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Boundary Value Problems in Thermo Viscoplasticity

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Abstract. In this work, we study two uncoupled quasistatic problems for thermo viscoplastic materials. In the model of the equation of generalised thermo viscoplasticity, both the elastic and the plastic rate of deformation depend on a parameter θ which may be interpreted as the absolute temperature. The boundary conditions considered here as displacement-traction conditions as well as unilateral contact conditions. We establish a variational formulation for the model and we prove the existence of a unique weak solution to the problem, reducing the isotherm problem to an ordinary differential equation in a Hilbert space.

1. Introduction

In this work, we analyze two models for thermo viscoplastic materials. The thermo viscoplasticity effect is characterized by the coupling between the mechanical, and the thermal properties of material. The thermo viscoplasticity laws have been studied by mathematians, physicists and engineers in order to model the effect of temperature in the behavior of some real bodies like metals, magmas, polymers and so on. For more details see [1, 2] and [12]. The constitutive laws with internal state variables has been used in various publications, see for example [6, 7] and [8, 11]. We are study some thermomechanical problems in [4, 5]. Examples and mechanical interpretation of thermo viscoplasticity can be found in [10]. In order to describe the behavior of real problems for materials, we consider a rate type constitutive equation of the form

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$$\dot{\sigma} = \xi\varepsilon(\dot{u}) + \mathcal{G}(\sigma, \varepsilon, \theta). \tag{1.1}$$

In (1.1) u, σ represent, respectively, the displacement field and the stress field, $\varepsilon(u) = (\varepsilon_{ij}(u))$ is the linearised strain tensor

$$\left[\varepsilon_{ij}(u) = \frac{1}{2}(\nabla u \ + \nabla^T u)\right],$$

θ the absolute temperature, ξ the fourth order elastic tensor and $\mathcal{G}$ is a nonlinear constitutive function, which describes the thermo-plastic behavior of the material. Situations of such problem are very common in industry, and geology. We consider for the heat flux q a constitutive classical Fourier law given by

$$q = K\nabla\theta. \tag{1.2}$$

Existence and uniqueness results for problems (1.1)- (1.2) were obtained by many authers using different functional methods (fixed point, monotony and other methods), see for example [12] in the case of classical displacement traction boundary conditions. In many researches, investigation are formulated on the basis of the generalized thermo viscoplastic theories whith temperature independent mechanical properties. The aim of this paper is to study the effects of temperature dependence of ξ on the behavior of the solution in generalized thermoviscoplastic. For this, we consider a rate type constitutive equations of the form

$$\dot{\sigma} = \xi(\theta)\varepsilon(\dot{u}) + \mathcal{G}(\sigma, \varepsilon, \theta). \tag{1.3}$$

In the future research, we are interested in the properties of the solution (dependence of the solution on the parameter θ, and the stability of the solution).

The paper is organized as follows. In Section 2 we describe the mathematical model for the problem. Also, we introduce some notations, list the assumptions on the problem's data, and derive the variational formulation of the model. In Section 3 we state our main existence and uniqueness result which is based on a Cauchy Lipschitz technique.

2. Problem Statement

Let Ω be a bounded domain in $\mathbb{R}^d (d = 1, 2, 3)$ with a smooth boundary Γ which partitioned into three disjoint measurable parts Γ_1, Γ_2 and Γ_3, such that $meas\Gamma_1 > 0$. Let $T > 0$ and let $[0, T]$ denote the time interval of interest.

We consider the following mixed problem:

Problem P

Find a displacement field $u : \Omega \times (0,T) \to \mathbb{R}^d$, a stress field $\sigma : \Omega \times (0,T) \to \mathbb{S}_d$, a temperature $\theta : \Omega \times (0,T) \to \mathbb{R}$, and the heat flux function $q : \Omega \times (0,T) \to \mathbb{R}^d$ such that

$$\dot{\sigma} = \xi(\theta)\varepsilon(\dot{u}) + \mathcal{G}(\sigma, \varepsilon, \theta), \quad \text{in } \Omega \times (0,T), \tag{2.1}$$

