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Similar generalized frames

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Abstract. Generalized frames are an extension of frames in Hilbert spaces and Hilbert C^*-modules. In this paper, the concept "Similar" for modular g-frames is introduced and all of operator duals (ordinary duals) of similar g-frames with respect to each other are characterized. Also, an operator dual of a given g-frame is studied where g-frame is constructed by a primary g-frame and an orthogonal projection. Moreover, a g-frame is obtained by two the g-frames and its operator duals are investigated. Finally, the dilation of g-frames is studied.

1. Introduction

Frames have many applications in various branches of mathematics, signal processing, filter bank, and etc. [4]. Generalizations of frames in different spaces have been presented by researchers until 2005. Sun [12] introduced the notion of g-frames as a generalization of frames for a sequence of bounded operators on Hilbert spaces. Afterwards, Frank-Larson [6] extended the concept of frames for the elements of (finitely or countably generated) Hilbert C^*-modules and the authors in [8] presented g-frames for Hilbert C^*-modules. It is well known that the theory of Hilbert C^*-modules has applications in the study of locally compact quantum groups, complete maps between C^*-algebras, non-commutative geometry, and KK-theory. There are many differences between Hilbert C^*-modules and Hilbert spaces. It is expected that problems about frames and g-frames for Hilbert C^*-modules to be more complicated than those for Hilbert spaces. This makes the study of the frames for Hilbert C^*-modules important and interesting. The main purpose of the

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paper is to study the similar g-frames and their operator duals (their ordinary duals) in Hilbert C^*-modules. The present results are valid for g-frames in Hilbert spaces, g-frames with real valued bounds and g-frames with C^*-valued bounds in Hilbert C^*-modules; g-frames with C^*-valued bounds have been studied in [1].

Throughout the paper, we fix the notations $\mathcal{A}$ and J for a unital C^*-algebra and a finite or countably infinite index set, respectively. Also, the sets $\mathcal{H}$, $\mathcal{K}$, and $\mathcal{K}_j$, for $j \in J$, are finitely or countably generated Hilbert $\mathcal{A}$-modules. The family of adjointable operators from $\mathcal{H}$ into $\mathcal{K}$ is denoted by $B_*(\mathcal{H}, \mathcal{K})$.

A g-frame for a given Hilbert $\mathcal{A}$-module $\mathcal{H}$ is a family of ordered pairs $\{(\Lambda_j, \mathcal{K}_j)\}_{j\in J}$ consisting of Hilbert $\mathcal{A}$-modules $\mathcal{K}_j$ and operators $\Lambda_j \in B_*(\mathcal{H}, \mathcal{K}_j)$ satisfying

$$A\langle f, f\rangle \leq \sum_{j\in J}\langle \Lambda_j f, \Lambda_j f\rangle \leq B\langle f, f\rangle, \tag{1.1}$$

for all $f \in \mathcal{H}$ and some positive constants A, B independent of f. The corresponding operators to a g-frame $\{(\Lambda_j, \mathcal{K}_j)\}_{j\in J}$ are the pre-frame operator, $\Theta_\Lambda(f) = \{\Lambda_j f\}_{j\in J}$ from $\mathcal{H}$ into $\oplus_{j\in J}\mathcal{K}_j$ and the frame operator S on $\mathcal{H}$, $Sf = \sum_{j\in J}\Lambda_j^*\Lambda_j f$. The pre-frame operator is adjointable and closed range (the range of Θ_Λ is complementable subspace), and the frame operator S is well defined, positive, and invertible [2].

A g-complete sequence $\{(\Lambda_j, \mathcal{K}_j)\}_{j\in J}$ is a modular g-Riesz basis if there exist two constants $0 < A < B < \infty$ such that for any finite subset $S \subset J$ and $g_j \in \mathcal{K}_j$, $j \in S$, the following inequalities are valid:

$$A\|\sum_{j\in S}|g_j|^2\| \leq \|\sum_{j\in S}\Lambda_j^* g_j\|^2 \leq B\|\sum_{j\in S}|g_j|^2\|.$$

Every element of a Hilbert space (and a Hilbert C^*-module) has a decomposition with respect to its frame. Dual frames have an important role in this decomposition because the coefficients are obtained from a dual frame. In [1] and [5], the authors extended dual frames to generalized dual frames in Hilbert spaces and Hilbert C^*-modules.

