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On Approximating Fixed Point in CAT(0) Spaces

Chanchal Garodia[1*] and Izhar Uddin[2]

Abstract. In this paper, we obtain a new modified iteration process in the setting of CAT(0) spaces involving generalized α-nonexpansive mapping. We prove strong and Δ convergence results for approximating fixed point via newly defined iteration process. Further, we reconfirm our results by non trivial example and tables.

1. Introduction

Nonlinear analysis is a beautiful mixture of Topology, Analysis and Linear Algebra. Fixed point theory is a very challenging and rapidly growing area of nonlinear functional analysis. Obviously, results dealing with the existence of fixed points are termed as fixed point theorems. Such theorems are very important tools for proving the existence and uniqueness of the solutions of various mathematical models representing phenomena arising in different fields such as: optimization theory, variational inequalities, equilibrium problems, economic theory, chemical equations, neutron transport theory, epidemics and flow of fluids besides facilitating existence and uniqueness theories of differential, integral and partial differential equations etc. Fixed point theory is relatively old but still a young area of research. There exists a vast literature on fixed point theory and this is still growing. Banach Contraction Principle [23] is one of the most celebrated result of fixed point theory. The early findings in fixed point theory revolve around generalization of Banach Contraction Principle. The entire mathematics community had to wait for the first fixed point theorem for nonexpansive mappings for 43 years. Let J be a nonempty closed convex subset of a uniformly

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convex Banach space P. Then, a mapping $G : J \to J$ is said to be nonexpansive if $\|Gc - Gb\| \leq \|c - b\|$ for all $c, b \in J$. A point $c \in J$ is said to be a fixed point of G if $Gc = c$. We will denote the set of fixed points of G by $F(G)$. The mapping G is called quasi-nonexpansive if $F(G) \neq \emptyset$ and $\|Gc - e\| \leq \|c - e\|$ for all $c \in J$ and $e \in F(G)$. It is worth mentioning that every nonexpansive mapping with a fixed point is a quasi-nonexpansive mapping. It is well known that Banach Contraction Principle does not hold good for nonexpansive mappings i.e. nonexpansive mapping need not admit a fixed point on complete metric space. Also, Picard iteration need not be convergent for a nonexpansive map in a complete metric space. This led to the beginning of a new era of fixed point theory for nonexpansive mappings by using geometric properties. In 1965, Browder [8], Göhde [6] and Kirk [28] gave three basic existence results involving nonexpansive mappings.

Following this, many mathematicians have introduced various generalizations and extensions of nonexpansive mappings. In 2008, Suzuki [27] introduced a new extension of nonexpansive mappings and called the defining condition as Condition (C) which is also referred as Suzuki generalized nonexpansive mappings. A mapping $G : J \to J$ defined on a nonempty subset J of a Banach space P is said to satisfy the Condition (C) if

$$\frac{1}{2}\|c - Gc\| \leq \|c - b\| \quad \Rightarrow \quad \|Gc - Gb\| \leq \|c - b\|,$$

for all c and $b \in J$. Suzuki proved that the mappings satisfying the Condition (C) is weaker than nonexpansive and also obtained few results regarding the existence of fixed points for such mappings.

Recently in 2017, Pant and Shukla [21] introduced the class of generalized α-nonexpansive mappings. A mapping $G : J \to J$ is said to be generalized α-nonexpansive if there exists $\alpha \in [0, 1)$ such that

$$\frac{1}{2}\|c-Gc\| \leq \|c-b\| \quad \Rightarrow \quad \|Gc-Gb\| \leq \alpha\|Gc-b\|+\alpha\|Gb-c\|+(1-2\alpha)\|c-b\|,$$

for all c and $b \in J$. They established some existence and convergence theorems for the newly introduced class of mappings. One can easily see that every mapping satisfying the Condition (C) is a generalized α-nonexpansive mapping.

Over the last few years many iterative processes have been obtained in different domains to approximate fixed points of various classes of mappings. Mann iteration [33], Ishikawa iteration [26] and Halpern iteration [2] are the few basic iteration processes.

In 2020, Garodia and Uddin [4] introduced a new iteration process as follows:

$$\begin{cases} c_1 \in J, \\ a_n = G((1-\kappa_n)c_n + \kappa_n G c_n), \\ b_n = G((1-\rho_n)Gc_n + \rho_n G a_n), \\ c_{n+1} = G b_n, \end{cases} \tag{1.1}$$

where $\{\kappa_n\}$ and $\{\rho_n\}$ are sequences in $(0,1)$ and $n \in \mathbb{N}$. Authors proved that their iteration process (1.1) has a better rate of convergence than a number of existing iteration processes. Further, they used their iteration process to obtain few convergence results involving generalized α-nonexpansive mappings in the setting of uniformly convex Banach space. The purpose of this paper is to study the convergence of iteration process (1.1) for generalized α-nonexpansive mappings in CAT(0) spaces.