$$Div\sigma + f_0 = 0, \quad \text{in } \Omega \times (0,T), \tag{2.2}$$

$$divq + r = 0, \quad \text{in } \Omega \times (0,T), \tag{2.3}$$

$$q = K\nabla\theta, \quad \text{in } \Omega \times (0,T), \tag{2.4}$$

$$u = 0 \quad \text{on } \Gamma_1 \times (0,T), \tag{2.5}$$

$$\sigma \cdot \nu = f_2, \quad \text{on } \Gamma_2 \times (0,T), \tag{2.6}$$

$$\theta = \alpha \quad \text{on } \tilde{\Gamma}_1 \times (0,T), \tag{2.7}$$

$$q \cdot \nu = \beta \quad \text{on } \tilde{\Gamma}_2 \times (0,T), \tag{2.8}$$

$$u(0) = u_0, \sigma(0) = \sigma_0, \quad \text{in } \Omega. \tag{2.9}$$

Here $\mathbb{S}_d$ is the set of second order symmetric tensors on $\mathbb{R}^d$, $\nu = (\nu_i)$ is the unit outward vector field normal to Ω and u_0, σ_0 are the initial data .

We consider the following boundary conditions

$$u_\nu \leq 0, \qquad \sigma_\nu \leq 0, \qquad \sigma_\tau = 0, \qquad \sigma_\nu.u_\nu = 0, \quad \text{on } \Gamma_3 \times (0,T), \tag{2.10}$$

In this way, we obtain two initial and boundary value problems $(\mathbf{P}_i)$, defined as follows:

Problem $\mathbf{P}_1$ Find the unknowns (u, σ, θ, q) such that (2.1)-(2.9) are satisfied. This problem represents a displacement traction problem, in this case $\Gamma_3 = \phi$.

Problem $\mathbf{P}_2$ Find the unknowns (u, σ, θ, q) such that (2.1)-(2.10) hold. This problem models the frictionless contact between the thermo viscoplastic body and the rigid foundation, (2.10) represents the Signorini's boundary conditions.

2.1. **Variational Formulation.** For a weak formulation, we list the assumptions on the data and derive variational formulations for the contact problems $(\mathbf{P}_i)$. To this end, we need to introduce some notations and preliminary material. For more details, we refer the reader to [3, 9]. We denote by $\mathbb{S}_d$ the space of second order symmetric tensors on $\mathbb{R}^d$ $(d = 2, 3)$, while $\|\cdot\|$ denotes the Euclidean norm.

Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with Lipschitz boundary Γ and let ν denote the unit outer normal on $\partial\Omega = \Gamma$. We shall use the notations

$$H = \tilde{H} = \left[L^2(\Omega)\right]^d,$$
$$\mathcal{H} = \tilde{\mathcal{H}} = \left[L^2(\Omega)\right]_s^{d\times d},$$

and

$$H_1 =\{u = (u_i) \in H : \varepsilon(u) \in \mathcal{H}\},$$
$$\tilde{H}_1 =\{\theta \in \tilde{H} : \nabla\theta \in \tilde{\mathcal{H}}\},$$

$$\mathcal{H}_1=\{\sigma \in \mathcal{H} : Div\sigma \in H\},$$
$$\mathcal{V} =\{\sigma \in \mathcal{H}_1 : Div\sigma = 0 \text{ in } \Omega, \sigma\nu = 0 \quad on\ \Gamma_1\},$$

$$\tilde{\mathcal{H}}_1=\left\{q \in \tilde{\mathcal{H}} : divq \in \tilde{H}\right\},$$
$$\tilde{\mathcal{V}} =\left\{q \in \tilde{\mathcal{H}}_1 : divq = 0 \text{ in } \Omega, q\nu = 0 \quad on\ \Gamma_1\right\},$$

where $\varepsilon : H \to \mathcal{H}$, $\nabla : \tilde{H} \to \tilde{\mathcal{H}}$, $Div : \mathcal{H}\to H$, and $div : \tilde{\mathcal{H}}\to\tilde{H}$ are the partial derivative operators of the first order, respectively, defined by

$$\left[\varepsilon(u) = (\varepsilon_{ij}(u)), \quad \varepsilon_{ij}(u) = \tfrac{1}{2}(u_{i,j} + u_{j,i})\right], \quad \left[\nabla\theta = (\nabla_i\theta), \quad \nabla_i\theta = \tfrac{\partial\theta}{\partial x_i}\right],$$
$$Div\sigma = \left(\tfrac{\partial\sigma_{ij}}{\partial x_j}\right), \qquad divq = \left(\tfrac{\partial q_i}{\partial x_i}\right).$$

