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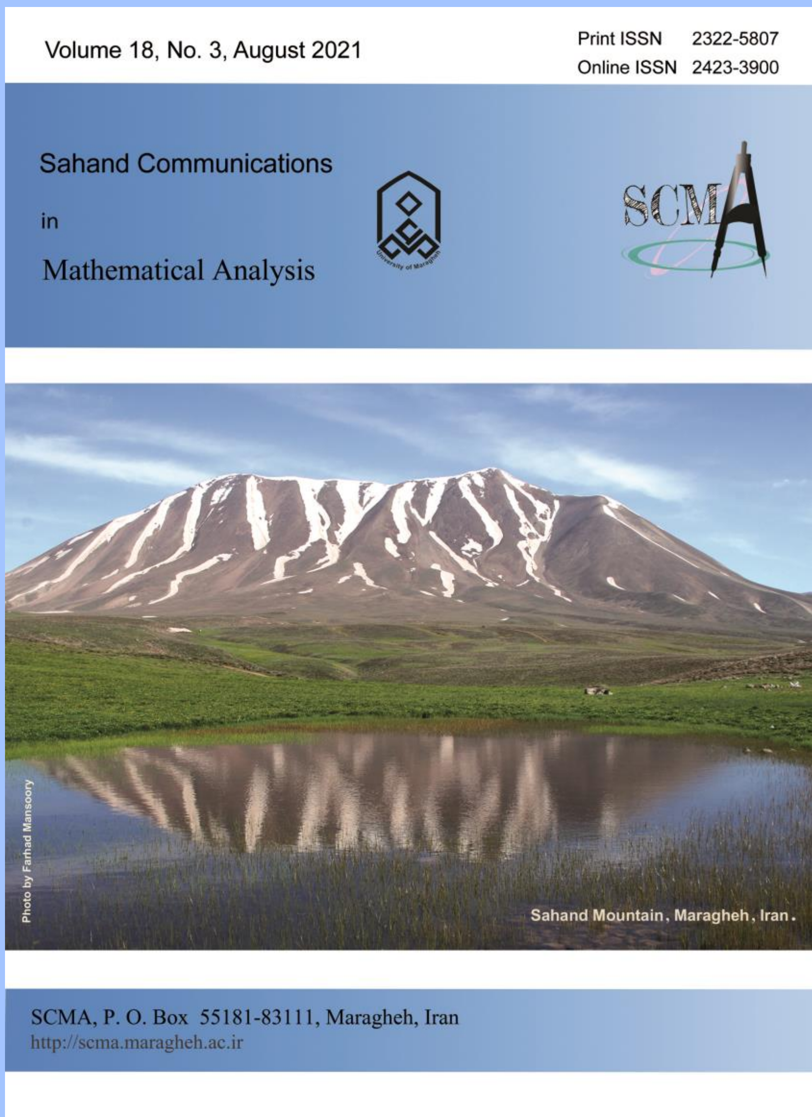

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Woven g-Fusion Frames in Hilbert Spaces

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Abstract. In this paper, we introduce the notion of woven g-fusion frames in Hilbert spaces. Then, we present sufficient conditions for woven g-fusion frames in terms of woven frames in Hilbert spaces. We extend some of the recent results of standard woven frames and woven fusion frames to woven g-fusion frames. Also, we study perturbations of woven g-fusion frames.

1. Introduction

Frames for a Hilbert space were first introduced by Duffin and Schaeffer [10] in 1952. Daubechies, Grossmann and Meyer [8] reintroduced frames, in 1986 [8] and considered from then. Frame theory has applications in signal processing, image processing, data compression and sampling theory.

Orthonormal bases are special case of frames in Hilbert space. Any element in a Hilbert space can be present as an infinite linear combination, not necessary unique, of the frame elements. For more information, readers can refer to [7, 11].

Some new types and generalizations of frame were introduced by researchers such as fusion frames, g-frames, woven frames, ... Frame of subspaces or fusion frames are a generalization of frames which were introduced by Cassaza and Kutyniok [4] in 2003 and were investigated in [1, 5, 12, 13]. Generalized frames or in abbreviation g-frames were

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introduced by Sun [17] in 2006. Most recently, g-fusion frames in Hilbert space were introduced by Sadri et.al. [16].

In other side, weaving frames were introduced by Bemrose et.al. [2] and [6] in 2016. Weaving frames are powerful tools for pre-processing signals and distributed data processing. Many researchers studied and generalized weaving frames. Some of these generalizations are weaving g-frames, weaving fusion frames [14], Weaving K-frames [9] and controlled weaving frames [15].

In this paper, motivated and inspired by the above-mentioned works we introduce the concept of weaving g-fusion frame. This frame includes weaving g-frames and weaving fusion frames. We extend some of the recent results of standard woven frames and woven fusion frames to woven g-fusion frames. Also, we study perturbations of woven g-fusion frames.

The paper is organized as follows: Section 2 contains the basic definitions about fusion frames, g-frames, g-fusion frames and woven frames. Section 3 is devoted to introducing the woven g-fusion frames and investigating their structuers. In section 4, we study perturbations of woven g-fusion frames.

2. Basic Definitions and Preliminaries

As a preliminary of frames, at the first, we mention fusion frames. Also we review g-frames, g-fusion frames and woven frames . Throughout this paper, $\mathbb{I}$ is the indexing set where it can be a finite or countably infinite set, and $[m]$ is the set consisting of the of natural numbers $\{1, 2, ..., m\}$. Also, $\mathcal{H}$ and $\mathcal{K}$ are separable Hilbert spaces and$B(\mathcal{H}, \mathcal{K})$ is the collection of all the bounded linear operators of $\mathcal{H}$ into $\mathcal{K}$. If $\mathcal{H} = \mathcal{K}$, then $B(\mathcal{H}, \mathcal{H})$ will be denoted by$B(\mathcal{H})$ and P is the orthogonal projection.

2.1. **Fusion Frames.** In 2003, a new type of generalization of frames were introduced by Cassaza and Kutyniok to the science world that today we know them as fusion frames. In this section, we briefly recall some basic notations, definitions and some important properties of fusion frames that are useful for our study. For more detailed information one can see [1, 4, 5, 12, 13].

Definition 2.1. Let $\{v_i\}_{i\in\mathbb{I}}$ be a family of real weights such that $v_i > 0$ for all $i \in \mathbb{I}$. A family of closed subspaces $\{W_i\}_{i\in\mathbb{I}}$ of a Hilbert space $\mathcal{H}$ is called a fusion frame (or frame of subspaces) for $\mathcal{H}$ with respect to weights $\{v_i\}_{i\in\mathbb{I}}$, if there exist constants $C, D > 0$ such that

$$C\,\|f\|^2 \leq \sum_{i\in\mathbb{I}} v_i^2\,\|P_{W_i}(f)\|^2 \leq D\,\|f\|^2\,, \quad \forall f \in \mathcal{H}, \tag{2.1}$$

where P_{W_i} is the orthogonal projection of $\mathcal{H}$ to W_i. The constants C and D are called the lower and upper fusion frame bounds, respectively. If the right inequality in (2.1) holds, the family of subspace $\{W_i\}_{i\in\mathbb{I}}$ is called a Bessel sequence of subspaces with respect to $\{v_i\}_{i\in\mathbb{I}}$ with Bessel bound D. Also it is called a tight fusion frame with respect to $\{v_i\}_{i\in\mathbb{I}}$, if $C = D$ and is called parseval fusion frame, if $C = D = 1$. We say $\{W_i\}_{i\in\mathbb{I}}$ an orthogonal fusion basis for $\mathcal{H}$, if $\mathcal{H} = \bigoplus_{i\in\mathbb{I}} W_i$.

Definition 2.2. The fusion frame $\{W_i\}_{i\in\mathbb{I}}$ with respect to some family of weights is called a Riesz decomposition of $\mathcal{H}$, if for every $f \in \mathcal{H}$, there is a unique choice of $f_i \in W_i$ so that $f = \sum\limits_{i\in\mathbb{I}} f_i$.

For each family of subspaces $\{W_i\}_{i\in\mathbb{I}}$ of $\mathcal{H}$, the representation space:

$$\left(\sum_{i\in\mathbb{I}} \oplus W_i\right)_{\ell^2} = \left\{ \{f_i\}_{i\in\mathbb{I}} | f_i \in W_i \text{ and } \sum_{i\in\mathbb{I}} \|f_i\|^2 < \infty \right\},$$

with inner product

$$\langle \{f_i\}_{i\in\mathbb{I}}, \{g_i\}_{i\in\mathbb{I}} \rangle = \sum_{i\in\mathbb{I}} \langle f_i, g_i \rangle ,$$

is a Hilbert space. This space is needed in the study of fusion systems.