A g-frame $\{(\Gamma_j, \mathcal{K}_j)\}_{j\in J}$ is a dual for a given g-frame $\{(\Lambda_j, \mathcal{K}_j)\}_{j\in J}$ if

$$\sum_{j\in J}\Lambda_j^*\Gamma_j = I.$$

Also, the ordered pair $(\{\Gamma_j\}_{j\in J}, \gamma)$ is called to be an operator dual of $\{(\Lambda_j, \mathcal{K}_j)\}_{j\in J}$ if the following equality satisfies:

$$f = \sum_{j\in J}\Lambda_j^*\Gamma_j\gamma(f), \quad \forall f \in \mathcal{H}, \tag{1.2}$$

where the sequence $\{(\Gamma_j, \mathcal{K}_j)\}_{j\in J}$ is a g-frame for $\mathcal{H}$ and γ is an invertible adjointable $\mathcal{A}$-module map on $\mathcal{H}$.

The paper is organized as follows. Section 2 introduces similar g-frames and obtains all ordinary duals of similar g-frames with respect to each other. A g-frame constructed by two given frame operators of two g-frames and its duals are studied. Moreover, some equivalent conditions are considered for similar g-frames and their duals. Section 3 characterizes operator duals of similar g-frames and all of operator duals of g-frame $\{\Lambda_j P\}_{j\in J}$ are investigated where P is an orthogonal projection. Some relations between the primary frame operator and the new frame operator are given. Also a necessary and sufficient conditions between operator duals and the pre-frame operator is presented. At the end of Section 3, operator duals of a special g-frame in the Hilbert C^*-module $\mathcal{A}$ over $\mathcal{A}$ are calculated. Finally, Section 4 contains dilation of g-frames in a Hilbert C^*-module.

2. Similar g-frames

The notion "similar" for g-frames has been defined in [11]. We use this concept for g-frames in Hilbert C^*-modules and we study some properties of them.

Definition 2.1. Let $\{(\Lambda_j, \mathcal{K}_j)\}_{j\in J}$ and $\{(\Gamma_j, \mathcal{K}_j)\}_{j\in J}$ be g-frames for $\mathcal{H}$. These two g-frames are similar if there exists an adjointable and invertible operator T on $\mathcal{H}$ such that $\Gamma_j = \Lambda_j T$, for $j \in J$.

For every g-frame, we have a family of g-frames by invertible and adjointable operators on $\mathcal{H}$. Now, the set of duals of these g-frames is characterized is characterized with respect to the primary g-frame. Firstly, we see the result about these g-frames and then we investigate their duals.

Theorem 2.2 ([2]). *Let $\{(\Lambda_j, \mathcal{K}_j)\}_{j\in J}$ be a g-frame for $\mathcal{H}$ with bounds A and B, and g-frame operator S. Let $T \in B_*(\mathcal{H})$ be invertible. Then $\{(\Lambda_j T^*, \mathcal{K}_j)\}_{j\in J}$ is a g-frame for $\mathcal{H}$ with bounds $A\|(T^*T)^{-1}\|^{-1}$ and $B\|T\|^2$ and frame operator TST^*.*

Remark 2.3. The frame operator S of a g-frame $\{(\Lambda_j, \mathcal{K}_j)\}_{j\in J}$ has the following properties:

- Since frame operator is invertible and positive, every real power of it is invertible and positive by Spectral Mapping Theorem. Corresponding to g-frame $\{(\Lambda_j, \mathcal{K}_j)\}_{j\in J}$ and by the properties of the frame operator and Theorem 2.2, the sequence

$$\left\{(\Lambda_j S^{\frac{t-1}{2}}, \mathcal{K}_j)\right\}_{j\in J}$$

is a g-frame for $\mathcal{H}$ with frame operator S^t, for all $t \in \mathbb{R}$. More precisely, $\left\{(\Lambda_j S^{-\frac{1}{2}}, \mathcal{K}_j)\right\}_{j\in J}$ is a Parseval g-frame.