Here , the indices i and j run from 1 to d, The spaces H, H_1, $\mathcal{H}$, $\mathcal{H}_1$, $\tilde{H}$, $\tilde{H}_1$, $\tilde{\mathcal{H}}$ and $\tilde{\mathcal{H}}_1$ are real Hilbert spaces endowed with the canonical inner products given by:

$$(u, v)_H = \int_\Omega u_i v_i dx,$$
$$(u, v)_{H_1} = (u, v)_H + (\varepsilon(u), \varepsilon(v))_{\mathcal{H}},$$
$$(\sigma, \tau)_{\mathcal{H}} = \int_\Omega \sigma_{ij}.\tau_{ij} dx,$$
$$(\sigma, \tau)_{\mathcal{H}_1} = (\sigma, \tau)_{\mathcal{H}} + (Div\sigma, Div\tau)_H,$$
$$(\theta, \varphi)_{\tilde{H}} = \int_\Omega \theta_i\varphi_i dx, ,$$
$$(\theta, \varphi)_{\tilde{H}_1} = (\theta, \varphi)_{\tilde{H}} + (\nabla\theta, \nabla\varphi)_{\tilde{\mathcal{H}}},$$
$$(q, p)_{\tilde{\mathcal{H}}} = \int_\Omega q_{ij}.p_{ij} dx, ,$$
$$(q, p)_{\tilde{\mathcal{H}}_1} = (q, p)_{\tilde{\mathcal{H}}} + (divq, divp)_{\tilde{H}}.$$

The associated norms are denoted by $\|\cdot\|_H$, $\|\cdot\|_{H^1}$, $\|\cdot\|_{\mathcal{H}}$, $\|\cdot\|_{\mathcal{H}_1}$, $\|\cdot\|_{\tilde{H}}$, $\|\cdot\|_{\tilde{H}_1}$, $\|\cdot\|_{\tilde{\mathcal{H}}}$, and $\|\cdot\|_{\tilde{\mathcal{H}}_1}$, respectively. Also, for any real normed space X, we denote by X' its strong dual, by $\|\cdot\|_X$, $\|\cdot\|_{X'}$ the norms on X and X', respectively and by $\langle,\rangle_{X',X}$ the canonical duality pairing between X and X'. If in addition X is a real Hilbert space and $A : X \to X$ is a continuous symmetric and positively definite linear operator, we denote by $\langle , \rangle_{A,X}$ and $\| \cdot \|_{A,X}$ the energetical product and the energetical norm induced by A on X.

Let

$$H_\Gamma = (H^{1/2}(\Gamma))^d, \qquad \tilde{H}_\Gamma = (\tilde{H}^{1/2}(\Gamma))^d,$$

and

$$\gamma : H_1(\Gamma)^d \to H_\Gamma, \qquad \tilde{\gamma} : \tilde{H}^1(\Gamma)^d \to \tilde{H}_\Gamma,$$

be the trace map. We introduce the following closed subspaces of H_1 and $\tilde{H}_1$

$$\begin{aligned} V =& \{u \in H_1 : \gamma u = 0 \quad on \quad \Gamma_1\}, \\ \tilde{V} =& \left\{\theta \in \tilde{H}_1 : \tilde{\gamma}\theta = 0 \quad on \quad \tilde{\Gamma}_1\right\}. \end{aligned}$$

We introduce the following notations for the problems (P_i)

$$L\left(t,v\right) = \left\langle f_0\left(t\right), v\right\rangle + \left\langle f_2\left(t\right), \gamma v\right\rangle_{L^2(\Gamma_2)}.$$

For the problem P_1, we have:

$$\begin{aligned} &U_{ad} = V, \\ &\sum_{ad}\left(t,v\right) = \{\tau \in \mathcal{H};\ \ \langle \tau, \varepsilon\left(w\right)\rangle = L\left(t,w\right); \quad \forall w \in V\}, \\ &\left(\sum_{ad} \text{ does not depend on } V\right). \end{aligned}$$