Definition 2.3. Let $\{W_i\}_{i\in\mathbb{I}}$ be a fusion frame family for $\mathcal{H}$ with respect to $\{v_i\}_{i\in\mathbb{I}}$. Then the analysis operator for $\{W_i\}_{i\in\mathbb{I}}$ with weights $\{v_i\}_{i\in\mathbb{I}}$ is defined by:

$$U_{W,v} : \mathcal{H} \to \left(\sum_{i\in\mathbb{I}} \oplus W_i\right)_{\ell^2}, \qquad U_{W,v}(f) = \{v_i P_{W_i}(f)\}_{i\in\mathbb{I}} .$$

The adjoint of $U_{W,v}$ is called the synthesis operator, we denote $T_{W,v} = U^*_{W,v}$.

By elementary calculation, we have

$$T_{W,v} : \left(\sum_{i\in\mathbb{I}} \oplus W_i\right)_{\ell^2} \to \mathcal{H}, \qquad T_{W,v}(\{f_i\}_{i\in\mathbb{I}}) = \sum_{i\in\mathbb{I}} v_i P_{W_i} f_i.$$

Like discrete frames, the fusion frame operator for $\{W_i\}_{i\in\mathbb{I}}$ with respect to $\{v_i\}_{i\in\mathbb{I}}$ is the composition of analysis and synthesis operators,

$$S_{W,v} : \mathcal{H} \to \mathcal{H}, \qquad S_{W,v}(f) = T_{W,v} U_{W,v}(f) = \sum_{i\in I} v_i^2 P_{W_i}(f), \quad \forall f \in \mathcal{H}.$$

The following theorem presents the equivalence conditions between the fusion frames and their operators.

Theorem 2.4. *Let $\{W_i\}_{i\in\mathbb{I}}$ be a family of subspaces in $\mathcal{H}$ and $\{v_i\}_{i\in\mathbb{I}}$ be a family of weights. Then the following conditions are equivalent:*

(i) $\{W_i\}_{i\in\mathbb{I}}$ *is a fusion frame with respect to* $\{v_i\}_{i\in\mathbb{I}}$.
(ii) *The synthesis operator* $T_{W,v}$ *is bounded, linear and onto.*
(iii) *The analysis operator* $U_{W,v}$ *is a (possibly into) isomorphism.*

2.2. **Generalized Frames.** Sun [17] introduced g-frames which are generalized frames and include ordinary frames and many recent generalizations of frames.

Definition 2.5. Let $\{\mathcal{H}_i\}_{i\in\mathbb{I}}$ be a family of Hilbert spaces. We call $\Lambda = \{\Lambda_i \in B(\mathcal{H},\mathcal{H}_i),\ i\in\mathbb{I}\}$ a g-frame for $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i\in\mathbb{I}}$, or simply, a g-frame for H, if there exist two positive constants C, D such that

$$C\|f\|^2 \le \sum_{i\in\mathbb{I}} \|\Lambda_i f\|^2 \le D\|f\|^2, \quad \forall f\in\mathcal{H}. \tag{2.2}$$

The positive numbers C and D are called the lower and upper g-frame bounds, respectively. We call Λ a tight g-frame, if $C = D$ and we call it a parseval g-frame, if $C = D = 1$. If only the second inequality holds, we call it g-Bessel sequence. If Λ is a g-frame, then the g-frame operator S_Λ is defined by

$$S_\Lambda f = \sum_{i\in\mathbb{I}} \Lambda_i^*\Lambda_i f, \quad f\in\mathcal{H},$$

which is a bounded, positive and invertible operator such that

$$AI \le S_\Lambda \le BI,$$

and for each $f\in\mathcal{H}$, we have

$$\begin{aligned} f &= S_\Lambda S_\Lambda^{-1} f \\ &= S_\Lambda^{-1} S_\Lambda f \\ &= \sum_{i\in\mathbb{I}} S_\Lambda^{-1}\Lambda_i^*\Lambda_i f \\ &= \sum_{i\in\mathbb{I}} \Lambda_i^*\Lambda_i S_\Lambda^{-1} f. \end{aligned}$$

The canonical dual g-frame for Λ is defined by $\{\Lambda_i S_\Lambda^{-1}\}_{i\in\mathbb{I}}$ with bounds $\frac{1}{B}, \frac{1}{C}$. In other words, $\{\Lambda_i S_\Lambda^{-1}\}_{i\in\mathbb{I}}$ and $\{\Lambda_i\}_{i\in\mathbb{I}}$ are dual g-frames with respect to each other.

It is easy to show that by letting $\mathcal{H}_i = W_i$, $\Lambda_i = P_{W_i}$ and $v_i = 1$, a fusion frame is a g-frame.

2.3. **Generalized Fusion frames.** Generalized fusion frames (g-fusion frames) in Hilbert space were introduced by Sadri et.al. [16].

Let

$$\left(\sum_{i\in\mathbb{I}} \oplus\mathcal{H}_i\right)_{\ell^2} = \left\{\{f_i\}_{i\in\mathbb{I}} | f_i \in \mathcal{H}_i \text{ and } \sum_{i\in\mathbb{I}} \|f_i\|^2 < \infty\right\},$$

with the inner product defined by

$$\langle \{f_i\}_{i\in\mathbb{I}}, \{g_i\}_{i\in\mathbb{I}}\rangle = \sum_{i\in\mathbb{I}} \langle f_i, g_i\rangle ,$$

is a Hilbert space.

Definition 2.6. Let $W = \{W_i\}_{i\in\mathbb{I}}$ be a family of closed subspaces of $\mathcal{H}$, $\{v_i\}_{i\in\mathbb{I}}$ be a family of weights, i.e. $v_i > 0$ and $\Lambda_i \in B(\mathcal{H}, \mathcal{H}_i)$ for all $i \in \mathbb{I}$. We say $\Lambda := (\Lambda_i, W_i, v_i)$ is a generalized fusion frame (or g-fusion frame) for $\mathcal{H}$, if there exist $0 < A \leq B < \infty$ such that for each $f \in \mathcal{H}$

$$A\|f\|^2 \leq \sum_{i\in\mathbb{I}} v_i^2 \|\Lambda_i P_{W_i} f\|^2 \leq B\|f\|^2 . \tag{2.3}$$

We call Λ a parseval g-fusion frame, if $A = B = 1$. When the right hand of (2.3) holds, Λ is called a g-fusion Bessel sequence for $\mathcal{H}$ with bound B. If $\mathcal{H}_i = \mathcal{H}$ for all $i \in \mathbb{I}$ and $\Lambda_i = I_{\mathcal{H}}$, then we get the fusion frame (W_i, v_i) for $\mathcal{H}$. Throughout this paper, Λ will be a triple (Λ_i, W_i, v_i) with $i \in \mathbb{I}$ unless otherwise stated.

Definition 2.7. Let Λ be a g-fusion frame for $\mathcal{H}$. Then, the analysis operator for Λ is defined by

$$U_\Lambda : \mathcal{H} \to \left(\sum_{i\in\mathbb{I}} \oplus\mathcal{H}_i\right)_{\ell^2}, \qquad T_\Lambda(f) = \{v_i \Lambda_i P_{W_i}(f)\}_{i\in\mathbb{I}} .$$

The adjoint of U_Λ is called the synthesis operator, and is denoted by $T_\Lambda = U_\Lambda^*$.

By the elementary calculation, we have

$$T_\Lambda : \left(\sum_{i\in\mathbb{I}} \oplus\mathcal{H}_i\right)_{\ell^2} \to \mathcal{H}, \qquad T_\Lambda(\{f_i\}_{i\in\mathbb{I}}) = \sum_{i\in\mathbb{I}} v_i P_{W_i} \Lambda_i^* f_i.$$

The g-fusion frame operator Λ is the composition of analysis and synthesis operators,

$$S_\Lambda : \mathcal{H} \to \mathcal{H}, \qquad S_\Lambda f = T_\Lambda U_\Lambda(f) = \sum_{i\in\mathbb{I}} v_i^2 P_{W_i} \Lambda_i^* \Lambda_i P_{W_i} f.$$

We have

$$\langle S_\Lambda f, f\rangle = \sum_{i\in\mathbb{I}} v_i^2 \|\Lambda_i P_{W_i} f\|^2 .$$

Therefore

$$CI \leq S_\Lambda \leq DI.$$

This means that S_Λ is bounded, positive and invertible operator (with adjoint inverse). So, we have the reconstruction formula for any $f \in \mathcal{H}$

$$\begin{aligned} f &= \sum_{i\in\mathbb{I}} v_i^2 P_{W_i}\Lambda_i^*\Lambda_i P_{W_i} S_\Lambda^{-1} f \\ &= \sum_{i\in\mathbb{I}} v_i^2 S_\Lambda^{-1} P_{W_i}\Lambda_i^*\Lambda_i P_{W_i} f. \end{aligned}$$

The following theorem gives the equivalence conditions between the g-fusion frames and their operators.

