
\end{aligned}
$$

Therefore, $x \in S$.

Step 4. Finally we shall show that $\delta M + \rho < 1$ holds.
Since

$$
\begin{aligned}
M &= \|B(S)\| \\
&= \sup_{x \in S} \left\{ \sup_{t \in J} |Bx(t)| \right\} \\
&\leq \frac{M_1 T^\alpha}{\Gamma(\alpha+1)} + \frac{\|b\| M_2 T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)},
\end{aligned}
$$

and by (A_4) we have

$$
2\left[\sum_{i=1}^m \frac{(\|\lambda_i\| H_i + \|\psi_i\| K_i) T^{\gamma_i}}{\Gamma(\gamma_i+1)} \right] - \left[\sum_{i=1}^m \frac{(\|\lambda_i\| H_i + \|\psi_i\| K_i) T^{\gamma_i}}{\Gamma(\gamma_i+1)} \right]^2
$$

$$
\begin{aligned}
&+4\left(\sum_{i=1}^{m}\frac{(\|\lambda_i\|\|\psi_i\|T^{\gamma_i}}{\Gamma(\gamma_i+1)}\right)\sum_{i=1}^{m}\frac{H_iK_iT^{\gamma_i}}{\Gamma(\gamma_i+1)}\\
&+4\|g\|\left(\sum_{i=1}^{m}\frac{(\|\lambda_i\|\|\psi_i\|T^{\gamma_i}}{\Gamma(\gamma_i+1)}\right)\left(\frac{M_1T^{\alpha}}{\Gamma(\alpha+1)}+\frac{b_1M_2T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\right)\\
&<1,
\end{aligned}
$$

$$
\begin{aligned}
2&\left[\sum_{i=1}^{m}\frac{(\|\lambda_i\|H_i+\|\psi_i\|K_i)T^{\gamma_i}}{\Gamma(\gamma_i+1)}\right]-\left[\sum_{i=1}^{m}\frac{(\|\lambda_i\|H_i+\|\psi_i\|K_i)T^{\gamma_i}}{\Gamma(\gamma_i+1)}\right]^2\\
&+4\left(\sum_{i=1}^{m}\frac{\|\lambda_i\|\|\psi_i\|T^{\gamma_i}}{\Gamma(\gamma_i+1)}\right)\sum_{i=1}^{m}\frac{H_iK_iT^{\gamma_i}}{\Gamma(\gamma_i+1)}+4\|g\|\left(\sum_{i=1}^{m}\frac{(\|\lambda_i\|\|\psi_i\|T^{\gamma_i}}{\Gamma(\gamma_i+1)}\right)M\\
&<1,
\end{aligned}
$$

with $\delta=4\|g\|\left(\sum_{i=1}^{m}\frac{(\|\lambda_i\|\|\psi_i\|T^{\gamma_i}}{\Gamma(\gamma_i+1)}\right)$, and

$$
\begin{aligned}
\rho=2&\left[\sum_{i=1}^{m}\frac{(\|\lambda_i\|H_i+\|\psi_i\|K_i)T^{\gamma_i}}{\Gamma(\gamma_i+1)}\right]-\left[\sum_{i=1}^{m}\frac{(\|\lambda_i\|H_i+\|\psi_i\|K_i)T^{\gamma_i}}{\Gamma(\gamma_i+1)}\right]^2\\
&+4\left(\sum_{i=1}^{m}\frac{\|\lambda_i\|\|\psi_i\|T^{\gamma_i}}{\Gamma(\gamma_i+1)}\right)\sum_{i=1}^{m}\frac{H_iK_iT^{\gamma_i}}{\Gamma(\gamma_i+1)}.
\end{aligned}
$$

Then

$$\delta M+\rho<1.$$

Thus all the conditions of Dhage's hybrid fixed point theorem [5] are satisfied and hence the operator equation $x = AxBx+Cx$ has a solution in S. In consequence, problem (1.2) has a solution on J. This completes the proof. □