For the problem P_2, we have:

$$\begin{aligned} &U_{ad} = \left\{v \in V;\ v_\nu \le 0 \text{ on } \Gamma_3\right\}, \\ &\sum_{ad}\left(t,v\right) = \left\{\tau \in \mathcal{H};\ \ \langle \tau, \varepsilon\left(w\right) - \varepsilon\left(v\right)\rangle \ge L\left(t, w-v\right); \quad \forall w \in V\right\}. \end{aligned}$$

In the study of the Problem $(\mathbf{P}_i)$, we consider the following assumptions:

The *operator* $\xi : \mathbb{R}^d \times \mathbb{S}_d \to \mathbb{S}_d$ satisfies

$$(2.11) \qquad \left\{ \begin{array}{l} \text{(a) There exists } L_\xi > 0 \text{ such that} \\ \|\xi(\theta_1) - \xi(\theta_2)\| \le L_\xi \|\theta_1 - \theta_2\| \text{ ,for all } \theta_1, \theta_2 \in \mathbb{R}^d, \\ \text{(b) } \xi(\theta).\sigma.\tau = \sigma.\xi(\theta).\tau, \quad \forall \theta \in \mathbb{R}^d, \quad \forall \sigma, \tau \in \mathbb{S}_d, \\ \text{(c) There exists } \alpha > 0 \text{ such that } \ \xi(\theta).\sigma.\sigma \ge \alpha \|\sigma\|^2, \\ \quad \forall \theta \in \mathbb{R}^d, \forall \sigma \in \mathbb{S}_d, \\ \text{(d) } \xi(\theta) \text{ is Lebesgue measurable on } \Omega, \\ \text{(e) There exists } \beta > 0 \text{ such that } \ \ \|\xi(\theta)\| \le \beta. \end{array} \right.$$

The *operator* $\mathcal{G} : \mathbb{S}_d \times \mathbb{S}_d \times \mathbb{R}^d \to \mathbb{S}_d$ satisfies

$$\text{(2.12)} \quad \begin{cases} \text{(a) There exists } L_{\mathcal{G}} > 0 \text{ such that} \\ \|\mathcal{G}(\sigma_1, \varepsilon_1, \theta_1) - \mathcal{G}(\sigma_2, \varepsilon_2, \theta_2)\| \leq L_{\mathcal{G}} \left(\|\sigma_1 - \sigma_2\| + \|\varepsilon_1 - \varepsilon_2\| + \|\theta_1 - \theta_2\|\right) \\ \text{for all } \sigma_1, \sigma_2 \in \mathbb{S}_d, \quad \varepsilon_1, \varepsilon_2 \in \mathbb{S}_d, \quad \theta_1, \theta_2 \in \mathbb{R}^d, \\ \text{(b) The mapping } \mathcal{G}(\sigma, \varepsilon, \theta) \text{ is Lebesgue measurable on } \Omega. \end{cases}$$

The *tensor* $K : \Omega \times \mathbb{S}_d \to \mathbb{S}_d$ satisfies

$$\text{(2.13)} \qquad \begin{cases} \text{(a) } K(x).q.p = q.K(x).p, \quad \forall q, p \in \mathbb{S}_d, \quad \text{a.e in } \Omega, \\ \text{(b) There exists } \lambda > 0 \text{ such that } \ K(x).q.q \geq \lambda \|q\|^2, \\ \ \text{for all } q \in \mathbb{S}_d, \quad \text{a.e in } \Omega, \\ \text{(c)} K_{ij} \in L^\infty(\Omega), \quad \text{for all } i, j \in 1, 2, 3. \end{cases}$$

K is a symetric and positively definite bounded tensor.