2. Preliminaries and Lemmas

It was M. Gromov who gave the term CAT(0) space. A metric space P is said to be a CAT(0) space if it is geodesically connected and if every geodesic triangle in P is at least as thin as its comparison triangle in the Euclidean plane. Some well known examples include complete, simply connected Riemannian manifold having non-positive sectional curvature and the complex Hilbert ball with a hyperbolic metric [11]. Other examples include pre-Hilbert spaces, $\mathbb{R}$- trees [17] and Euclidean buildings [12]. For a thorough discussion of these spaces and the crucial role they play in geometry, one can refer Bridson and Haefliger [17]. Also, one can see Burago et al. [5] for more elementary information and Gromov [18] for comparatively deeper study about these spaces.

Study of fixed point theory in CAT(0) spaces was initiated by Kirk [29, 30]. He proved that one can always find a fixed point for every non-expansive (single valued) mapping defined on a bounded closed convex subset of a complete CAT(0) space. It can also be noted that the results in CAT(0) space can be applied to any CAT(k) space with $k \leq 0$, as any CAT(k) space is a CAT(k') space for every $k' \geq k$.

Next, we recall the basic lemmas to be used later on.

Lemma 2.1 ([24]). *Let (P,d) be a CAT(0) space. For $c, b \in P$ and $t \in [0,1]$, there exists a unique $a \in [c,b]$ such that*

$$d(c,a) = td(c,b), \qquad d(b,a) = (1-t)d(c,b).$$

We use the notation $(1-t)c \oplus tb$ for the unique point a of the above lemma.

Lemma 2.2 ([24])**.** *For $c, b, a \in P$ and $t \in [0,1]$ we have*

$$d((1-t)c \oplus tb, a) \leq (1-t)d(c,a) + td(b,a).$$

Now, we collect some basic geometric properties, which are instrumental throughout the discussions.

Let $\{c_n\}$ be a bounded sequence in a complete CAT(0) space P. For $c \in P$ we write:

$$r(c, \{c_n\}) = \limsup_{n\to\infty} d(c, c_n).$$

The asymptotic radius $r(\{c_n\})$ is given by

$$r(\{c_n\}) = \inf\{r(c, \{c_n\}) : c \ \in P\},$$

and the asymptotic center $A(\{c_n\})$ of $\{c_n\}$ is defined as:

$$A(\{c_n\}) = \{c \in P : r(c, \{c_n\}) = r(\{c_n\})\}.$$

In 2006, Dhompongsa, Kirk and Sims showed that $A(\{c_n\})$ consists of exactly one point if P is a CAT(0) space (Proposition 5 of [25]).

In 2008, Kirk and Panyanak [31] gave the following definition of Δ-convergence.

Definition 2.3 ([31])**.** A sequence $\{c_n\}$ in P is said to be Δ-convergent to $c \in P$ if c is the unique asymptotic center of u_n for every subsequence $\{u_n\}$ of $\{c_n\}$. In this case, we write $\Delta - \lim_n c_n = c$ and read as c is the Δ-limit of $\{c_n\}$.

Definition 2.4. A Banach space P is said to satisfy Opial's condition if for any sequence $\{c_n\}$ in P with $c_n \rightharpoonup c$ ($\rightharpoonup$ denotes weak convergence) implies that $\limsup_{n\to\infty} \|c_n - c\| < \limsup_{n\to\infty} \|c_n - b\|$ for all $b \in P$ with $b \neq c$.

From the definition of Δ-convergence it is trivial that every CAT(0) space satisfies Opial's property.

Definition 2.5. A mapping $G : J \to P$ is demiclosed at $b \in P$ if for each sequence $\{c_n\}$ in J and each $c \in P$, $c_n \rightharpoonup c$ and $Gc_n \to b$ imply that $c \in J$ and $Gc = b$.

The following lemma is a consequence of Lemma 2.9 of [32] which will be used to prove our main result.

Lemma 2.6. *Let (P, d) be a complete CAT(0) space and $e \in P$. Suppose $\{t_n\}$ is a sequence in $[b, c]$ for some $b, c \in (0, 1)$ and $\{u_n\}$, $\{v_n\}$ are sequences in P such that $\limsup_{n\to\infty} d(u_n, e) \leq r$, $\limsup_{n\to\infty} d(v_n, e) \leq r$ and $\lim_{n\to\infty} d(t_n v_n \oplus (1-t_n)u_n, e) = r$ hold for some $r \geq 0$, then $\lim_{n\to\infty} d(u_n, v_n) = 0$.*

Now, we define generalized α-nonexpansive mapping for CAT(0) spaces.