By a similar way as done above, we can prove an existence result for the following fractional hybrid differential equation

(2.7)
$$
\begin{cases}
D^{\alpha}\left(\dfrac{x(t)-\sum_{i=1}^{m}k_i(t,x(t)).I^{\gamma_i}h_i(t,x(t))}{g(t,x(t))}\right)=f_1(t,I^{\beta}f_2(t,x(t))), & t\in J,\\
x(0)=0.
\end{cases}
$$

Lemma 2.4. *Assume that hypotheses* $(A_1)-(A_4)$ *hold,* $\alpha,\beta,$ *and* $\gamma_i\in(0,1)$*,* $i=1,2,\dots,m$*. If a function* $x\in C(J,R)$ *is a solution of the FHDE* (2.7)*, then it satisfies the quadratic fractional integral equation* (2.2)*.*

Theorem 2.5. *Assume that the hypotheses* $(A_1) - (A_4)$ *of Theorem 2.3 hold. Then the FHDE* (2.7) *has at least one solution defined on* J.

3. Particular Cases and Remarks

The problem (1.2) considered in this paper includes many particular well-known classes of initial value problems of fractional differential equations appearing in the literature and it is equivalent to a multi-term quadratic integral equation of fractional order. This multi-term quadratic functions leads to cover many fractional dynamical systems and special cases:

(i) When $k_i(t, x) = 1, f_2(t, x) = x$ and letting $\beta \to 0$, we have the following hybrid fractional integro-differential equation

$$\begin{cases} D^{\alpha}\left(\frac{x(t) - \sum_{i=1}^{m} I^{\gamma_i} h_i(t, x(t))}{g(t, x(t))}\right) = f_1(t, x(t)), & t \in J, \\ x(0) = 0, \end{cases}$$

which is studied in [19] in case of Caputo fractional derivative.

(ii) When $f_1(t, x) = 0$, we have the $m-$term quadratic fractional integral equation

$$x(t) = \sum_{i=1}^{m} k_i(t, x(t)) I^{\gamma_i} h_i(t, x(t)),$$

which is studied in [12].

(iii) When $f_2(t, x) = x, m = 1, h_i(t, x) = 1$ and $\beta, \gamma_i \to 0$ we have the following quadratic fractional integral equation

$$x(t) = k(t, x(t)) + g(t, x(t)) I^{\alpha} f_1(t, x(t)),$$

which is studied in [12] and when $k(t, x) = p(t) \in C(J, R)$ we obtain a quadratic integral equation of fractional order which is studied in [4] and [11]. Also, taking $g(t, x) = 1$, we obtain the fractional order integral equation

$$x(t) = p(t) + I^{\alpha} f_1(t, x(t)),$$

which is studied in [9].

(iv) Taking $m = 1, h_1(t, x) = 1$ and $\gamma_1 \to 0$, we have the following quadratic fractional integral equation

$$x(t) = k_1(t, x(t)) + g(t, x(t)) I^{\alpha} f_1(t, I^{\beta} f_2(t, x(t))),$$

which is studied in [3] and proves the existence of solution of a quadratic integral equations of fractional orders in Banach algebra.

(v) When $k_i(t,x) = 0, f_2(t,x) = x$ and letting $\beta \to 0$, we have the following hybrid fractional differential equation

$$\begin{cases} D^\alpha \left(\frac{x(t)}{g(t,x(t))}\right) = f_1(t,x(t)), & t \in J, \\ x(0) = 0, \end{cases}$$

which is studied in [21].