The tensor $K^{-1}A$ is a symetric and positively definite bounded tensor i.e.

$$\text{(2.14)} \quad \begin{cases} \text{(a) } K^{-1}A(x).q.p = q.K^{-1}A(x).p, \quad \forall q, p \in \mathbb{S}_d, \quad \text{a.e in } \Omega, \\ \text{(b) There exists } \delta > 0 \text{ such that } \ K^{-1}A(x).q.q \geq \delta \|q\|^2, \\ \ \text{for all } q \in \mathbb{S}_d \quad \text{a.e in } \Omega, \\ \text{(c) } \left(K^{-1}A\right)_{ij} \in L^\infty(\Omega), \quad \text{for all } i, j \in 1, 2, 3. \end{cases}$$

We, also suppose that

$$f_0 \in C^1(0, T; H), \tag{2.15}$$
$$r \in L^2(\Omega), \tag{2.16}$$
$$f_2 \in C^1(0, T; H'_\Gamma), \tag{2.17}$$
$$\alpha \in L^2(\tilde{\Gamma}_1), \tag{2.18}$$
$$\beta \in L^2(\tilde{\Gamma}_2), \tag{2.19}$$
$$u_0 \in H_1, \sigma_0 \in \mathcal{H}_1, \tag{2.20}$$

and we suppose

$$Div\sigma_0 + f_0(0) = 0, \quad \text{in } \Omega, \tag{2.21}$$
$$\sigma_0.\nu = f_2(0), \quad \text{on } \Gamma_2. \tag{2.22}$$

By using standard arguments, we obtain the following variational formulation of the problem (2.1)-(2.10).

2.2. **Problem $\mathcal{P}_V$.** Find the displacement field $u : [0,T] \to \mathbb{R}^d$, the stress field $\boldsymbol{\sigma} : [0,T] \to \mathbb{S}_d$, the temperature function $\theta : [0,T] \to \mathbb{R}$, and the heat flux $q : [0,T] \to \mathbb{R}^d$ such that

$$u(t) = U_{ad}, \sigma(t) \in \sum_{ad}(t, w(t)), \quad \forall t \in [0,T], \tag{2.23}$$

$$\dot{\sigma}\,(t) = \xi(\theta\,(t)).\varepsilon(\dot{u}(t)) + \mathcal{G}(\sigma(t), \varepsilon(u(t)), \theta(t)), \tag{2.24}$$

$$u(0) = u_0, \qquad \sigma(0) = \sigma_0, \tag{2.25}$$

$$(K\nabla\theta, \nabla\eta)_H = (r, \eta) + (\alpha, \tilde{\gamma}\eta) + (\beta, \tilde{\gamma}\eta). \tag{2.26}$$

We notice that the variational formulation PV is formulated in terms of a displacements field, a stress field, a temperature and heat. The existence of the unique solution of the problem PV is proved in the next section.

3. Existence and Uniqueness of a Solution

Now, we propose our existence and uniqueness result.

Theorem 3.1. *Assume that* (2.11)-(2.20) *hold. Then there exists a unique weak solution of the problems* (2.23)-(2.26)*, such that*

$$\theta \in \tilde{H}_1, \tag{3.1}$$

$$q \in \tilde{\mathcal{H}}_1, \tag{3.2}$$

$$u \in C^1(0,T;V), \tag{3.3}$$

$$\sigma \in C^1(0,T;\mathcal{H}_1). \tag{3.4}$$

The proof of the Theorem 3.1 will be carried in several steps. It is based on parabolic equations and a Cauchy Lipschitz technique.

In the first step, we consider the following variational problem:

Problem $\mathbf{PV}_\theta$

Find a temperature function $\theta : [0,T] \to \mathbb{R}$, such that

$$(K\nabla\theta, \nabla\eta)_H = (r, \eta) + (\alpha, \tilde{\gamma}\eta) + (\beta, \tilde{\gamma}\eta). \tag{3.5}$$

The existence and uniqueness of the functions (θ, q) satisfying (3.1),(3.2) can be obtained by using Lax-Miligram lemma. For the problem $\mathbf{PV}_\theta$ we have the following lemma.

Lemma 3.2. $\mathbf{PV}_\theta$ *has a unique solution satisfying*

$$\theta \in \tilde{V}, \tag{3.6}$$

$$q \in \tilde{\mathcal{H}}_1. \tag{3.7}$$

Proof. Applying Lax-Miligram lemma for the coercive bilinear and continuous form

$$a(\theta,\eta) = (K\nabla\theta, \nabla\eta),$$

and for the continuous linear form

$$l(x) = (r,\eta) + (\alpha, \tilde{\gamma}\eta) + (\beta, \tilde{\gamma}\eta),$$

we get the existence and uniqueness of $\theta : [0,T] \to \mathbb{R}$, such that

$$(K\nabla\theta, \nabla\eta)_H = (r,\eta) + (\alpha, \tilde{\gamma}\eta) + (\beta, \tilde{\gamma}\eta).$$

□

Now, the existence and uniqueness of the solution (u,σ) of the mechanical problem with the regularity(3.3), (3.4) can be proved by considering θ as a known function and the existence and uniqueness of the solution of the mechanical problem can be proved by reducing the studied problem to an ordinary differential equation in a Hilbert space.