Theorem 2.8 ([16]). *Let Λ be the triple (Λ_i, W_i, v_i) with $i \in \mathbb{I}$. Then the following conditions are equivalent:*

(i) *Λ is a g-fusion frame for $\mathcal{H}$.*
(ii) *The synthesis operator T_Λ is bounded, linear and onto.*
(iii) *The analysis operator S_Λ is well-defined, bounded, surjectire.*

2.4. **Woven Frames.** Woven frames in Hilbert spaces, were introduced in 2015 by Bemrose et.al. [2, 3, 6], after that, Vashisht, Deepshikha, and others. have done more research [9, 18–20]. They have studied a variety of different types of generalized weaving frames, such as g-frame, K-frame, and continuous frame. In the following, we mention the definition of woven frames.

Definition 2.9. Let $F = \{f_{ij}\}_{i\in\mathbb{I}}$ for $j \in [m]$ (where $[m]$ is the set $\{1, 2, \ldots, m\}$) be a family of frames for separable Hilbert space $\mathcal{H}$. If there exist universal constants A' and B' such that for every partition $\{\sigma_j\}_{j\in[m]}$, the family $F_j = \{f_{ij}\}_{i\in\sigma_j}$ is a frame for $\mathcal{H}$ with bounds A' and B', then F is said Woven frames and for every $j \in [m]$, the frames F_j are called Weaving frame.

3. Woven g-Fusion Frames and their Structures

In this section, we introduce woven g-fusion frames by extending and improving the notions of g-fusion frames and weaving frames. We investigate the structure of woven g-fusion frames and characterize them.

Definition 3.1. A family of g-fusion frames $\{(\Lambda_{ij}, W_{ij}, v_{ij})\}_{i\in\mathbb{I}}$ for $j \in [m]$, is said woven g-fusion frames if there exist universal constants A and B, such that for every partition $\{\sigma_j\}_{j\in[m]}$ of $\mathbb{I}$, the family $\{(\Lambda_{ij}, W_{ij}, v_{ij})\}_{i\in\sigma_j, j\in[m]}$ is a g-fusion frame for $\mathcal{H}$ with lower and upper frame bounds A and B. Each family $\{(\Lambda_{ij}, W_{ij}, v_{ij})\}_{i\in\sigma_j, j\in[m]}$ is called a Weaving g-fusion frame.

The following theorem states the equivalence conditions between woven frames and woven g-fusion frames.

Theorem 3.2. *Suppose* $\{\Lambda_i \in B(\mathcal{H}, \mathcal{H}_i)\}_{i\in\mathbb{I}}$, $\{\Gamma_i \in B(\mathcal{H}, \mathcal{H}_i)\}_{i\in\mathbb{I}}$ *and for every* $i \in \mathbb{I}$, $\mathbb{J}_i$ *is a subset of index set* $\mathbb{I}$ *and* $\nu_i, \mu_i > 0$. *Let* $\{f_{i,j}\}_{j\in\mathbb{J}_i}$ *and* $\{g_{i,j}\}_{j\in\mathbb{J}_i}$ *be frame sequences in* $\mathcal{H}_i$ *with frame bounds* (A_{f_i}, B_{f_i}) *and* (A_{g_i}, B_{g_i}), *respectively. Define*

$$W_i = \overline{\operatorname{span}}\{\Lambda_i^* f_{i,j}\}_{j\in\mathbb{J}_i}, \qquad V_i = \overline{\operatorname{span}}\{\Gamma_i^* g_{i,j}\}_{j\in\mathbb{J}_i}, \quad \forall i \in \mathbb{I},$$

and choose orthogonal basis $\{e_{i,j}\}_{j\in\mathbb{J}_i}$ *for subspace* $\mathcal{H}_i$. *Suppose that*

$$0 < A_f = \inf_{i\in\mathbb{I}} A_{f_i} \le B_f = \sup_{i\in\mathbb{I}} B_{f_i} < \infty,$$

and

$$0 < A_g = \inf_{i\in\mathbb{I}} A_{g_i} \le B_g = \sup_{i\in\mathbb{I}} B_{g_i} < \infty.$$

Then the following conditions are equivalent:

(i) $\{\nu_i\Lambda_i^* f_{i,j}\}_{i\in\mathbb{I},j\in\mathbb{J}_i}$ *and* $\{\mu_i\Gamma_i^* g_{i,j}\}_{i\in\mathbb{I},j\in\mathbb{J}_i}$ *are woven frames in* $\mathcal{H}$.
(ii) $\{\nu_i\Lambda_i^* e_{i,j}\}_{i\in\mathbb{I},j\in\mathbb{J}_i}$ *and* $\{\mu_i\Gamma_i^* e_{i,j}\}_{i\in\mathbb{I},j\in\mathbb{J}_i}$ *are woven frames in* $\mathcal{H}$.
(iii) $\{(\Lambda_i, W_i, \nu_i)\}_{i\in\mathbb{I}}$ *and* $\{(\Gamma_i, V_i, \mu_i)\}_{i\in\mathbb{I}}$ *are woven g-fusion frames in* $\mathcal{H}$ *with respect to* $\{\mathcal{H}_i\}_{i\in\mathbb{I}}$.

Proof. Since for every $i \in \mathbb{I}$, $\{f_{i,j}\}_{j\in\mathbb{J}_i}$ and $\{g_{i,j}\}_{j\in\mathbb{J}_i}$ are frames for $\mathcal{H}_i$ with frame bounds (A_{f_i}, B_{f_i}) and (A_{g_i}, B_{g_i}), then for $\sigma \subset \mathbb{I}$;

$$\begin{aligned}
A_f \sum_{i\in\sigma} \nu_i^2 \|\Lambda_i P_{W_i}(f)\|^2 &+ A_g \sum_{i\in\sigma^c} \mu_i^2 \|\Gamma_i P_{V_i}(f)\|^2 \\
&\le \sum_{i\in\sigma} A_{f_i}\nu_i^2 \|\Lambda_i P_{W_i}(f)\|^2 + \sum_{i\in\sigma^c} A_{g_i}\mu_i^2 \|\Gamma_i P_{V_i}(f)\|^2 \\
&= \sum_{i\in\sigma} A_{f_i} \|\nu_i\Lambda_i P_{W_i}(f)\|^2 + \sum_{i\in\sigma^c} A_{g_i} \|\mu_i\Gamma_i P_{V_i}(f)\|^2 \\
&\le \sum_{i\in\sigma}\sum_{j\in\mathbb{J}_i} |\langle \nu_i\Lambda_i P_{W_i}(f), f_{i,j}\rangle|^2 + \sum_{i\in\sigma^c}\sum_{j\in\mathbb{J}_i} |\langle \mu_i\Gamma_i P_{V_i}(f), g_{i,j}\rangle|^2 \\
&\le \sum_{i\in\sigma} B_{f_i} \|\nu_i\Lambda_i P_{W_i}(f)\|^2 + \sum_{i\in\sigma^c} B_{g_i} \|\mu_i\Gamma_i P_{V_i}(f)\|^2 \\
&\le B_f \sum_{i\in\sigma} \|\nu_i\Lambda_i P_{W_i}(f)\|^2 + B_g \sum_{i\in\sigma^c} \|\mu_i\Gamma_i P_{V_i}(f)\|^2 .
\end{aligned}$$