(vi) When $k_i(t,x) = 0, g(t,x) = 1$. Taking $f_1(t,x(t)) = p(t) + x(t)$. we can deduce existence results for the following FHDE

$$\begin{cases} D^\alpha x(t) = p(t) + I^\beta f_2(t,x(t)), & t \in J, \\ x(0) = 0. \end{cases} \tag{3.1}$$

(vii) Taking $m = 1, k_1(t,x) = f_1(t,x), \alpha \to 1$ and $h_1(t,x) = g(t,x)$, we obtain the two term quadratic integral equation

$$x(t) = f_1(t,x(t)) \int_0^t g(s,x(s))ds + g(t,x(t)) \int_0^t f_1(s,x(s))ds,$$

which is studied in [15] and proves the existence of solution of a quadratic integro-differential equation

$$\begin{aligned} x(t) = & \int_0^t g(s,x'(s))ds. \int_0^t f_1(s,x'(s))ds, x(0) \\ & = x_0. \end{aligned} \tag{3.2}$$

Differentiating both sides of (3.2), we

$$x'(t) = f_1(t,x'(t)) \int_0^t g(s,x'(s))ds + g(t,x'(t)) \int_0^t f_1(s,x'(s))ds,$$

then

$$u(t) = f_1(t,u(t)) \int_0^t g(s,u(s))ds + g(t,u(t)) \int_0^t f_1(s,u(s))ds.$$

(viii) Taking $g(t,x) = 1$, we obtain a class of neutral fractional order differential equations

(3.3)
$$\begin{cases} D^\alpha \left(x(t) - \sum\limits_{i=1}^m k_i(t,x(t)).I^{\gamma_i} h_i(t,x(t))\right) = f_1(t, I^\beta f_2(t,x(t))), & t \in J, \\ x(0) = 0. \end{cases}$$

Remark 3.1. The existence results for the FHDE (1.2) can be proved under another sequence of assumptions.

Let the assumptions of Theorem 2.3 be satisfied, and replace assumptions $(A_1) - (A_2)$ and (A_4) by the following assumptions:

(A_1^*) The functions $g : J \times R \to R\backslash\{0\}$, $k_i, : J \times R \to R$, and $h_i : J \times R \to R, h_i(0,0) = 0, i = 1, 2, \dots, m$, are continuous and there exist positive functions $\lambda_i(t)$, $\psi_i(t)$, and $\omega(t)$ with norms $\|\lambda_i\|, \|\psi_i\|$, and $\|\omega\|$ respectively such that

$$\begin{aligned} &|k_i(t,x)| \le p_i(t)\Phi(|x|), \\ &|h_i(t,x) - h_i(t,y)| \le \psi_i(t)|x-y|, \\ &|g(t,x) - g(t,y)| \le \omega(t)|x-y|. \end{aligned}$$

(A_4^*) There exists a number $r > 0$ such that

$$\begin{aligned} &\sum_{i=1}^{m} \frac{(\|p\|\Phi(r))(\|\psi_i\|r + H_i)T^{\gamma_i}}{\Gamma(\gamma_i+1)} \\ &\quad + (\|\omega\|r + G)\left(\frac{M_1T^{\alpha}}{\Gamma(\alpha+1)} + \frac{b_1M_2T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\right) \\ &\le r, \end{aligned}$$

where $G = \sup_{t\in J}|g(t,0)|$, and $H_i = \sup_{t\in J}|h_i(t,0)|$,

$$\begin{aligned} &\sum_{i=1}^{m} \frac{[\|p\|\Phi(r) + \|\lambda_i\|H_i + \|\psi_i\|K_i]T^{\gamma_i}}{\Gamma(\gamma_i+1)} \\ &\quad + \frac{\|\omega\|M_1T^{\alpha}}{\Gamma(\alpha+1)} + \frac{\|\omega\|b_1M_2T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} \\ &< 1. \end{aligned}$$

4. Continuous Dependence

In this section, we give sufficient conditions for the uniqueness of the solution of the quadratic functional integral equation (1.2) and study the continuous dependence of solution on the function f_1.