Definition 2.7. A mapping G defined on a subset J of a CAT(0) space (P, d) is said to be a generalized α-nonexpansive if there exists $\alpha \in [0, 1)$ such that

$$\frac{1}{2}d(c, Gc) \leq d(c, b) \quad \Rightarrow \quad d(Gc, Gb) \leq \alpha d(Gc, b) + \alpha d(Gb, c) + (1 - 2\alpha)d(c, b),$$

for all c and $b \in J$.

Since every CAT(0) space is a hyperbolic space, we have the following two important results due to Mebawondu and Izuchukwu [1].

Lemma 2.8. *Let J be a nonempty closed and convex subset of a complete CAT(0) space (P, d). Let $G : J \to J$ be a generalized α-nonexpansive mapping with $F(G) \neq \emptyset$, then*

(i) *$F(G)$ is closed and convex.*
(ii) *G is quasi-nonexpansive.*

The next important lemma is related to demiclosedness principle for generalized α-nonexpansive mappings in CAT(0) spaces.

Lemma 2.9. *Let $G : J \to J$ be a generalized α-nonexpansive mapping defined on a nonempty closed and convex subset of a complete CAT(0) space (P, d) such that $\lim_{n\to\infty} d(Gc_n, c_n) = 0$ and $\Delta\text{-}\lim_{n\to\infty} c_n = c$. Then, $c \in F(G)$ or $I - G$ is demiclosed at zero.*

We now modify (1.1) in a CAT(0) space as follows:
Let J be a nonempty closed convex subset of a complete CAT(0) space P and $G : J \to J$ be a mapping. Let $c_1 \in J$ be an arbitrary element, then the sequence $\{c_n\}$ is generated iteratively by:

$$\begin{cases} c_1 \in J, \\ a_n = G((1 - \kappa_n)c_n \oplus \kappa_n Gc_n), \\ b_n = G((1 - \rho_n)Gc_n \oplus \rho_n Ga_n), \\ c_{n+1} = Gb_n, \end{cases} \tag{2.1}$$

where $\{\kappa_n\}$ and $\{\rho_n\}$ are sequences in $(0, 1)$ and $n \in \mathbb{N}$.

3. Convergence Results

Let us begin with the following important lemma.

Lemma 3.1. *Let $G : J \to J$ be a generalized α-nonexpansive mapping defined on a nonempty closed convex subset J of a complete CAT(0) space P such that $F(G) \neq \emptyset$. If $\{c_n\}$ is a sequence defined by (2.1), then $\lim_{n\to\infty} d(c_n, e)$ exists for all $e \in F(G)$.*

Proof. Let $e \in F(G)$. By Lemma 2.8(ii), G is quasi-nonexpansive, so we have

$$\begin{aligned} d(a_n, e) &= d(G((1-\kappa_n)c_n \oplus \kappa_n Gc_n), e) \\ &\le d((1-\kappa_n)c_n \oplus \kappa_n Gc_n, e) \\ &\le (1-\kappa_n)d(c_n, e) + \kappa_n d(Gc_n, e) \\ &\le d(c_n, e), \end{aligned} \tag{3.1}$$

and

$$\begin{aligned} d(b_n, e) &= d(G((1-\rho_n)Gc_n \oplus \rho_n Ga_n), e) \\ &\le d((1-\rho_n)Gc_n \oplus \rho_n Ga_n, e) \\ &\le (1-\rho_n)d(Gc_n, e) + \rho_n d(Ga_n, e) \\ &\le (1-\rho_n)d(c_n, e) + \rho_n d(a_n, e) \\ &\le d(c_n, e). \end{aligned} \tag{3.2}$$

Using (3.1) and (3.2), we get

$$\begin{aligned} d(c_{n+1}, e) &= d(Gb_n, e) \\ &\le d(b_n, e) \\ &\le d(c_n, e). \end{aligned}$$

Thus, $\{d(c_n, e)\}$ is a bounded and non-increasing sequence of reals and hence $\lim_{n\to\infty} d(c_n, e)$ exists. □

Lemma 3.2. *Let G be a generalized α-nonexpansive mapping defined on a nonempty closed convex subset J of a complete CAT(0) space P such that $F(G) \neq \emptyset$. Let $\{c_n\}$ be the iterative sequence defined by the iteration process (2.1), then* $\lim_{n\to\infty} d(Gc_n, c_n) = 0$.