(i) $\Rightarrow$ *(iii)* Let $\{\nu_i\Lambda_i^* f_{i,j}\}_{i\in\mathbb{I},j\in\mathbb{J}_i}$ and $\{\mu_i\Gamma_i^* g_{i,j}\}_{i\in\mathbb{I},j\in\mathbb{J}_i}$ be Woven frames for $\mathcal{H}$, with universal frame bounds C and D. The above calculation shows that for every $f \in \mathcal{H}$,

$$\sum_{i\in\sigma} \nu_i^2 \|\Lambda_i P_{W_i}(f)\|^2 + \sum_{i\in\sigma^c} \mu_i^2 \|\Gamma_i P_{V_i}(f)\|^2$$

$$\leq \frac{1}{A}\left(\sum_{i\in\sigma}\sum_{j\in\mathbb{J}_i}|\langle\Lambda_i P_{W_i}(f),\nu_i f_{i,j}\rangle|^2+\sum_{i\in\sigma^c}\sum_{j\in\mathbb{J}_i}|\langle\Gamma_i P_{V_i}(f),\mu_i g_{i,j}\rangle|^2\right)$$
$$=\frac{1}{A}\left(\sum_{i\in\sigma}\sum_{j\in\mathbb{J}_i}|\langle f,\nu_i\Lambda_i^* f_{i,j}\rangle|^2+\sum_{i\in\sigma^c}\sum_{j\in\mathbb{J}_i}|\langle f,\mu_i\Gamma_i^* g_{i,j}\rangle|^2\right)$$
$$\leq \frac{D}{A}\|f\|^2,$$

where $A=\min\{A_f,A_g\}$. For lower frame bound,

$$\sum_{i\in\sigma}\nu_i^2\|\Lambda_i P_{W_i}(f)\|^2+\sum_{i\in\sigma^c}\mu_i^2\|\Gamma_i P_{V_i}(f)\|^2$$
$$\geq \frac{1}{B}\left(\sum_{i\in\sigma}\sum_{j\in\mathbb{J}_i}|\langle\Lambda_i P_{W_i}(f),\nu_i f_{i,j}\rangle|^2+\sum_{i\in\sigma^c}\sum_{j\in\mathbb{J}_i}|\langle\Gamma_i P_{V_i}(f),\mu_i g_{i,j}\rangle|^2\right)$$
$$=\frac{1}{B}\left(\sum_{i\in\sigma}\sum_{j\in\mathbb{J}_i}|\langle f,\nu_i\Lambda_i^* f_{i,j}\rangle|^2+\sum_{i\in\sigma^c}\sum_{j\in\mathbb{J}_i}|\langle f,\mu_i\Gamma_i^* g_{i,j}\rangle|^2\right)$$
$$\geq \frac{C}{B}\|f\|^2,$$

for every $f\in\mathcal{H}$, $B=\max\{B_f,B_g\}$.This calculations implies (iii).

$(iii)\Rightarrow(i)$ Let $\{(\Lambda_i,W_i,\nu_i)\}_{i\in\mathbb{I}}$ and $\{(\Gamma_i,V_i,\mu_i)\}_{i\in\mathbb{I}}$ be woven g-fusion frames with universal frame bounds C and D. Then for every $f\in H$, we have

$$\sum_{i\in\sigma}\sum_{j\in\mathbb{J}_i}|\langle f_i\nu_i\Lambda_i^* f_{i,j}\rangle|^2+\sum_{i\in\sigma^c}\sum_{j\in\mathbb{J}_i}|\langle f,\mu_i\Gamma_i^* g_{i,j}\rangle|^2$$
$$=\sum_{i\in\sigma}\sum_{j\in\mathbb{J}_i}|\langle\nu_i\Lambda_i P_{W_i}(f),f_{i,j}\rangle|^2+\sum_{i\in\sigma^c}\sum_{j\in\mathbb{J}_i}|\langle\mu_i\Gamma_i P_{\nu_i}(f),g_{i,j}\rangle|^2$$
$$\geq\sum_{i\in\sigma}A_{f_i}\nu_i^2\|\Lambda_i P_{W_i}(f)\|^2+\sum_{i\in\sigma^c}A_{g_i}\mu_i^2\|\Gamma_i P_{V_i}(f)\|^2$$
$$\geq A\left(\sum_{i\in\sigma}\nu_i^2\|\Lambda_i P_{W_i}(f)\|^2+\sum_{i\in\sigma^c}\mu_i^2\|\Gamma_i P_{V_i}(f)\|^2\right)$$
$$\geq AC\|f\|^2,$$

and similarly

$$\sum_{i\in\sigma}\sum_{j\in\mathbb{J}_i}|\langle f,\nu_i\Lambda_i^* f_{i,j}\rangle|^2+\sum_{i\in\sigma^c}\sum_{j\in\mathbb{J}_i}|\langle f,\mu_i\Gamma_i^* g_{i,j}\rangle|^2\leq BD\|f\|^2.$$

So (i) holds. $(ii) \iff (iii)$ since $\{e_{i,j}\}_{j\in\mathbb{J}_i}$ is an orthonormal basis for subspace $\mathcal{H}_i$, then for $f \in \mathcal{H}$, we have :

$$\begin{aligned}
&\sum_{i\in\sigma} \nu_i^2 \left\|\Lambda_i P_{W_i}(f)\right\|^2 + \sum_{i\in\sigma^c} \mu_i^2 \left\|\Gamma_i P_{V_i}(f)\right\|^2 \\
&\quad= \sum_{i\in\sigma} \nu_i^2 \left\| \sum_{j\in\mathbb{J}_i} \left\langle \Lambda_i P_{W_i} f, e_{i,j}\right\rangle e_{i,j} \right\|^2 \\
&\qquad+ \sum_{i\in\sigma^c} \mu_i^2 \left\| \sum_{j\in\mathbb{J}_i} \left\langle \Gamma_i P_{V_i}(f), e_{i,j}\right\rangle e_{i,j} \right\|^2 \\
&\quad= \sum_{i\in\sigma} \nu_i^2 \sum_{j\in\mathbb{J}_i} \left|\left\langle \Lambda_i P_{W_i}(f), e_{i,j}\right\rangle\right|^2 + \sum_{i\in\sigma^c} \mu_i^2 \sum_{j\in\mathbb{J}_i} \left|\left\langle \Gamma_i P_{V_i}(f), e_{i,j}\right\rangle\right|^2 \\
&\quad= \sum_{i\in\sigma} \sum_{j\in\mathbb{J}_i} \left|\left\langle f, \nu_i \Lambda_i^* e_{i,j}\right\rangle\right|^2 + \sum_{i\in\sigma^c} \sum_{j\in\mathbb{J}_i} \left|\left\langle f, \mu_i \Gamma_i^* e_{i,j}\right\rangle\right|^2 .
\end{aligned}$$

So (ii) is equivalent to (iii). □

In the following theorem, we show that the intersection of components of a woven g-fusion frames with the other subspace, is a woven g-fusion frames for the smaller space.

Theorem 3.3. *Let $\mathcal{K}$ be a closed subspace of $\mathcal{H}$ and let $\{(\Lambda_i, W_i, \nu_i)\}_{i\in\mathbb{I}}$ and $\{(\Gamma_i, V_i, \mu_i)\}_{i\in\mathbb{I}}$ be woven g-fusion frames for $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i\in\mathbb{I}}$ with woven bounds A and B. Then $\{(\Lambda_i, W_i \cap \mathcal{K}, \nu_i)\}_{i\in\mathbb{I}}$ and $\{(\Gamma_i, V_i \cap \mathcal{K}, \mu_i)\}_{i\in\mathbb{I}}$ are woven g-fusion frames for $\mathcal{H}$ with universal bounds A and B.*

Proof. Let the operators $P_{W_i\cap\mathcal{K}} = P_{W_i}(P_{\mathcal{K}})$ and $P_{V_i\cap\mathcal{K}} = P_{V_i}(P_k)$ be orthognal projections of $\mathcal{H}$ onto $W_i \cap \mathcal{K}$ and $V_i \cap \mathcal{K}$, respectively. Then for every $f \in \mathcal{K}$, we can write:

$$\begin{aligned}
&\sum_{i\in\sigma} \nu_i^2 \left\|\Lambda_i P_{W_i}(f)\right\|^2 + \sum_{i\in\sigma^c} \mu_i^2 \left\|\Gamma_i P_{V_i}(f)\right\|^2 \\
&\quad= \sum_{i\in\sigma} \nu_i^2 \left\|\Lambda_i P_{W_i}(P_{\mathcal{K}}(f))\right\|^2 + \sum_{i\in\sigma^c} \mu_i^2 \left\|\Gamma_i P_{V_i}(P_{\mathcal{K}}(f))\right\|^2 \\
&\quad= \sum_{i\in\sigma} \nu_i^2 \left\|\Lambda_i P_{W_i\cap\mathcal{K}}(f)\right\|^2 + \sum_{i\in\sigma^c} \mu_i^2 \left\|\Gamma_i P_{V_i\cap\mathcal{K}}(f)\right\|^2 ,
\end{aligned}$$

which implies the result. □

The next proposition shows that every weaving of g-fusion Bessele, automatically has upper Bessel bound.