4.1. **Uniqueness of the solution.** Let us assume the following assumption

(A_2^*) Let $f_j : J\times R \to R, j = 1, 2$, be a continuous functions satisfying the Lipschitz condition and there exists two positive functions $\varphi(t), \theta(t)$ with norms $\|\varphi\|$ and $\|\theta\|$, such that

$$\begin{aligned} &|f_1(t,x) - f_1(t,y)| \le \varphi(t)|x-y|, \\ &|f_2(t,x) - f_2(t,y)| \le \theta(t)|x-y|, \end{aligned}$$

with $F_1 = \sup_{t\in J}|f_1(t,0)|$, and $F_2 = \sup_{t\in J}|f_2(t,0)|$.

Theorem 4.1. *Let the assumptions of Theorem 2.3 hold, with replace assumption* (A_2) *by* (A_2^*), *if*

$$\sum_{i=1}^{m} \frac{[\|\lambda_i\|r + K_i]\|\psi_i\| + \|\lambda_i\|[\|\psi_i\|r + H_i]}{\Gamma(\gamma_i+1)} T^{\gamma_i} + \|L\| \left[\frac{r\|\theta\|T^{\alpha+\beta}F_2}{\Gamma(\alpha+\beta+1)} + \frac{T^{\alpha}F_1}{\Gamma(\alpha+1)}\right] + [\|L\|r + G] \frac{\|\varphi\|\|\theta\|T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} < 1.$$

Then the solution $x \in C[0,T]$ *of FHDE(1.2) is unique.*

Proof. Firstly, we notice that condition (A_2^*) implies condition (A_2) for functions f_j, $j = 1, 2$. Let x, y be two solutions of (1.2). Then

$$\begin{aligned}
&|x(t) - y(t)| \\
&\leq \left|\sum_{i=1}^{m} k_i(t,x(t)) I^{\gamma_i} h_i(t,x(t)) - \sum_{i=1}^{m} k_i(t,y(t)) I^{\gamma_i} h_i(t,y(t))\right| \\
&\quad + |g(t,x(t)) I^{\alpha} f_1(t, I^{\beta} f_2(t,x(t))) - g(t,y(t)) I^{\alpha} f_1(t, I^{\beta} f_2(t,y(t)))| \\
&\leq \left|\sum_{i=1}^{m} k_i(t,x(t)) I^{\gamma_i} h_i(t,x(t)) - \sum_{i=1}^{m} k_i(t,x(t)) I^{\gamma_i} h_i(t,y(t))\right| \\
&\quad + \left|\sum_{i=1}^{m} k_i(t,x(t)) I^{\gamma_i} h_i(t,y(t)) - \sum_{i=1}^{m} k_i(t,y(t)) I^{\gamma_i} h_i(t,y(t))\right| \\
&\quad + |g(t,x(t)) - g(t,y(t))| \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |f_1(s, I^{\beta} f_2(s,x(s))| ds \\
&\quad + |g(t,x(t))| \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |f_1(s, I^{\beta} f_2(s,x(s)) - f_1(s, I^{\beta} f_2(s,y(s))| ds \\
&\leq \sum_{i=1}^{m} |k_i(t,x(t))| |I^{\gamma_i} h_i(t,x(t)) - I^{\gamma_i} h_i(t,y(t))| \\
&\quad + \sum_{i=1}^{m} |k_i(t,x(t)) - k_i(t,y(t))| |I^{\gamma_i} h_i(t,y(t))| \\
&\quad + L(t)|x(t) - y(t)| \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} [|f_1(s, I^{\beta} f_2(s,x(s)) - f_1(s,0)| + |f_1(s,0)|] ds \\
&\quad + [[L(t)|x(t)| + G]] \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \varphi(s) |I^{\beta} f_2(s,x(s)) - I^{\beta} f_2(s,y(s))| ds \\
&\leq \sum_{i=1}^{m} [|k_i(t,x(t)) - k_i(t,0)| + |k_i(t,0)|] \int_0^t \frac{(t-s)^{\gamma_i-1}}{\Gamma(\gamma_i)} \psi_i(s) |x(s) - y(s)| ds \\
&\quad + \sum_{i=1}^{m} \lambda_i(t)|x(t) - y(t)| \int_0^t \frac{(t-s)^{\gamma_i-1}}{\Gamma(\gamma_i)} [|h_i(s,y(s)) - h_i(s,0)| + h_i(s,0)|] ds \\
&\quad + |L(t)||x(t) - y(t)| \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} [|\varphi(s)| I^{\beta} |f_2(s,x(s))| + F_1] ds \\
&\quad + [L(t)|x(t)| + G] \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |\varphi(s)| I^{\beta} |f_2(s,x(s)) - f_2(s,y(s))| ds
\end{aligned}$$