Proof. Let $e \in F(G)$. Then, by Lemma 3.1, $\lim_{n\to\infty} d(c_n, e)$ exists. Let

$$\lim_{n\to\infty} d(c_n, e) = x. \tag{3.3}$$

From (3.1) and (3.2), we have

$$\limsup_{n\to\infty} d(b_n, e) \le x, \tag{3.4}$$

and

$$\limsup_{n\to\infty} d(a_n, e) \le x. \tag{3.5}$$

Now,

$$\begin{aligned} x &= \lim_{n\to\infty} d(c_{n+1}, e) \\ &= \lim_{n\to\infty} d(Gb_n, e), \end{aligned}$$

and

$$d(Gb_n, e) \leq d(b_n, e).$$

So,

$$x \leq \liminf_{n\to\infty} d(b_n, e),$$

which along with (3.4) implies

$$\lim_{n\to\infty} d(b_n, e) = x. \tag{3.6}$$

Now, consider

$$\begin{aligned} d(b_n, e) &= d(G((1-\rho_n)Gc_n \oplus \rho_n Ga_n), e) \\ &\leq d((1-\rho_n)Gc_n \oplus \rho_n Ga_n, e) \\ &\leq (1-\rho_n)d(Gc_n, e) + \rho_n d(Ga_n, e) \\ &\leq (1-\rho_n)d(c_n, e) + \rho_n d(a_n, e) \\ &= d(c_n, e) + \rho_n(d(a_n, e) - d(c_n, e)). \end{aligned}$$

Since $\{\rho_n\} \in (0,1)$, we have

$$\begin{aligned} d(b_n, e) - d(c_n, e) &\leq \rho_n(d(a_n, e) - d(c_n, e)) \\ &\leq d(a_n, e) - d(c_n, e), \end{aligned} \tag{3.7}$$

which gives $d(b_n, e) \leq d(a_n, e)$ and using (3.6) we get

$$x \leq \liminf_{n\to\infty} d(a_n, e). \tag{3.8}$$

Owing to (3.5) and (3.8), we have

$$\lim_{n\to\infty} d(a_n, e) = x. \tag{3.9}$$

Also, using the fact that G is quasi-nonexpansive we have $d(Gc_n, e) \leq d(c_n, e)$, which gives

$$\limsup_{n\to\infty} d(Gc_n, e) \leq x. \tag{3.10}$$

From (3.1), we have

$$d(a_n, e) \leq d((1-\kappa_n)c_n \oplus \kappa_n Gc_n, e) \leq d(c_n, e),$$

which by using (3.3) and (3.9) gives

$$\lim_{n\to\infty} d((1-\kappa_n)c_n \oplus \kappa_n Gc_n, e) = x. \tag{3.11}$$

Using (3.3), (3.10), (3.11) and Lemma 2.6, we conclude that

$$\lim_{n\to\infty} d(Gc_n, c_n) = 0.$$

□

Theorem 3.3. *Let G be a generalized α-nonexpansive mapping defined on a nonempty closed convex subset J of a complete CAT(0) space P with $F(G) \neq \emptyset$. If $\{c_n\}$ is the iterative sequence defined by the iteration process (2.1), then $\{c_n\}$ Δ-converges to a fixed point of G.*

Proof. Let $e \in F(G)$. Then, from Lemma 3.1 $\lim_{n\to\infty} d(c_n, e)$ exists. In order to show the Δ convergence of the iteration process (2.1) gives a fixed point of G, we will prove that $\{c_n\}$ has a unique Δ subsequential limit in $F(G)$. For this, let $\{c_{n_j}\}$ and $\{c_{n_k}\}$ be two subsequences of $\{c_n\}$ which converge weakly to u and v respectively. By Lemma 3.2, we have $\lim_{n\to\infty} d(Gc_n, c_n) = 0$ and using Lemma 2.9, we have $I - G$ is demiclosed at zero. So $u, v \in F(G)$.

Next, we show the uniqueness. Since $u, v \in F(G)$, so $\lim_{n\to\infty} d(c_n, u)$ and $\lim_{n\to\infty} d(c_n, v)$ exists. Let $u \neq v$. Then, by Opial's condition, we obtain

$$\begin{aligned}
\lim_{n\to\infty} d(c_n, u) &= \lim_{j\to\infty} d(c_{n_j}, u) \\
&< \lim_{j\to\infty} d(c_{n_j}, v) \\
&= \lim_{n\to\infty} d(c_n, v) \\
&= \lim_{k\to\infty} d(c_{n_k}, v) \\
&< \lim_{k\to\infty} d(c_{n_k}, u) \\
&= \lim_{n\to\infty} d(c_n, u),
\end{aligned}$$

which is a contradiction, so $u = v$. Thus, $\{c_n\}$ Δ-converges to a fixed point of G. ☐

Now, we establish some strong convergence results.