Proposition 3.4. *Let* $\{(\Lambda_{ij}, W_{ij}, \nu_{ij})\}_{i\in\mathbb{I}}$ *be a g-fusion Bessele sequence of subspaces for* $\mathcal{H}$ *with bounds* B_j *for all* $j \in [m]$*. Then every weaving of this sequence is a Bessel sequence.*

Proof. For every partition $\{\sigma_j\}_{j\in[m]}$, such that $\sigma_j \subset \mathbb{I}$ for $j \in [m]$ and for $f \in \mathcal{H}$, we have

$$\begin{aligned}\sum_{j=1}^{m}\sum_{i\in\sigma_j}\nu_i^2\left\|\Lambda_{ij}P_{W_{ij}}(f)\right\|^2 &\leq \sum_{j=1}^{m}\sum_{i=1}^{\infty}\nu_i^2\left\|\Lambda_{ij}P_{W_{ij}}(f)\right\|^2\\ &\leq \sum_{j=1}^{m}B_j\left\|f\right\|^2.\end{aligned}$$

□

Next theorem is a generalization of Lemma 4.3 in [2], for g-fusion frames.

Theorem 3.5. *Suppose that* $\{(\Lambda_i, W_i, \nu_i)\}_{i\in\mathbb{I}}$ *and* $\{(\Gamma_i, V_i, \mu_i)\}_{i\in\mathbb{I}}$ *are g-fusion frames for* $\mathcal{H}$ *and also let for every two disjoint finite sets* $I, J \subseteq \mathbb{I}$ *and every* $\epsilon > 0$*, there exist subsets* $\sigma, \delta \subseteq \mathbb{I}\ (I \cup J)$ *such that the lower g-fusion frame bound of* $\{(\Lambda_i, W_i, \nu_i)\}_{i\in(I\cup\sigma)} \cup \{(\Gamma_i, V_i, \mu_i)\}_{i\in(J\cup\delta)}$ *is less than* ϵ*. Then there exists* $\mathcal{M} \subseteq I$ *such that* $\{(\Lambda_i, W_i, \nu_i)\}_{i\in\mathcal{M}} \cup \{(\Gamma_i, V_i, \mu_i)\}_{i\in\mathcal{M}^c}$ *is not a g-fusion frame. Hence* $\{(\Lambda_i, W_i, \nu_i)\}_{i\in\mathbb{I}}$ *and* $\{(\Gamma_i, V_i, \mu_i)\}_{i\in\mathbb{I}}$ *are not woven g-fusion frames.*

Proof. Let $\epsilon > 0$ be arbitrary. By hypothesis, for $I_0 = J_0 = \emptyset$, we can choose $\sigma_1 \subset \mathbb{I}$, so that if $\delta_1 = \sigma_1^c$, then the lower g-fusion frame bound of $\{(\Lambda_i, W_i, \nu_i)\}_{i\in(I\cup\sigma)} \cup \{(\Gamma_i, V_i, \mu_i)\}_{i\in(J\cup\delta)}$ is less than ϵ. Thus there exists $f_1 \in \mathcal{H}$, with $\|f_1\| = 1$ such that

$$\sum_{i\in\sigma_1}\nu_i^2\|\Lambda_i P_{W_i}(f_1)\|^2 + \sum_{i\in\delta_1}\mu_i^2\|\Gamma_i P_{V_i}(f_1)\|^2 < \epsilon.$$

Since$\{(\Lambda_i, W_i, \nu_i)\}_{i\in\mathbb{I}}$ and $\{(\Gamma_i, V_i, \mu_i)\}_{i\in\mathbb{I}}$ are g-fusion frames, so

$$\sum_{i=1}^{\infty}\nu_i^2\|\Lambda_i P_{W_i}(f_1)\|^2 + \sum_{i=1}^{\infty}\mu_i^2\|\Gamma_i P_{V_i}(f_1)\|^2 < \infty,$$

therefor there is a positive integer k_1 such that

$$\sum_{i=k_1+1}^{\infty}\nu_i^2\|\Lambda_i P_{W_i}(f_1)\|^2 + \sum_{i=k_1+1}^{\infty}\mu_i^2\|\Gamma_i P_{V_i}(f_1)\|^2 < \infty.$$

Let $I_1 = \sigma_1 \bigcap [k_1]$ and $J_1 = \delta_1 \bigcap [k_1]$. Then $I_1 \bigcap J_1 = \emptyset$ and $I_1 \bigcup J_1 = [k_1]$. By assumption, there are subsets $\sigma_2, \delta_2 \subset [k_1]^c$ with $\delta_2 = [k_1]^c\ \sigma_2$ such that the lower fusion frame bound of

$$\{(\Lambda_i, W_i, \nu_i)\}_{i\in(I\cup\sigma)} \cup \{(\Gamma_i, V_i, \mu_i)\}_{i\in(J\cup\delta)},$$

is less than $\frac{\epsilon}{2}$, so there exists a vector $f_2 \in \mathcal{H}$ with $\|f\|^2 = 1$, such that

$$\sum_{i \in I_1 \cup \sigma_2} \nu_i^2 \|\Lambda_i P_{W_i}(f_2)\|^2 + \sum_{i \in J_1 \cup \delta_2} \mu_i^2 \|\Gamma_i P_{V_i}(f_2)\|^2 < \frac{\epsilon}{2}.$$

Similarly, there is $k_2 > k_1$ such that

$$\sum_{i=k_2+1}^{\infty} \nu_i^2 \|\Lambda_i P_{W_i}(f_2)\|^2 + \sum_{i=k_2+1}^{\infty} \mu_i^2 \|\Gamma_i P_{V_i}(f_2)\|^2 < \frac{\epsilon}{2}.$$

Set $I_2 = I_1 \bigcup (\sigma_2 \bigcap [k_2])$ and $J_2 = J_1 \bigcup (\delta_2 \bigcap [k_2])$. Note that $I_2 \bigcap J_2 = \emptyset$ and $I_2 \bigcup J_2 = [k_2]$. Thus by induction, there are:

(i) a sequence of natural numbers $\{k_i\}_{i \in \mathbb{I}}$ with $k_i < k_{i+1}$ for all $i \in \mathbb{I}$,

(ii) a sequence of vectors $\{f_i\}_{i \in \mathbb{I}}$ from $\mathcal{H}$ with $\|f_i\| = 1$ for all $i \in \mathbb{I}$,

(iii) subsets $\sigma_i \subset [k_{i-1}]^c, \delta_i = [k_{i-1}]^c \setminus \sigma_i, i \in \mathbb{I}$ and

(iv) $I_i = I_{i-1} \bigcup (\sigma_i \bigcap [k_i]), J_i = J_{i-1} \bigcup (\delta_i \bigcap [k_i]), i \in \mathbb{I}$ which are abiding both:

$$\sum_{i \in I_{n-1} \cup \sigma_n} \nu_i^2 \|\Lambda_i P_{W_i}(f_n)\|^2 + \sum_{i \in J_{n-1} \cup \delta_n} \mu_i^2 \|\Gamma_i P_{V_i}(f_n)\|^2 < \frac{\epsilon}{n},$$

and

$$\sum_{i=k_n+1}^{\infty} \nu_i^2 \|\Lambda_i P_{W_i}(f_n)\|^2 + \sum_{i=k_n+1}^{\infty} \mu_i^2 \|\Gamma_i P_{V_i}(f_n)\|^2 < \frac{\epsilon}{n}.$$