$$\begin{aligned}
&\leq \sum_{i=1}^{m}[|\lambda_i(t)||x(t)| + K_i]\int_0^t \frac{(t-s)^{\gamma_i-1}}{\Gamma(\gamma_i)}|\psi_i(s)||x(s)-y(s)|ds\\
&\quad + \sum_{i=1}^{m}|\lambda_i(t)|\|x-y\|\int_0^t \frac{(t-s)^{\gamma_i-1}}{\Gamma(\gamma_i)}[|\psi_i(s)||x(s)| + H_i]ds\\
&\quad + \|L\|\|x-y\|\|\varphi\|\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\left[I^{\beta}[|f_2(s,x(s)) - f_2(s,0)| + |f_2(s,0)|] + F_1\right]ds\\
&\quad + \|L\|\|x\| + G]\|\varphi\|\|\theta\|\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}I^{\beta}|x(s)-y(s)|ds\\
&\leq \|x-y\|\sum_{i=1}^{m}[\|\lambda_i\|\|x\| + K_i]\frac{\|\psi_i\|T^{\gamma_i}}{\Gamma(\gamma_i+1)}\\
&\quad + \sum_{i=1}^{m}\|\lambda_i\|[\|\psi_i\|\|x\| + H_i]\|x-y\|\frac{T^{\gamma_i}}{\Gamma(\gamma_i+1)}\\
&\quad + \|L\|\|x-y\|\|\varphi\|\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\left[I^{\beta}[|\theta(s)||x(s)| + F_2] + F_1\right]ds\\
&\quad + \|L\|\|x\| + G]\|\varphi\|\|\theta\|\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}I^{\beta}\|x-y\|\\
&\leq \sum_{i=1}^{m}\frac{[\|\lambda_i\|\|x\| + K_i]\|\psi_i\| + \|\lambda_i\|[\|\psi_i\|\|x\| + H_i]}{\Gamma(\gamma_i+1)}T^{\gamma_i}\|x-y\|\\
&\quad + \|L\|\|x-y\|[|\ \theta\|\|x\|\frac{T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}F_2 + \frac{T^{\alpha}}{\Gamma(\alpha+1)}F_1]\\
&\quad + [\|L\|\|x\| + G]\,\|\varphi\|\|\theta\|\|x-y\|\frac{T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}.
\end{aligned}$$

Taking the supremum for $t \in J$, we have

$$\begin{aligned}
\|x-y\| \leq \Bigg[&\sum_{i=1}^{m}\frac{[\|\lambda_i\|r + K_i]\|\psi_i\| + \|\lambda_i\|[\|\psi_i\|r + H_i]}{\Gamma(\gamma_i+1)}T^{\gamma_i}\\
&+ \|L\|\left[\frac{r\|\theta\|T^{\alpha+\beta}F_2}{\Gamma(\alpha+\beta+1)} + \frac{T^{\alpha}F_1}{\Gamma(\alpha+1)}\right]\\
&+ [\|L\|r + G]\,\frac{\|\varphi\|\|\theta\|T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\Bigg]\|x-y\|.
\end{aligned}$$