- Since the inverse of an operator is unique and by the reconstruction formula, in the reconstruction formula the inverse of frame operator, S^{-1}, is unique.

In the next theorem, we show that duals of g-frames $\{\Lambda_j\}_{j\in J}$ and $\{\Lambda_j T^*\}_{j\in J}$ are similar.

Theorem 2.4. *Let $\{(\Lambda_j, \mathcal{K}_j)\}_{j\in J}$ be a g-frame for $\mathcal{H}$ with pre-frame operator Θ_Λ, and let $T \in B_*(\mathcal{H})$ be an invertible operator. Then every dual of g-frame $\{\Lambda_j T^*\}_{j\in J}$ is similar to a dual of $\{\Lambda_j\}_{j\in J}$. The converse is also satisfied.*

Proof. Assume that $\Gamma = \{\Gamma_j\}_{j\in J}$ is a dual of $\{\Lambda_j T^*\}_{j\in J}$ with pre-frame operator Θ_Λ. If $\Theta_{\Lambda T^*}$ is the pre-frame operator $\{\Lambda_j T^*\}_{j\in J}$, then for $f \in \mathcal{H}$,

$$\begin{aligned} \Theta_{\Lambda T^*}(f) &= \{(\Lambda_j T^*) f\}_{j\in J} \\ &= \{\Lambda_j (T^* f)\}_{j\in J} \\ &= \Theta_\Lambda T^*(f). \end{aligned} \tag{2.1}$$

For $f \in \mathcal{H}$,

$$\begin{aligned} f &= \sum_{j\in J} (\Lambda_j T^*)^* \Gamma_j f \\ &= \sum_{j\in J} T \Lambda_j^* \Gamma_j f \\ &= T \left(\sum_{j\in J} \Lambda_j^* \Gamma_j f \right), \end{aligned}$$

and by the definition of the pre-frame operator,

$$\begin{aligned} \Theta^*_{\Lambda T^*} \Theta_\Gamma(f) &= \Theta^*_{\Lambda T^*} \{\Gamma_j f\}_{j\in J} \\ &= \sum_{j\in J} (\Lambda_j T^*)^* \Gamma_j f \\ &= id_{\mathcal{H}}(f). \end{aligned} \tag{2.2}$$

From (2.1) and (2.2) and the invertibility of T, we obtain that

$$\begin{aligned} T\Theta^*_\Lambda \Theta_\Gamma = id_{\mathcal{H}} \quad &\Rightarrow \quad T^{-1} = \Theta^*_\Lambda \Theta_\Gamma \\ &\Rightarrow \quad \Theta^*_\Lambda \Theta_\Gamma T = id_{\mathcal{H}} \\ &\Rightarrow \quad \Theta^*_\Lambda \Theta_{\Gamma T} = id_{\mathcal{H}}. \end{aligned}$$

It shows that $\{\Gamma_j T\}_{j\in J}$ is a dual for $\{\Lambda_j\}_{j\in J}$ that is similar to $\{\Gamma_j\}_{j\in J}$. For the converse, suppose that $\{\Gamma_j\}_{j\in J}$ is a dual of $\{\Lambda_j\}_{j\in J}$ with the pre-frame operator Θ_Γ. Similar to the above equalities

$$\begin{aligned}\Theta_\Lambda^*\Theta_\Gamma = id_{\mathcal{H}} \quad &\Rightarrow \quad T\Theta_\Lambda^*\Theta_\Gamma T^{-1} = id_{\mathcal{H}}\\ &\Rightarrow \quad \Theta_{\Lambda T^*}^*\Theta_{\Gamma T^{-1}} = id_{\mathcal{H}},\end{aligned}$$

and so $\{\Gamma_j T^{-1}\}_{j\in J}$ is a dual for $\{\Lambda_j T^*\}_{j\in J}$ and is similar to g-frame to the dual g-frame $\{\Gamma_j\}_{j\in J}$ of $\{\Lambda_j\}_{j\in J}$. □

The following theorem constructs a similar g-frame by a combination of frame operators of two given g-frames. The resulting frame is similar to one of the given g-frames and it has the same frame operator as another g-frame. Also, its duals are introduced.