Theorem 3.4. *Let $G : J \to J$ be a generalized α-nonexpansive mapping defined on a nonempty closed convex subset J of a complete CAT(0) space P such that $F(G) \neq \emptyset$. If $\{c_n\}$ is a sequence defined by (2.1), then $\{c_n\}$ converges to a fixed point of G if and only if $\liminf_{n\to\infty} d(c_n, F(G)) = 0$.*

Proof. If the sequence $\{c_n\}$ converges to a point $e \in F(G)$, then

$$\liminf_{n\to\infty} d(c_n, e) = 0,$$

so

$$\liminf_{n\to\infty} d(c_n, F(G)) = 0.$$

For converse part, assume that $\liminf_{n\to\infty} d(c_n, F(G)) = 0$. From Lemma 3.1, we have

$$d(c_{n+1}, e) \leq d(c_n, e) \text{ for any } e \in F(G),$$

so we have

$$d(c_{n+1}, F(G)) \leq d(c_n, F(G)). \tag{3.12}$$

Thus, $d(c_n, F(G))$ forms a decreasing sequence which is bounded below by zero as well, thus $\lim_{n\to\infty} d(c_n, F(G))$ exists. As, $\liminf_{n\to\infty} d(c_n, F(G)) = 0$ so $\lim_{n\to\infty} d(c_n, F(G)) = 0$.

Now, there exists a subsequence $\{c_{n_j}\}$ of $\{c_n\}$ and a sequence $\{u_j\}$ in $F(G)$ such that $d(c_{n_j}, u_j) \leq \frac{1}{2^j}$ for all $j \in \mathbb{N}$. From the proof of Lemma 3.1, we have

$$\begin{aligned} d(c_{n_{j+1}}, u_j) &\leq d(c_{n_j}, u_j) \\ &\leq \frac{1}{2^j}. \end{aligned}$$

Using triangle inequality, we get

$$\begin{aligned} d(u_{j+1}, u_j) &\leq d(u_{j+1}, c_{n_{j+1}}) + d(c_{n_{j+1}}, u_j) \\ &\leq \frac{1}{2^{j+1}} + \frac{1}{2^j} \\ &\leq \frac{1}{2^{j-1}} \\ &\to 0 \text{ as } j \to \infty. \end{aligned}$$

So, $\{u_j\}$ is a cauchy sequence in $F(G)$. By Lemma 2.8(i) $F(G)$ is closed, so $\{u_j\}$ converges to some $u \in F(G)$.

Again, owing to triangle inequality, we have

$$d(c_{n_j}, u) \leq d(c_{n_j}, u_j) + d(u_j, u).$$

Letting $j \to \infty$, we have $\{c_{n_j}\}$ converges strongly to $u \in F(G)$.

Since $\lim_{n\to\infty} d(c_n, u)$ exists by Lemma 3.1, therefore $\{c_n\}$ converges to $u \in F(G)$. □

We recall (see [9]), a mapping $G : J \to J$ is said to satisfy the Condition (A) if there exists a nondecreasing function $q : [0, \infty) \to [0, \infty)$ with $q(0) = 0$ and $q(r) > 0$ for all $r \in (0, \infty)$ such that $d(c, Gc) \geq q(d(c, F(G)))$ for all $c \in J$.

Now, we present the following convergence result using above Condition (A).

Theorem 3.5. *Let $G : J \to J$ be a generalized α-nonexpansive mapping defined on a nonempty closed convex subset J of a complete CAT(0) space P with $F(G) \neq \emptyset$. If $\{c_n\}$ is a sequence defined by (2.1) and G satisfies the Condition (A), then $\{c_n\}$ converges strongly to a fixed point of G.*

Proof. From (3.12), $\lim_{n\to\infty} d(c_n, F(G))$ exists.

Also, by Lemma 3.2 we have $\lim_{n\to\infty} d(c_n, Gc_n) = 0$.

It follows from the Condition (A) that

$$\begin{aligned}\lim_{n\to\infty} q(d(c_n, F(G))) &\leq \lim_{n\to\infty} d(c_n, Gc_n)\\ &= 0,\end{aligned}$$

so $\lim_{n\to\infty} q(d(c_n, F(G))) = 0$.