Then

$$\begin{aligned}
&\Bigg[1 - \Bigg(\Big[\sum_{i=1}^{m}\frac{[\|\lambda_i\|r + K_i]\|\psi_i\| + \|\lambda_i\|[\|\psi_i\|r + H_i]}{\Gamma(\gamma_i+1)}T^{\gamma_i} + \|L\|\left[\frac{r\|\theta\|T^{\alpha+\beta}F_2}{\Gamma(\alpha+\beta+1)}\right.\\
&\quad\left. + \frac{T^{\alpha}F_1}{\Gamma(\alpha+1)}\right] + [\|L\|r + G]\,\frac{\|\varphi\|\|\theta\|T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\Big]\Bigg)\Bigg]\|x-y\|\\
&\leq 0.
\end{aligned}$$

Since

$$\sum_{i=1}^{m} \frac{[\|\lambda_i\| r + K_i]\|\psi_i\| + \|\lambda_i\|[\|\psi_i\| r + H_i]}{\Gamma(\gamma_i + 1)} T^{\gamma_i}$$
$$+ \|L\| \left[\frac{r\|\theta\| T^{\alpha+\beta} F_2}{\Gamma(\alpha+\beta+1)} + \frac{T^{\alpha} F_1}{\Gamma(\alpha+1)}\right] + [\|L\| r + G] \frac{\|\varphi\|\|\theta\| T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}$$
$$< 1,$$

then, $x(t) = y(t)$ and the solution of the FHDE (1.2) is unique. □

4.2. **Continuous dependence.** Next, we prove the continuous dependence of the unique solutions on the functions f_1.

Definition 4.2. The solution of FHDE (1.2) depends continuously on the functions f_1 if $\forall \epsilon > 0$, $\exists \delta > 0$, such that

$$|f_1(t, x(t)) - f_1^*(t, x(t))| \leq \delta \quad \Rightarrow \quad ||x - x^*|| \leq \epsilon.$$

Theorem 4.3. *Let the assumptions of Theorem 4.1 hold. Then the solution of FHDE (1.2) depends continuously on the function f_1.*

Proof. Let x, x^* be two solutions of the FHDE (1.2).

Let $\delta > 0$ be given such that $|f_1(t, x(t)) - f_1^*(t, x(t))| \leq \delta$, $\forall \delta \geq 0$. Then