In the second step, we consider the following variationnal problem.

Problem $\mathbf{PV_u}$

Find a displacement field $u : [0,T] \to \mathbb{R}^d$, and the stress $\sigma : [0,T] \to \mathbb{S}_d$ such that

$$u(t) = U_{ad}, \sigma(t) \in \sum_{ad}(t, w(t)), \quad \forall t \in [0,T], \tag{3.8}$$

$$\dot{\sigma}(t) = \xi(\theta(t)).\varepsilon(\dot{u}(t)) + \mathcal{G}\big(\sigma(t), \varepsilon(u(t)), \theta(t)\big), \tag{3.9}$$

$$u(0) = u_0, \qquad \sigma(0) = \sigma_0. \tag{3.10}$$

The existence and uniqueness of the functions (u,σ) satisfying (3.3),(3.4) is given by the following lemma.

Lemma 3.3. $\mathbf{PV_u}$ *has a unique solution satisfying*

$$u \in C^1(0,T;V), \tag{3.11}$$

$$\sigma \in C^1(0,T;\mathcal{H}_1), \tag{3.12}$$

In order to prove Lemma 3.3, we need some preliminaries given by the following lemma, whose proof can be easily obtained

Lemma 3.4. *Let $\theta \in \tilde{V}$, then for all $t \in [0,T]$, we have*

$$\|\xi(\theta(t)).\sigma\|_{\mathcal{H}} \le \beta\|\sigma\|_{\mathcal{H}},$$

$$\langle \xi(\theta(t))\sigma, \sigma\rangle_{\mathcal{H}} \ge \alpha\|\sigma\|^2_{\mathcal{H}},$$

$$\left\|\xi^{-1}(\theta(t)).\sigma\right\|_{\mathcal{H}} \le \frac{1}{\alpha}\|\sigma\|_{\mathcal{H}},$$

$$\left\langle \xi^{-1}(\theta(t))\sigma, \sigma\right\rangle_{\mathcal{H}} \ge \frac{\alpha}{\beta^2}\|\sigma\|^2_{\mathcal{H}}.$$

Using the properties of the trace maps, from (2.15) and (2.17), we obtain the existence of the function $\tilde{\sigma} \in C^1(0,T;\mathcal{H}_1)$, such that

$$\begin{aligned} &\tilde{\sigma}\nu(t) = f_2(t), \qquad \Gamma_2 \times [0,T], \\ &Div\ \tilde{\sigma}(t) + f_0 = 0. \end{aligned}$$

Now, let $X = V \times \mathcal{V}$, we consider

$$\begin{aligned} &a : [0,T] \times X \times X \to \mathbb{R}, \\ &F : [0,T] \times X \to X, \end{aligned}$$

given by

$$a(t,x,y) = \langle \xi(\theta(t))\varepsilon(u), \varepsilon(v)\rangle_{\mathcal{H}} + \left\langle \xi^{-1}(\theta(t))\sigma, \tau\right\rangle_{\mathcal{H}}, \tag{3.13}$$

$$\begin{aligned} \langle F(t,x), y\rangle_X = &\left\langle \xi^{-1}(\theta(t))\mathcal{G}(\sigma + \tilde{\sigma}(t), \varepsilon(u), \theta(t)), \tau\right\rangle_{\mathcal{H}} \\ &- \langle \mathcal{G}(\sigma + \tilde{\sigma}(t), \varepsilon(u)), \theta(t), \varepsilon(v)\rangle_{\mathcal{H}} \\ &- \left\langle \dot{\tilde{\sigma}}(t), \varepsilon(v)\right\rangle_{\mathcal{H}} - \left\langle \xi^{-1}(\theta(t))\dot{\tilde{\sigma}}(t), \tau\right\rangle_{\mathcal{H}}, \end{aligned} \tag{3.14}$$

for all $x = (u,\sigma)$, $y = (v,\tau) \in X$ and $t \in [0,T]$.