Theorem 2.5. *If $\{(\Lambda_j, \mathcal{K}_j)\}_{j\in J}$ and $\{(\Gamma_j, \mathcal{K}_j)\}_{j\in J}$ are g-frames with frame operators S_Λ and S_Γ, respectively, then there exists a similar g-frame to $\{\Gamma_j\}_{j\in J}$ with frame operator S_Λ, and its duals are $\{\eta_j S_\Gamma^{\frac{1}{2}} S_\Lambda^{-\frac{1}{2}}\}_{j\in J}$ where $\{\eta_j\}_{j\in J}$ is a dual of $\{\Gamma_j\}_{j\in J}$.*

Proof. Set $T = S_\Lambda^{\frac{1}{2}} S_\Gamma^{-\frac{1}{2}}$. Then by Theorem 2.2 the g-frame $\{\Gamma_j T^*\}_{j\in J}$ is similar to $\{\Gamma_j\}_{j\in J}$ with frame operator $S_{\Gamma T^*} = T S_\Gamma T^*$. So

$$\begin{aligned}S_{\Gamma T^*} &= T S_\Gamma T^*\\ &= \left(S_\Lambda^{\frac{1}{2}} S_\Gamma^{-\frac{1}{2}}\right) S_\Gamma \left(S_\Lambda^{\frac{1}{2}} S_\Gamma^{-\frac{1}{2}}\right)^*\\ &= S_\Lambda.\end{aligned}$$

The set of duals of $\{\Gamma_j T^*\}_{j\in J}$ is a direct result of Theorem 2.4. □

Now, the attractive question is that "Are there any relations between two given g-frames in the Hilbert space H?". Up to now, duals or operator duals corresponding to a given g-frame are defined that have a special relationship with a primary g-frame. Here, we give a different answer for the question, and characterize a lager family than duals or operator duals. On the other hand, in [1], it is shown that every ordinary dual of a g-frame is an operator dual. But we now see another relation between dual g-frames and operator dual frames of a g-frame by the similarity relation between g-frames.

Theorem 2.6. *Let $\{(\Lambda_j, \mathcal{K}_j)\}_{j\in J}$ and $\{(\Gamma_j, \mathcal{K}_j)\}_{j\in J}$ be g-frames for $\mathcal{H}$ with pre-frame operators Θ_Λ and Θ_Γ, respectively. Then the following statements are equivalent:*

(i) *The g-frame $\{\Gamma_j\}_{j\in J}$ is similar to a dual g-frame of $\{\Lambda_j\}_{j\in J}$.*

(ii) *There exists an invertible operator T of $B_*(\mathcal{H})$ such that*

$$Tf = \sum_{j\in J} \Gamma_j^* \Lambda_j f,$$

for $f \in \mathcal{H}$.

(iii) *There is an invertible operator γ of $B_*(\mathcal{H})$ such that $(\{\Gamma_j\}_{j\in J}, \gamma)$ is an operator dual for $\{\Lambda_j\}_{j\in J}$.*

3. Operator Duals of Constructed g-frames

This section studies duals of some constructed g-frames with respect to a primary g-frame. In Theorem 2.2, a family of g-frames have been obtained by a given g-frame and an invertible operator. Now, we can characterize all operator duals for every element of this family. This characterization is given in the following.

Theorem 3.1. *Let $\{\Lambda_j\}_{j\in J}$ be a g-frame for $\mathcal{H}$ with pre-frame operator Θ_Λ, and let $T \in B_*(\mathcal{H})$ be an invertible operator. The set of operator duals of g-frame $\{\Lambda_j\}_{j\in J}$ is one-to-one corresponding to the set of operator duals of g-frame $\{\Lambda_j T^*\}_{j\in J}$.*