Since q is a non decreasing function satisfying $q(0) = 0$ and $q(r) > 0$ for all $r \in (0, \infty)$, therefore $\lim_{n\to\infty} d(c_n, F(G)) = 0$.

By Theorem 3.4, the sequence $\{c_n\}$ converges strongly to a point of $F(G)$. □

4. Numerical Example

Now, we will construct an example of a generalized α-nonexpansive mapping which is neither a Suzuki generalized nonexpansive mapping nor a nonexpansive mapping. Then, using this example, we will show the convergence of iteration scheme (2.1).

Example 4.1. Let $P = \mathbb{R}$ with the distance metric $d(c, b) = |c - b|$ and $J = [0, \infty)$. Let $G : J \to J$ be a mapping defined as

$$G(c) = \begin{cases} 0, & c \in \left[0, \frac{3}{2}\right), \\ \frac{8c}{17}, & c \in \left[\frac{3}{2}, \infty\right), \end{cases}$$

for all $c \in J$.

Clearly $c = 0$ is the fixed point of G. Then,

(i) Since G is not continuous at $c = \frac{3}{2}$, so G is not a nonexpansive map.

(ii) Let $c = 1$ and $b = \frac{3}{2}$, then

$$\begin{aligned}\frac{1}{2} d(c, Gc) &= \frac{1}{2}|c - Gc|\\ &= \frac{1}{2}\\ &\leq \frac{1}{2}\\ &= |c - b|\\ &= d(c, b).\end{aligned}$$

But

$$\begin{aligned}d(Gc, Gb) &= |Gc - Gb|\\ &= \frac{8b}{17}\end{aligned}$$

$$= \frac{24}{34}$$
$$> \frac{1}{2}$$
$$= |c-b|$$
$$= d(c,b).$$

So, G is not a Suzuki generalized nonexpansive mapping.

(iii) Now, we prove that G is a generalized α-nonexpansive mapping. For this, let $\alpha = \frac{1}{3}$ and consider the following cases:

Case (A). When $c \in \left[\frac{3}{2}, \infty\right)$ and $b \in \left[0, \frac{3}{2}\right)$ then,

$$\begin{aligned} d(Gc, Gb) &= |Gc - Gb| \\ &= \frac{8c}{17}. \end{aligned}$$

Now,

$$\begin{aligned} \alpha d(Gc,b) + \alpha d(Gb,c) + (1-2\alpha) d(c,b) &= \frac{1}{3}|Gc-b| + \frac{1}{3}|Gb-c| + \frac{1}{3}|c-b| \\ &= \frac{1}{3}\left|\frac{8c}{17} - b\right| + \frac{1}{3}|c| + \frac{1}{3}|c-b| \\ &\geq \frac{1}{3}\left|\frac{8c}{17} - b\right| + \frac{1}{3}|c-b| \\ &> d(Gc, Gb). \end{aligned}$$

Case (B). When $c, b \in [\frac{3}{2}, \infty)$ then,

$$\begin{aligned} d(Gc, Gb) &= \frac{8}{17} d(c,b) \\ &= \frac{8}{17}|c-b|. \end{aligned}$$

Now,

$$\begin{aligned} \alpha d(Gc,b) + \alpha d(Gb,c) + (1-2\alpha) d(c,b) &= \frac{1}{3}|Gc-b| + \frac{1}{3}|Gb-c| + \frac{1}{3}|c-b| \\ &= \frac{1}{3}\left|\frac{8c}{17} - b\right| + \frac{1}{3}\left|c - \frac{8b}{17}\right| + \frac{1}{3}|c-b| \\ &\geq \frac{1}{3}\left|\frac{25c}{17} - \frac{25b}{17}\right| + \frac{1}{3}|c-b| \\ &> d(Gc, Gb). \end{aligned}$$

Case (C). When $c, b \in [0, \frac{3}{2})$ then,

$$d(Gc, Gb) = 0.$$

So,

$$\alpha d(Gc,b)+\alpha d(Gb,c)+(1-2\alpha)d(c,b)=\frac{1}{3}|Gc-b|+\frac{1}{3}|Gb-c|+\frac{1}{3}|c-b|$$
$$\geq d(Gc,Gb).$$

Therefore, G is a generalized α-nonexpansive mapping with $\alpha=\frac{1}{3}$.

Now, we will examine the influence of parameters κ_n, ρ_n and initial value. For this, we will consider the two following cases.