By construction $I_i \bigcap J_i = \emptyset$ and $I_i \bigcup J_i = [k_i]$, if we suppose that $\mathcal{M} = \bigcup_{i=1}^{\infty} I_i$ then $\mathcal{M}^c = \bigcup_{i=1}^{\infty} J_i$ such that $\mathcal{M} \bigcup \mathcal{M}^c = \mathbb{I}$, then we conclude from the above inequalities:

$$\begin{aligned}
&\sum_{i \in \mathcal{M}} \nu_i^2 \|\Lambda_i P_{W_i}(f_i)\|^2 + \sum_{i \in \mathcal{M}^c} \mu_i^2 \|\Gamma_i P_{V_i}(f_i)\|^2 \\
&\quad = \left(\sum_{i \in I_n} \nu_i^2 \|\Lambda_i P_{W_i}(f_n)\|^2 + \sum_{i \in J_n} \mu_i^2 \|\Gamma_i P_{V_i}(f_n)\|^2 \right) \\
&\qquad + \left(\sum_{i \in M \cap [k_n]^c} \nu_i^2 \|\Lambda_i P_{W_i}(f_i)\|^2 + \sum_{i \in M^c \cap [k_n]^c} \mu_i^2 \|\Gamma_i P_{V_i}(f_i)\|^2 \right) \\
&\quad \leq \left(\sum_{i \in I_{n-1} \cup \sigma_n} \nu_i^2 \|\Lambda_i P_{W_i}(f_n)\|^2 + \sum_{i \in J_{n-1} \cup \delta_n} \mu_i^2 \|\Gamma_i P_{V_i}(f_n)\|^2 \right) \\
&\qquad + \left(\sum_{i=k_n+1}^{\infty} \nu_i^2 \|\Lambda_i P_{W_i}(f_i)\|^2 + \sum_{i=k_n+1}^{\infty} \mu_i^2 \|\Gamma_i P_{V_i}(f_i)\|^2 \right)
\end{aligned}$$

$$< \frac{\epsilon}{n} + \frac{\epsilon}{n}$$
$$= \frac{2\epsilon}{n}.$$

Therefore the lower g-fusion frame of

$$\{(\Lambda_i, W_i, \nu_i)\}_{i\in\mathcal{M}} \bigcup \{(\Gamma_i, V_i, \mu_i)\}_{i\in\mathcal{M}^c},$$

is zero, that is a contradiction. Thus $\{(\Lambda_i, W_i, \nu_i)\}_{i\in\mathbb{I}} \bigcup \{(\Gamma_i, V_i, \mu_i)\}_{i\in\mathbb{I}}$ can not be a woven g-fusion frame. □

This section is concluded by showing that the upper bound in Proposition 3.4 can not be optimal for woven g-fusion frames.

Proposition 3.6. *Suppose that $\{(\Lambda_i, W_i, \nu_i)\}_{i\in\mathbb{I}}$ and $\{(\Gamma_i, V_i, \mu_i)\}_{i\in\mathbb{I}}$ are g-fusion frames for $\mathcal{H}$ with optimal upper g-fusion frame bounds B_1 and B_2 such that they be woven g-fusion frames. Then $B_1 + B_2$ can not be the optimal upper woven bound.*

Proof. Assume on the contrary, which is $B_1 + B_2$ is the smallest upper weaving bound for all possible weavings. Then by definition of optimal upper bound, we can choose $\sigma \subset \mathbb{I}$ and $\|f\| = 1$, such that

$$sup_{\|f\|=1}\left(\sum_{i\in\sigma} \nu_i^2 \|P_{W_i}(f)\|^2 + \sum_{i\in\sigma^c} \mu_i^2 \|P_{V_i}(f)\|^2\right) = B_1 + B_2.$$

Using of supreme property, for every $\epsilon > 0$, there exists $f \in \mathcal{H}$, such that

$$\sum_{i\in\sigma} \nu_i^2 \|\Lambda_i P_{W_i}(f)\|^2 + \sum_{i\in\mathbb{I}} \mu_i^2 \|\Gamma_i P_{V_i}(f)\|^2 \geq B_1 + B_2 - \varepsilon,$$

and using of upper fusion frame property, we have

$$\sum_{i\in\sigma} \nu_i^2 \|\Lambda_i P_{W_i}(f)\|^2 + \sum_{i\in\mathbb{I}} \mu_i^2 \|\Gamma_i P_{V_i}(f)\|^2 \leq B_1 + B_2.$$

So:

$$\sum_{i\in\mathbb{I}\setminus\sigma} \nu_i^2 \|\Lambda_i P_{W_i}(f)\|^2 + \sum_{i\in\mathbb{I}\setminus\sigma^c} \mu_i^2 \|\Gamma_i P_{V_i}(f)\|^2 \leq \varepsilon.$$

Now, if we assume $\sigma_1 = \mathbb{I} \setminus \sigma$, then $\sigma_1^c = \mathbb{I} \setminus \sigma^c$. Therefore

$$\sum_{i\in\sigma_1} \nu_i^2 \|\Lambda_i P_{W_i}(f)\|^2 + \sum_{i\in\sigma_1^c} \mu_i^2 \|\Gamma_i P_{V_i}(f)\|^2 \leq \varepsilon,$$

and this shows that there is a weaving for which the lower frame bound approaches zero. Theorem 3.5 gives that $\{(\Lambda_i, W_i, \nu_i)\}_{i\in\mathbb{I}}$ and $\{(\Gamma_i, V_i, \mu_i)\}_{i\in\mathbb{I}}$ are not woven g-fusion frame, which is a contradiction. □

Proposition 3.7. *Let* $\{(\Lambda_i, W_i, \nu_i)\}_{i\in J}$ *and* $\{(\Gamma_i, V_i, \mu_i)\}_{i\in J}$ *be g-fusion frames, such that* $J \subset \mathbb{I}$*. Then* $\{(\Lambda_i, W_i, \nu_i)\}_{i\in \mathbb{I}}$ *and* $\{(\Gamma_i, V_i, \mu_i)\}_{i\in \mathbb{I}}$ *are woven g-fusion frames.*

Proof. Let the positive constants A be the lower woven bound for $\{(\Lambda_i, W_i, \nu_i)\}_{i\in J}$ and $\{(\Gamma_i, V_i, \mu_i)\}_{i\in J}$. Then for every $\sigma \subset \mathbb{I}$ and $f \in \mathcal{H}$, we have

$$\begin{aligned} A\|f\|^2 &\leq \sum_{i\in\sigma\cap J} \nu_i^2 \|\Lambda_i P_{W_i}(f)\|^2 + \sum_{i\in\sigma^c\cap J} \mu_i^2 \|\Gamma_i P_{V_i}(f)\|^2 \\ &\leq \sum_{i\in\sigma} \nu_i^2 \|\Lambda_i P_{W_i}(f)\|^2 + \sum_{i\in\sigma^c} \mu_i^2 \|\Gamma_i P_{V_i}(f)\|^2 \\ &\leq (B_\Lambda + B_\Gamma)\|f\|^2, \end{aligned}$$

where B_Λ and B_Γ are upper fusion frame bounds for $\{(\Lambda_i, W_i, \nu_i)\}_{i\in J}$ and $\{(\Gamma_i, V_i, \mu_i)\}_{i\in J}$ respectively. ☐

Proposition 3.8. *Suppose* $\{(\Lambda_i, W_i, \nu_i)\}_{i\in \mathbb{I}}$ *and* $\{(\Gamma_i, V_i, \mu_i)\}_{i\in \mathbb{I}}$ *are g-fusion frames for* $\mathcal{H}$ *with universal woven bounds* A *and* B*. For some constants* $0 < D < A$ *and* $J \subset \mathbb{I}$*, if we have:*

$$\sum_{i\in J} \nu_i^2 \|\Lambda_i P_{W_i}(f)\|^2 \leq D\|f\|^2, \quad \forall f \in \mathcal{H}.$$

Then $\{(\Lambda_i, W_i, \nu_i)\}_{i\in \mathbb{I}\setminus J}$ *and* $\{(\Gamma_i, V_i, \mu_i)\}_{i\in \mathbb{I}\setminus J}$ *are g-fusion frames for* $\mathcal{H}$ *and are woven g-fusion frames with universal lower and upper woven bounds* $A - D$ *and* B*, respectively.*

Proof. Assume $\sigma \subset \mathbb{I} \setminus J$. Then for all $f \in \mathcal{H}$

$$\begin{aligned} &\sum_{i\in\sigma} \nu_i^2 \|\Lambda_i P_{W_i}(f)\|^2 + \sum_{i\in(\mathbb{I}\setminus J)\setminus\sigma} \mu_i^2 \|\Gamma_i P_{V_i}(f)\|^2 \\ &\quad= \left(\sum_{i\in\sigma\cup J} \nu_i^2 \|\Lambda_i P_{W_i}(f)\|^2 - \sum_{i\in J} \nu_i^2 \|\Lambda_i P_{W_i}(f)\|^2\right) \\ &\qquad+ \sum_{i\in(\mathbb{I}\setminus J)\setminus\sigma} \mu_i^2 \|\Gamma_i P_{V_i}(f)\|^2 \\ &\quad= \left(\sum_{i\in\sigma\cup J} \nu_i^2 \|\Lambda_i P_{W_i}(f)\|^2 + \sum_{i\in\mathbb{I}\setminus(J\cup\sigma)} \mu_i^2 \|\Gamma_i P_{V_i}(f)\|^2\right) \\ &\qquad- \sum_{i\in J} \nu_i^2 \|\Lambda_i P_{W_i}(f)\|^2 \\ &\quad\geq (A - D)\|f\|^2. \end{aligned}$$