$$\begin{aligned}
&|x(t) - x^*(t)| \\
&\quad \leq |\sum_{i=1}^{m} k_i(t, x(t)) I^{\gamma_i} h_i(t, x(t)) - \sum_{i=1}^{m} k_i(t, x^*(t)) I^{\gamma_i} h_i(t, x^*(t))| \\
&\qquad + \big|g(t, x(t)) \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f_1(s, I^{\beta} f_2(s, x(s)) ds \\
&\qquad - g(t, x^*(t)) \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f_1^*(s, I^{\beta} f_2(s, x^*(s)) ds\big| \\
&\quad \leq \sum_{i=1}^{m} |k_i(t, x(t))||I^{\gamma_i} h_i(t, x(t)) - I^{\gamma_i} h_i(t, x^*(t))| \\
&\qquad + \sum_{i=1}^{m} |k_i(t, x(t)) - k_i(t, x^*(t))||I^{\gamma_i} h_i(t, x^*(t))| \\
&\qquad + |g(t, x(t))| \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f_1(s, I^{\beta} f_2(s, x(s)) ds \\
&\qquad - g(t, x^*(t)) \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f_1(s, I^{\beta} f_2(s, x(s)) ds| \\
&\qquad + |g(t, x^*(t)) \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f_1(s, I^{\beta} f_2(s, x(t)) ds \\
&\qquad - g(t, x^*(t)) \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f_1^*(s, I^{\beta} f_2(s, x^*(t)) ds| \\
&\quad \leq \sum_{i=1}^{m} [|k_i(t, x(t)) - k_i(t, 0)| + |k_i(t, 0)|] \int_0^t \frac{(t-s)^{\gamma_i - 1}}{\Gamma(\gamma_i)} \psi_i(t) |x(s) - x^*(s)| ds
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{i=1}^{m}\lambda_i(t)|x(t)-x^*(t)|\int_0^t\frac{(t-s)^{\gamma_i-1}}{\Gamma(\gamma_i)}[|h_i(s,x^*(s))-h_i(s,0)|+h_i(s,0)|]ds\\
&+|g(t,x(t))-g(t,x^*(t))|\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}|f_1(s,I^\beta f_2(s,x(s))|ds\\
&+|g(t,x^*(t))|\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}|f_1(s,I^\beta f_2(s,x(s))-f_1^*(s,I^\beta f_2(s,x^*(s))|ds\\
\le&\sum_{i=1}^{m}[|\lambda_i(t)||x(t)|+K_i]\int_0^t\frac{(t-s)^{\gamma_i-1}}{\Gamma(\gamma_i)}|\psi_i(s)||x(s)-x^*(s)|ds\\
&+\sum_{i=1}^{m}|\lambda_i(t)||x(t)-x^*(t)|\int_0^t\frac{(t-s)^{\gamma_i-1}}{\Gamma(\gamma_i)}[|\psi_i(s)||x^*(s)|+H_i]ds\\
&+L(t)|x(t)-x^*(t)|\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}[|f_1(s,I^\beta f_2(s,x(s))-f_1(s,0)|+|f_1(s,0)|]ds\\
&+(\|L\||x^*(t)|+G)\bigg[\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}|f_1(s,I^\beta f_2(s,x(s)))-f_1(s,I^\beta f_2(s,x^*(s)))|ds\\
&+\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}|f_1(s,I^\beta f_2(s,x^*(s)))-f_1^*(s,I^\beta f_2(s,x^*(s)))|ds\bigg]\\
\le&\|x-x^*\|\sum_{i=1}^{m}[\|\lambda_i\|r+K_i]\frac{\|\psi_i\|T^{\gamma_i}}{\Gamma(\gamma_i+1)}+\sum_{i=1}^{m}\|\lambda_i\|[\|\psi_i\|r+H_i]\|x-x^*\|\frac{T^{\gamma_i}}{\Gamma(\gamma_i+1)}\\
&+\|L\|\|x-x^*\|\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}[\varphi(s)|I^\beta|f_2(s,x(s))|+F_1]ds\\
&+(\|L\|\|x^*\|+G)\bigg[\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}|\varphi(s)||I^\beta f_2(s,x(s))-I^\beta f_2(s,x^*(s))|ds\\
&+(\|L\|\|x^*\|+G)\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\delta ds\\
\le&\|x-x^*\|\bigg[\sum_{i=1}^{m}[\|\lambda_i\|r+K_i]\frac{\|\psi_i\|T^{\gamma_i}}{\Gamma(\gamma_i+1)}+\sum_{i=1}^{m}\|\lambda_i\|[\|\psi_i\|r+H_i]\frac{T^{\gamma_i}}{\Gamma(\gamma_i+1)}\bigg]\\
&+\|L\|\|x-x^*\|\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}[\varphi(s)|I^\beta|[|\theta(s)||x(s)|+F_2]|+F_1]ds\\
&+(\|L\|\|x^*\|+G)\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}||\varphi(s)||I^\beta[|\theta(s)||x(s)-x^*(s)|]ds\\
&+(\|L\|\|x^*\|+G)\delta\frac{T^\alpha}{\Gamma(\alpha+1)}\\
\le&\|x-x^*\|\bigg[\sum_{i=1}^{m}[\|\lambda_i\|r+K_i]\frac{\|\psi_i\|T^{\gamma_i}}{\Gamma(\gamma_i+1)}+\sum_{i=1}^{m}\|\lambda_i\|[\|\psi_i\|r+H_i]\frac{T^{\gamma_i}}{\Gamma(\gamma_i+1)}\bigg]\\
&+\|L\|\|x-x^*\|\|\varphi\|\frac{T^\alpha}{\Gamma(\alpha+1)}[\frac{T^\beta}{\Gamma(\beta+1)}|[\|\theta\|\|x\|+F_2]|+F_1]\\
&+(\|L\|\|x^*\|+G)\frac{T^\alpha}{\Gamma(\alpha+1)}\|\varphi\|\frac{T^\beta}{\Gamma(\beta+1)}\|\theta\|\|x-x^*\|+(\|L\|\|x^*\|+G)\delta\frac{T^\alpha}{\Gamma(\alpha+1)}\\
\le&\|x-x^*\|\bigg[\sum_{i=1}^{m}[\|\lambda_i\|r+K_i]\frac{\|\psi_i\|T^{\gamma_i}}{\Gamma(\gamma_i+1)}+\sum_{i=1}^{m}\|\lambda_i\|[\|\psi_i\|r+H_i]\frac{T^{\gamma_i}}{\Gamma(\gamma_i+1)}
\end{aligned}
$$