Let us set the following natations

$$\sigma = \bar{\sigma} + \tilde{\sigma}, \quad x = (u, \bar{\sigma}), \tag{3.15}$$

$$\sigma_0 = \bar{\sigma}_0 + \tilde{\sigma}(0), \quad x_0 = (u_0, \bar{\sigma}_0). \tag{3.16}$$

We have:

Lemma 3.5. *The pair $(u,\sigma) \in C^1(0,T;V \times \mathcal{H}_1)$ is a solution of the problem P_u if and only if $x \in C^1(0,T;X)$ is a solution of the problem*

$$a(t, \dot{x}(t), y) = \langle F(t,x), y\rangle_X, \tag{3.17}$$

$$x(0) = x_0, \tag{3.18}$$

where $X = V \times \mathcal{V}$, $x = (u, \bar{\sigma})$.

Proof. Using (3.9)-(3.13), (3.16), (3.17) it is easy to see that $(u,\sigma) \in C^1(0,T;V \times \mathcal{H}_1)$ is a solution of the viscoplastic problem if and only if $x \in C^1(0,T;X)$ and

$$\dot{\bar{\sigma}} = \xi(\theta).\varepsilon(\dot{u}) + \mathcal{G}(\bar{\sigma} + \tilde{\sigma}, \varepsilon(u), \theta) - \dot{\tilde{\sigma}}, \tag{3.19}$$

$$u(0) = u_0, \qquad \bar{\sigma}(0) = \bar{\sigma}_0. \tag{3.20}$$

Let us suppose (3.19)-(3.20) are fullfield. Since $\varepsilon(v)$ is the orthogonal complement of $\mathcal{V}$ in $\mathcal{H}$, we have (3.17).

Conversely, let (3.17) hold and let

$$z(t) = \dot{\bar{\sigma}}(t) - \xi(t)\varepsilon(\dot{u}) - \mathcal{G}(\bar{\sigma} + \tilde{\sigma}, \varepsilon(u), \theta) - \dot{\tilde{\sigma}}. \tag{3.21}$$

Taking $y = (v, 0) \in X$ in (3.17) and using the orthogonality of $\varepsilon(v)$ and v, we get

$$\langle z(t), \boldsymbol{\varepsilon}(v)\rangle_{\mathcal{H}} = 0. \tag{3.22}$$

Now, we take $y = (0, \tau) \in X$ in (3.17) and using the orthogonality of $\varepsilon(v)$ and v, we get

$$\left\langle \xi^{-1}(\theta(t)) z(t), \tau \right\rangle_{\mathcal{H}} = 0. \tag{3.23}$$

Since $\varepsilon(v)$ is orthogonal to v in $\mathcal{H}$, from (3.22) we get $z(t) \in V$, thus we may put $\tau = z(t)$ in (3.23), and from (3.8) we deduce $z(t) = 0$. Hence, we proved that (3.19) is equivalent to (3.18). □

The following lemma can be easily obtained

Lemma 3.6. *For every $x \in X$ and $t \in [0, T]$ there exists a unique element $z \in X$, such that*

$$a(t, z, y) = \langle F(t, x), y\rangle_X, \tag{3.24}$$

where $X = V \times \mathcal{V}$, $x = (u, \bar{\sigma})$.

Proof. Let $x \in X$ and $t \in [0, T]$. Using the properties of ξ, ξ^{-1} and Korn's equality, we get that $a(t, ., .)$ is bilinear, continuous and coercive, hence the existence and uniqueness of z satisfying (3.24) follows from Lax Miligram's lemma. □

The previous lemma allows us to consider the operator $A : [0, T] \times X \to X$ defined by : $A(t, x) = z$, Moreover, we have:

Lemma 3.7. *The operator A is continuous and there exists $C > 0$, such that:*

$$|A(t, x_1) - A(t, x_2)|_X \le C |x_1 - x_2|_X, \ \forall x_1, x_2 \in X. \tag{3.25}$$

Proof. Let us consider $t_1, t_2 \in [0, T]$; $x_i = (u_i, \sigma_i) \in X$ and let $z_i = (w_i, \tau_i) \in X$ defined by $z_i = A(t_i, x_i)$.