For upper woven bound, we have

$$\sum_{i\in\sigma}\nu_i^2\left\|\Lambda_i P_{W_i}(f)\right\|^2+\sum_{i\in(\mathbb{I}\setminus J)\setminus\sigma}\mu_i^2\left\|\Gamma_i P_{V_i}(f)\right\|^2$$
$$\leq\sum_{i\in\sigma\cup J}\nu_i^2\left\|\Lambda_i P_{W_i}(f)\right\|^2+\sum_{i\in\mathbb{I}\setminus(\sigma\cup J)}\mu_i^2\left\|\Gamma_i P_{V_i}(f)\right\|^2$$
$$\leq B\left\|f\right\|^2.$$

Thus $\{(\Lambda_i, W_i, \nu_i)\}_{i\in\mathbb{I}\setminus J}$ and $\{(\Gamma_i, V_i, \mu_i)\}_{i\in\mathbb{I}\setminus J}$ are woven g-fusion frames. Now, if we take $\sigma=\mathbb{I}$ and $\sigma^c=\phi$, then $\{(\Lambda_i, W_i, \nu_i)\}_{i\in\mathbb{I}\setminus J}$ is a g-fusion frame:

$$\sum_{i\in\mathbb{I}\setminus J}\nu_i^2\left\|\Lambda_i P_{W_i}(f)\right\|^2$$
$$=\sum_{i\in\mathbb{I}}\nu_i^2\left\|\Lambda_i P_{W_i}(f)\right\|^2-\sum_{i\in J}\nu_i^2\left\|\Lambda_i P_{W_i}(f)\right\|^2$$
$$=\sum_{i\in\sigma}\nu_i^2\left\|\Lambda_i P_{W_i}(f)\right\|^2+\sum_{i\in\sigma^c}\mu_i^2\left\|\Gamma_i P_{V_i}(f)\right\|^2-\sum_{i\in J}\nu_i^2\left\|P_{W_i}(f)\right\|^2$$
$$\geq(A-D)\left\|f\right\|^2.$$

Similar to above, we can demonstrate that $\{(\Gamma_i, V_i, \mu_i)\}_{i\in\mathbb{I}\setminus J}$ is a g-fusion frame with same bounds. □

4. Perturbation of Woven g-fusion Frames

In this section, we show that those of g-fusion frames that are small perturbations of each other, constitute woven g-fusion frame. We start this section with Paley-Wiener perturbation of weaving g-fusion frames and continue two results of perturbations in the sequel.

Theorem 4.1. *Let $\{(\Lambda_i, W_i, \nu_i)\}_{i\in\mathbb{I}}$ and $\{(\Gamma_i, V_i, \mu_i)\}_{i\in\mathbb{I}}$ be g-fusion frames for $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i\in\mathbb{I}}$ with g-fusion frame bounds (A_Λ, B_Λ) and (A_Γ, B_Γ), respectively. If there exist constants $0<\lambda_1,\lambda_2,\mu<1$ such that:*

$$\frac{2}{A_\Lambda}\left(\sqrt{B_\Lambda}+\sqrt{B_\Gamma}\right)\left(\lambda_1\sqrt{B_\Lambda}+\lambda_2\sqrt{B_\Gamma}+\mu\right)\leq 1$$

and

$$\left\|T_\Lambda(f)-T_\Gamma(f)\right\|\leq\lambda_1\left\|T_\Lambda(f)\right\|+\lambda_2\left\|T_\Gamma(f)\right\|+\mu, \tag{4.1}$$

where T_Λ, T_Γ are the analysis operators for these g-fusion frames, then $\{(\Lambda_i, W_i, \nu_i)\}_{i\in\mathbb{I}}$ and $\{(\Gamma_i, V_i, \mu_i)\}_{i\in\mathbb{I}}$ are woven g-fusion frames.

Proof. For each $\sigma \subset \mathbb{I}$, we define the bounded operators

$$T_\Lambda^\sigma : \left(\sum_{i\in\sigma} \oplus \mathcal{H}_i\right) \to \mathcal{H}, \qquad T_\Lambda^\sigma(f) = \sum_{i\in\sigma} \nu_i P_{W_i} \Lambda_i^* f_i,$$

and

$$T_\Gamma^\sigma : \left(\sum_{i\in\sigma} \oplus \mathcal{H}_i\right) \to \mathcal{H}, \qquad T_\Gamma^\sigma(f) = \sum_{i\in\sigma} \mu_i P_{V_i} \Gamma_i^* f_i,$$

for every $f = \{f_i\}_{i\in\mathbb{I}} \in \left(\sum\limits_{i\in\sigma} \oplus \mathcal{H}_i\right)$. Note that $\|T_\Lambda^\sigma(f)\| \leq \|T_\Lambda(f)\|$, $\|T_\Gamma^\sigma(f)\| \leq \|T_\Gamma(f)\|$ and $\|T_\Lambda^\sigma(f) - T_\Gamma^\sigma(f)\| \leq \|T_\Lambda(f) - T_\Gamma^\sigma(f)\|$ because for $f = \{f_i\}_{i\in\mathbb{I}} \in \left(\sum\limits_{i\in\sigma} \oplus \mathcal{H}_i\right)$,

$$\begin{aligned}
\|T_\Lambda^\sigma(f)\|^2 &= \sum_{i\in\sigma} \nu_i^2 \left\|P_{W_i}\Lambda_i^* f_i\right\|^2 \\
&\leq \sum_{i\in\mathbb{I}} \nu_i^2 \left\|P_{W_i}\Lambda_i^* f_i\right\|^2 \\
&= \|T_\Lambda(f)\|^2 .
\end{aligned}$$

Using statement (4.1), for every $f \in \mathcal{H}$ and $\sigma \in \mathbb{I}$, we have

$$\begin{aligned}
\|T_\Lambda^\sigma U_\Lambda^\sigma(f) - T_\Gamma^\sigma U_\Gamma^\sigma(f)\| &= \|T_\Lambda^\sigma U_\Lambda^\sigma(f) - T_\Lambda^\sigma U_\Gamma^\sigma(f) + T_\Lambda^\sigma U_\Gamma^\sigma(f) - T_\Gamma^\sigma U_\Gamma^\sigma(f)\| \\
&\leq \|T_\Lambda^\sigma(U_\Lambda^\sigma - U_\Gamma^\sigma)(f)\| + \|(T_\Lambda^\sigma - T_\Gamma^\sigma)U_\Gamma^\sigma(f)\| \\
&\leq \|T_\Lambda^\sigma\| \|U_\Lambda^\sigma - U_\Gamma^\sigma\| \|f\| + \|T_\Lambda^\sigma - T_\Gamma^\sigma\| \|U_\Gamma^\sigma\| \|f\| \\
&\leq \|T_\Lambda\| \|T_\Lambda - T_\Gamma\| \|f\| + \|T_\Lambda - T_\Gamma\| \|T_\Gamma\| \|f\| \\
&= \|T_\Lambda - T_\Gamma\| (\|T_\Lambda\| - \|T_\Gamma\|) \|f\| \\
&\leq \left(\lambda_1\sqrt{B_\Lambda} + \lambda_2\sqrt{B_\Gamma} + \mu\right)\left(\sqrt{B_\Lambda} + \sqrt{B_\Gamma}\right)\|f\| \\
&\leq \frac{A_\Lambda}{2}\|f\| .
\end{aligned}$$

Now by using above calculation, we have

$$\begin{aligned}
S_\Lambda^{\sigma^c} + S_\Gamma^\sigma &= S_\Lambda + S_\Gamma^\sigma - S_\Lambda^\sigma \\
&\geq A_\Lambda I - \|S_\Lambda^\sigma - S_\Gamma^\sigma\| I \\
&\geq A_\Lambda I - \frac{A_\Lambda}{2} I = \frac{A_\Lambda}{2} I.
\end{aligned}$$