Case (I). In this case, we will show the convergence of our iteration scheme for three different set of parameters with same initial value. We take the following set of parameters:

1. $\kappa_n=\frac{n}{n+1}$, $\rho_n=\frac{n}{n+5}$ for all $n\in\mathbb{N}$ and $c_1=1000$,
2. $\kappa_n=\frac{2n}{5n+2}$, $\rho_n=\frac{1}{n+5}$ for all $n\in\mathbb{N}$ and $c_1=1000$,
3. $\kappa_n=\frac{1}{\sqrt{n+5}}$, $\rho_n=\sqrt{\frac{n+1}{5n+1}}$ for all $n\in\mathbb{N}$ and $c_1=1000$.

We get the following Table 1 and Figure 1 for the initial value 1000.

TABLE 1. Tabular Values for Case(I)

Iteration Number	*Parameter Set* 1	*Parameter Set* 2	*Parameter Set* 3
1	1000	1000	1000
2	92.85442	93.78168	66.24031
3	7.75377	8.918196	4.655107
4	0.000000	0.8576755	0.000000
5	0.000000	0.000000	0.000000

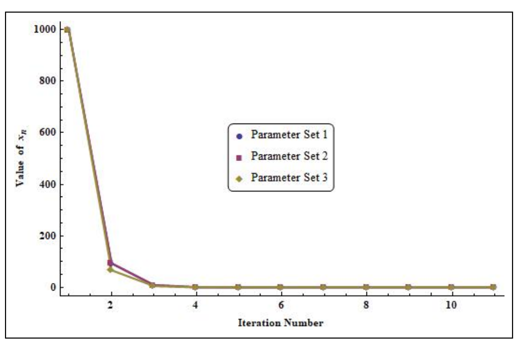

FIGURE 1. Graph corresponding to Table 1.

Case (II). In this case, we will show the convergence of our iteration scheme for three different initial values with same parameters. We take the following set of parameters:

1. $\kappa_n = \frac{n}{n+1}$, $\rho_n = \frac{1}{\sqrt{2n+1}}$ for all $n \in \mathbb{N}$ and $c_1 = 500$,
2. $\kappa_n = \frac{n}{n+1}$, $\rho_n = \frac{1}{\sqrt{2n+1}}$ for all $n \in \mathbb{N}$ and $c_1 = 1000$,
3. $\kappa_n = \frac{n}{n+1}$, $\rho_n = \frac{1}{\sqrt{2n+1}}$ for all $n \in \mathbb{N}$ and $c_1 = 1500$.

We get the following Table 2 and Figure 2 for three different initial values.

TABLE 2. Tabular Values for Case(II)

Iteration Number	*Parameter Set* 1	*Parameter Set* 2	*Parameter Set* 3
1	500	1000	1500
2	32.43248	64.86496	97.29744
3	2.328621	4.657241	6.985862
4	0.000000	0.000000	0.000000

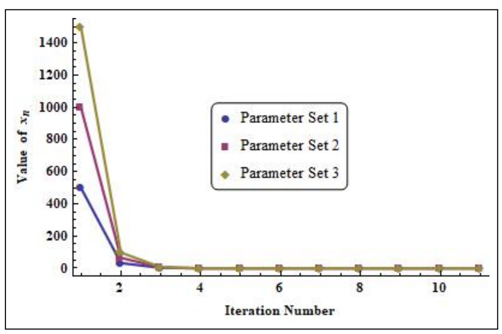

FIGURE 2. Graph corresponding to Table 2.

Next, we show that the iteration process (2.1) converges faster than a number of existing iteration processes in the literature by using above example. Let $\kappa_n = \rho_n = \frac{n}{n+10}$ and $c_1 = 70000.5$. In the following tables, comparison of the convergence of (2.1) with the Noor iteration [19], Agarwal iteration [22], Abbas iteration [16], Thakur et al. iterations [3, 7], M iteration [13], M^* iteration [14], K iteration [20], K^* iteration [15], Piri et al. iteration [10] processes are given. Graphical representation is given in the Figure 3 and Figure 4.