Using (3.24), we have

$$a(t_1, z_1, z_1 - z_2) - a(t_2, z_2, z_1 - z_2) = \langle F(t_1, z_1) - F(t_2, z_2), z_1 - z_2\rangle_X, \tag{3.26}$$

and from Lemma 3.4 and Korn's inequality, we get

$$\begin{aligned} a(t_1, z_1, z_{1-}z_2) - a(t_2, z_2, z_{1-}z_2) &\\ \ge C\|z_{1-}z_2\|_X^2 - \|[\xi(\theta(t_1)) - \xi(\theta(t_2))]\,\varepsilon(w_2)\|_H \|z_{1-}z_2\|_X &\\ - \left\|\left[\xi^{-1}(\theta(t_1)) - \xi^{-1}(\theta(t_2))\right]\tau_2\right\|_H \|z_{1-}z_2\|_X. & \end{aligned} \tag{3.27}$$

In a similar way, from (2.12) and Lemma 3.4, we get

(3.28)

$$\langle F(t_1, x_1) - F(t_2, x_2), z_{1-}z_2\rangle_X$$

$$\leq C(\|x_{1-}x_2\|_X + \|\tilde{\sigma}(t_1) - \tilde{\sigma}(t_2)\|_H$$
$$+ \|\tilde{\sigma}(t_1) - \tilde{\sigma}(t_2)\|_H + \|\theta(t_1) - \theta(t_2)\|_{L^2(\Omega)}$$
$$+ \left\| \left[\xi^{-1}(\theta(t_1)) - \xi^{-1}(\theta(t_2))\right] \right\|_H F(\sigma_2 + \tilde{\sigma}(t), \varepsilon(u), \theta(t_2))$$
$$+ \left\| \left[\xi^{-1}(\theta(t_1)) - \xi^{-1}(\theta(t_2))\right] \sigma(t_2) \right\|_H). \|z_{1-}z_2\|_X .$$

So, from (3.26)-(3.28), we have

(3.29)

$$\|z_{1-}z_2\|_X \leq \mathcal{C}(\|[\xi(\theta(t_1)) - \xi(\theta(t_2))]\,\varepsilon(w_2)\|_H$$
$$+ \left\| \left[\xi^{-1}(\theta(t_1)) - \xi^{-1}(\theta(t_2))\right] \tau_2 \right\|_H$$
$$+ \|x_{1-}x_2\|_X + \|\tilde{\sigma}(t_1) - \tilde{\sigma}(t_2)\|_H$$
$$+ \|\tilde{\sigma}(t_1) - \tilde{\sigma}(t_2)\|_H + \|\theta(t_1) - \theta(t_2)\|_{L^2(\Omega)}$$
$$+ \left\| \left[\xi^{-1}(\theta(t_1)) - \xi^{-1}(\theta(t_2))\right] \right\|_H F(\sigma_2 + \tilde{\sigma}(t), \varepsilon(u), \theta(t_2))$$
$$+ \left\| \left[\xi^{-1}(\theta(t_1)) - \xi^{-1}(\theta(t_2))\right] \sigma(t_2) \right\|_H). \|z_{1-}z_2\|_X .$$

Using the properties of ξ, ξ^{-1} and the regularity of $\tilde{\sigma}$, u, from (3.29) we get $z_1 \longrightarrow z_2$ in X when $t_1 \longrightarrow t_2$ in $[0,T]$, and $x_1 \longrightarrow x_2$ in X. Hence A is a continuous operator, moreover taking $t_1 = t_2$ in (3.29), we get (3.25). □

Now, we have all the ingredients needed to prove Theorem 3.1.

Proof of theorem 3.1. Using the hypothesis on u_0, σ_0 we get that $x_0 \in X$, and by Lemma 3.7 and the classical Cauchy Lipschitz theorem we get that, there exists a unique solution $x \in C^1(0,T;X)$ of the Cauchy problem

$$\dot{x}(t) = A(t, x(t)),$$
$$x(0) = x_0.$$

Theorem 3.1 follows now from the definition of the operator A, and Lemma 3.5 . □

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