$$+ \|L\|\|\varphi\|\frac{T^{\alpha}}{\Gamma(\alpha+1)}[\frac{T^{\beta}}{\Gamma(\beta+1)}|[\|\theta\|\|x\| + F_2]| + F_1]$$
$$+ (\|L\|\|x^*\| + G)\frac{T^{\alpha}}{\Gamma(\alpha+1)}\|\varphi\|\frac{T^{\beta}}{\Gamma(\beta+1)}\|\theta\|\Big] + (\|L\|\|x^*\| + G)\delta\frac{T^{\alpha}}{\Gamma(\alpha+1)}.$$

Taking the supremum $t \in J$, we have

$$\begin{aligned}
&\|x - x^*\| \\
&\le \Big(1 - \Big[\sum_{i=1}^{m}[\|\lambda_i\|r + K_i]\frac{\|\psi_i\|T^{\gamma_i}}{\Gamma(\gamma_i+1)} + \sum_{i=1}^{m}\|\lambda_i\|[\|\psi_i\|r + H_i]\frac{T^{\gamma_i}}{\Gamma(\gamma_i+1)} \\
&\quad + \|L\|\|\varphi\|\frac{T^{\alpha}}{\Gamma(\alpha+1)}[\frac{T^{\beta}}{\Gamma(\beta+1)}|[\|\theta\|r + F_2]| + F] \\
&\quad + (\|L\|r + G)\frac{T^{\alpha}}{\Gamma(\alpha+1)}\|\varphi\|\frac{T^{\beta}}{\Gamma(\beta+1)}\|\theta\|\Big]\Big)^{-1}(\|L\|r + G)\frac{\delta T^{\alpha}}{\Gamma(\alpha+1)} \\
&\le \epsilon.
\end{aligned}$$

This means that the solution of the FHDE (1.2) depends continuously on f_1. This completes the proof. □

By a similar way as done above, the continuous dependence of the solution of the FHDE (1.2) on functions k_i, h_i, $i = 1, 2, \ldots, m$, f_2 and g can be studied.

5. Conclusion

It is known that, various forms of fractional differential equations model most natural phenomena. This variety in the study of complicated fractional differential equations increases our ability to model various phenomena precisely. That helps develop modern software that allows us to allow more cost-free testing and less consumption of materials. In this work, we have proven an auxiliary lemma related to the linear variant of the FHDEs (1.2) and stated sufficient conditions that guarantee the existence of solutions in a Banach algebra due to Dhage [5], some particular cases, remarks are added. Results on the existence and continuous dependence of solutions for FHDE (1.2) on function f_1 were also studied. In the same way, the reader can get the continuous dependence of solutions on the other functions.

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