This shows that $\dfrac{A_\Lambda}{2}$ is the universal lower woven bound. Finally, for universal upper bound, we have

$$\sum_{i\in\sigma^c} \nu_i^2 \|\Lambda_i P_{W_i} f\|^2 + \sum_{i\in\sigma} \mu_i^2 \|\Gamma_i P_{V_i} f\|^2 \leq \sum_{i\in\mathbb{I}} \nu_i^2 \|\Lambda_i P_{W_i} f\|^2 + \sum_{i\in\mathbb{I}} \mu_i^2 \|\Gamma_i P_{V_i} f\|^2$$

$$\leq (B_\Lambda + B_\Gamma) \|f\| .$$

□

Theorem 4.2. *Let $\{(\Lambda_i, W_i, \nu_i)\}_{i\in\mathbb{I}}$ and $\{(\Gamma_i, V_i, \mu_i)\}_{i\in\mathbb{I}}$ be g-fusion frames for $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i\in\mathbb{I}}$ and g-fusion frame bounds (A_Λ, B_Λ) and (A_Γ, B_Γ), respectively. If there exist constants $0 < \lambda, \mu, \gamma < 1$, such that $\lambda B_\Lambda + \mu B_\Gamma + \gamma\sqrt{B_\Lambda} < A_\Lambda$. We have*

$$S_\Lambda^\sigma < \lambda S_\Lambda^\sigma + \mu S_\Gamma^\sigma + \gamma U_\Lambda^\sigma,$$

where S_Λ, U_Λ are g-fusion frame operators of $\{(\Lambda_i, W_i, \nu_i)\}_{i\in\mathbb{I}}$. Then $\{(\Lambda_i, W_i, \nu_i)\}_{i\in\mathbb{I}}$ and $\{(\Gamma_i, V_i, \mu_i)\}_{i\in\mathbb{I}}$ are woven g-fusion frame with universal woven bounds

$$\left(A_\Lambda - \lambda B_\Lambda - \mu B_\Gamma - \gamma\sqrt{B_\Lambda}\right), \qquad \left(B_\Gamma + \lambda B_\Lambda + \mu B_\Gamma + \gamma\sqrt{B_\Lambda}\right).$$

Proof. First, for lower frame bound, we have

$$\begin{aligned} S_\Lambda^\sigma + S_\Gamma^{\sigma^c} &= S_\Lambda + S_\Gamma^{\sigma^c} - S_\Lambda^{\sigma^c} \\ &= S_\Lambda - \left(S_\Lambda^{\sigma^c} - S_\Gamma^{\sigma^c}\right) \\ &\geq A_\Lambda I - \left(\lambda S_\Lambda^{\sigma^c} + \mu S_\Gamma^{\sigma^c} + \gamma U_\Lambda^{\sigma^c}\right) \\ &\left(A_\Lambda - \lambda B_\Lambda - \mu B_\Gamma - \gamma\sqrt{B_\Lambda}\right) I. \end{aligned}$$

Also, for upper frame bound, we have

$$\begin{aligned} S_\Lambda^\sigma + S_\Gamma^{\sigma^c} &= S_\Gamma + S_\Lambda^\sigma - S_\Gamma^\sigma \\ &\leq \left(B_\Gamma + \lambda B_\Lambda + \mu B_\Gamma + \gamma\sqrt{B_\Lambda}\right) I. \end{aligned}$$

Therefore g-fusion frames $\{(\Lambda_i, W_i, \nu_i)\}_{i\in\mathbb{I}}$ and $\{(\Gamma_i, V_i, \mu_i)\}_{i\in\mathbb{I}}$ are woven g-fusion frame with considered bounds. □

Theorem 4.3. *Let $\{(\Lambda_i, W_i, \nu_i)\}_{i\in\mathbb{I}}$ and $\{(\Gamma_i, V_i, \nu_i)\}_{i\in\mathbb{I}}$ be g-fusion frames of $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i\in\mathbb{I}}$ and g-fusion frame bounds (A_Λ, B_Λ) and (A_Γ, B_Γ), respectively. Also, if there exists a constant $K > 0$, such that for every $\sigma \subseteq \mathbb{I}$:*

$$\sum_{i\in\sigma} \nu_i^2 \|\Lambda_i P_{W_i} f - \Gamma P_{\nu_i} f\| \leq K \min\left\{\sum_{i\in\sigma} \nu_i^2 \|\Lambda_i P_{W_i} f\|, \sum_{i\in\sigma} \nu_i^2 \|\Gamma_i P_{\nu_i} f\|\right\},$$

then $\{(\Lambda_i, W_i, \nu_i)\}_{i\in\mathbb{I}}$ and $\{(\Gamma_i, V_i, \nu_i)\}_{i\in\mathbb{I}}$ are woven g-fusion frame.

Proof. Let $\sigma \subseteq \mathbb{I}$ be an arbitrary set. By hypothesis for every $f \in \mathcal{H}$, we have

$$\begin{aligned} &(A_\Lambda + A_\Gamma) \|f\|^2 \\ &\quad\leq \sum_{i\in II} \nu_i^2 \|\Lambda_i P_{W_i}(f)\|^2 + \sum_{i\in II} \nu_i^2 \|\Gamma_i P_{V_i}(f)\|^2 \end{aligned}$$

$$
\begin{aligned}
&= \left(\sum_{i\in\sigma} \nu_i^2 \left\| \Lambda_i P_{W_i}(f) \right\|^2 + \sum_{i\in\sigma^c} \left\| \Lambda_i P_{W_i}(f) - \Gamma_i P_{V_i}(f) + \Gamma_i P_{V_i}(f) \right\|^2 \right) \\
&\quad + \left(\sum_{i\in\sigma} \nu_i^2 \left\| \Gamma_i P_{V_i}(f) - \Lambda_i P_{W_i}(f) + \Lambda_i P_{W_i}(f) \right\|^2 + \sum_{i\in\sigma^c} \nu_i^2 \left\| \Gamma_i P_{\nu_i} f \right\|^2 \right) \\
&\leq \sum_{i\in\sigma} \nu_i^2 \left\| \Lambda_i P_{W_i}(f) \right\|^2 + 2 \sum_{i\in\sigma^c} \nu_i^2 \left\| \Lambda_i P_{W_i}(f) - \Gamma_i P_{V_i}(f) \right\|^2 \\
&\quad + 2 \sum_{i\in\sigma^c} \nu_i^2 \left\| \Gamma_i P_{V_i}(f) \right\|^2 + 2 \sum_{i\in\sigma} \nu_i^2 \left\| \Lambda_i P_{W_i}(f) - \Gamma_i P_{V_i}(f) \right\|^2 \\
&\quad + 2 \sum_{i\in\sigma} \nu_i^2 \left\| \Lambda_i P_{W_i}(f) \right\|^2 + \sum_{i\in\sigma^c} \left\| \Gamma_i P_{V_i}(f) \right\|^2 \\
&\leq \sum_{i\in\sigma} \nu_i^2 \left\| \Lambda_i P_{W_i}(f) \right\|^2 + 2 \left(K \sum_{i\in\sigma^c} \nu_i^2 \left\| \Gamma_i P_{V_i}(f) \right\|^2 + \sum_{i\in\sigma^c} \nu_i^2 \left\| \Gamma_i P_{V_i}(f) \right\|^2 \right) \\
&\quad + 2 \left(K \sum_{i\in\sigma} \nu_i^2 \left\| \Lambda_i P_{W_i}(f) \right\| + \sum_{i\in\sigma} \nu_i^2 \left\| \Lambda_i P_{W_i}(f) \right\|^2) + \sum_{i\in\sigma^c} \nu_i^2 \left\| \Gamma_i P_{V_i}(f) \right\|^2 \right) \\
&\quad \times (2K+3) \left(\sum_{i\in\sigma} \nu_i^2 \left\| \Lambda_i P_{W_i}(f) \right\|^2 + \sum_{i\in\sigma^c} \nu_i^2 \left\| \Gamma_i P_{V_i}(f) \right\|^2 \right),
\end{aligned}
$$

then

$$
\begin{aligned}
\frac{A_\Lambda + A_\Gamma}{2K+3} \left\| f \right\|^2 &\leq \sum_{i\in\sigma} \nu_i^2 \left\| \Lambda_i P_{W_i}(f) \right\|^2 + \sum_{i\in\sigma^c} \nu_i^2 \left\| \Gamma_i P_{V_i}(f) \right\|^2 \\
&\leq (B_\Lambda + B_\Gamma) \left\| f \right\|^2.
\end{aligned}
$$

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