TABLE 3. Sequence generated by Agarwal, Abbas, Thakur, M^*, K^* and New iteration

Step	*Agarwal*	*Abbas*	*Thakur*	M^*	K^*	*New*
1	70000.5	70000.5	70000.5	70000.5	70000.5	70000.5
2	32797.3	16881.5	32660.1	14723.8	14045.6	6928.87
3	15207.	4276.78	14937.4	2950.38	2585.76	653.37
4	6954.5	1115.71	6657.19	564.878	441.254	58.8678
5	3131.27	296.177	2882.49	103.628	70.3914	5.08209
6	1386.86	79.4145	1210.96	18.2638	10.5721	0.0000
7	604.051	21.4054	493.437	3.0999	1.50389	0.0000
8	258.744	5.78169	195.063	0.0000	0.0000	0.0000
9	109.029	1.56165	74.8542	0.0000	0.0000	0.0000
10	45.2128	0.0000	27.9061	0.0000	0.0000	0.0000

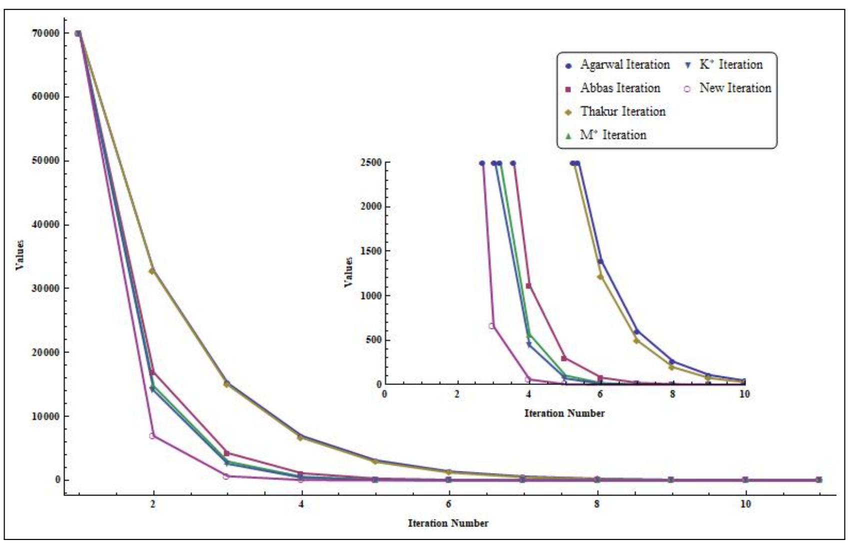

FIGURE 3. Graph corresponding to Table 3.

TABLE 4. Sequence generated by Noor, Thakur New, K, M, Piri et al. and New iteration

Step	*Noor*	*ThakurNew*	*K*	*M*	Piri et al.	*New*
1	70000.5	70000.5	70000.5	70000.5	70000.5	70000.5
2	66481.2	15434.	7263.07	14755.8	14045.6	6928.87
3	60119.	3367.65	745.777	2979.38	2585.76	653.37
4	51889.9	724.751	75.5287	579.186	441.254	58.8678
5	42843.8	153.562	7.53093	108.862	70.3914	5.08209
6	33911.1	32.0065	0.738657	19.8534	10.5721	0.0000
7	25781.1	6.56025	0.0000	3.52375	1.50389	0.0000
8	18860.9	1.32238	0.0000	0.0000	0.0000	0.0000
9	13300.8	0.0000	0.0000	0.0000	0.0000	0.0000
10	9056.02	0.0000	0.0000	0.0000	0.0000	0.0000

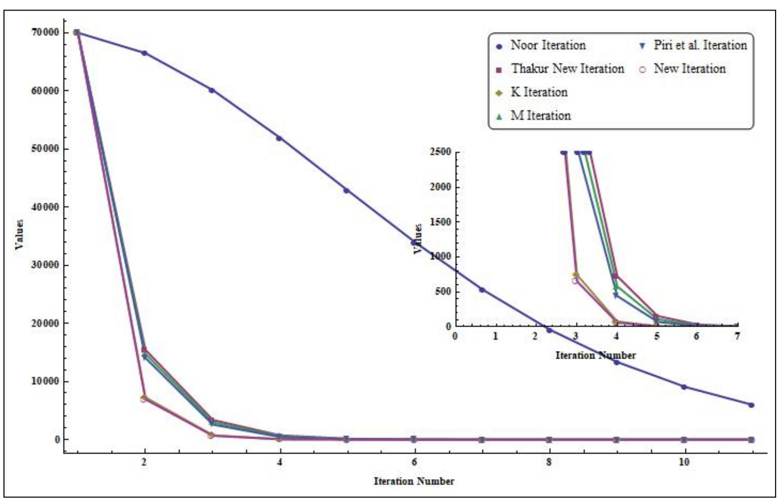

FIGURE 4. Graph corresponding to Table 4].

5. **Conclusion**

In this paper, we proposed a new modified iteration process in the framework of CAT(0) spaces. We proved some strong and Δ-convergence results under mild conditions by using the proposed process. Also, with the help of an example it is showed that our process is stable with respect to parameters κ_n, ρ_n, initial values and it converges faster than a number of existing iteration processes in the literature.

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