Proof. Let $\Theta_{\Lambda T^*}$ be the pre-frame operator of $\{\Lambda_j T^*\}_{j\in J}$. If $(\{\Gamma_j\}_{j\in J}, \gamma)$ is an operator dual for $\{\Lambda_j\}_{j\in J}$ with the pre-frame operator Θ_Γ, then

$$\Theta_\Lambda^* \Theta_\Gamma \gamma = id_{\mathcal{H}} \quad \Rightarrow \quad T\Theta_\Lambda^* \Theta_\Gamma T^* (T^*)^{-1} \gamma T^{-1} = id_{\mathcal{H}},$$

and

$$\Theta_{\Lambda T^*}^* \Theta_{\Gamma T^*} (T^*)^{-1} \gamma T^{-1} = id_{\mathcal{H}},$$

so $\left(\{\Gamma_j T^*\}_{j\in J}, (T^*)^{-1}\gamma T^{-1}\right)$ is an operator dual for $\{\Lambda_j T^*\}_{j\in J}$. Now, suppose $(\{\Gamma_j\}_{j\in J}, \gamma)$ is an operator dual for $\{\Lambda_j T^*\}_{j\in J}$ with the pre-frame operator Θ_Γ. Then

$$\begin{aligned} \Theta_{\Lambda T^*}^* \Theta_\Gamma \gamma = id_{\mathcal{H}} \quad &\Rightarrow \quad T\Theta_\Lambda^* \Theta_\Gamma \gamma = id_{\mathcal{H}} \\ &\Rightarrow \quad \Theta_\Lambda^* \Theta_\Gamma T^* (T^*)^{-1} \gamma = T^{-1} \\ &\Rightarrow \quad \Theta_\Lambda^* \Theta_{\Gamma T^*} (T^*)^{-1} \gamma T = id_{\mathcal{H}}, \end{aligned}$$

and $(\{\Gamma_j T^*\}_{j\in J}, (T^*)^{-1}\gamma T)$ is an operator dual of $\{\Lambda_j\}_{j\in J}$. □

A family of constructed g-frames in Hilbert spaces is obtained by orthogonal projections [11]. We study these g-frames in Hilbert C^*-modules and their operator duals with respect to a primary g-frame. Also, a relation is given for these orthogonal projections.

Proposition 3.2. *Let $\{(\Lambda_j, \mathcal{K}_j)\}_{j\in J}$ be a g-frame for $\mathcal{H}$, and suppose P is an orthogonal projection onto an orthogonally complemented submodule $\mathcal{H}_0$ of $\mathcal{H}$. Then the following statements hold.*

(i) *The sequence $\{\Lambda_j P\}_{j\in J}$ is a g-frame for $\mathcal{H}_0$.*

(ii) *If* $(\{\Gamma_j\}_{j\in J}, \gamma)$ *is an operator dual of* $\{\Lambda_j\}_{j\in J}$ *and* $\gamma(\mathcal{H}_0) \subseteq \mathcal{H}_0$ *, then* $(\{\Gamma_j P\}_{j\in J}, \gamma|_{\mathcal{H}_0})$ *is an operator dual for* $\{\Lambda_j P\}_{j\in J}$*, where* $\gamma|_{\mathcal{H}_0}$ *is the restriction of operator* γ *to* $\mathcal{H}_0$.

(iii) *If* S *and* S_P *are operator duals of* $\{\Lambda_j\}_{j\in J}$ *and* $\{\Lambda_j P\}_{j\in J}$*, respectively, and for* $j \in J$*,* $S_P^{-1} P \Lambda_j^* = P S^{-1} \Lambda_j^*$*, then* $S_P^{-1} P = P S^{-1}$ *on* $\mathcal{H}$.

Proof. (i) We have $\Lambda_j P f = \Lambda_j f$, for all $f \in \mathcal{H}_0$. Therefore, $\{\Lambda_j P\}_{j\in J}$ is clearly a g-frame for $\mathcal{H}_0$.

(ii) If $(\{\Gamma_j\}_{j\in J}, \gamma)$ is an operator dual of $\{\Lambda_j\}_{j\in J}$ such that $\gamma(\mathcal{H}_0) \subseteq \mathcal{H}_0$, then for $f \in \mathcal{H}_0$, $P\gamma f = \gamma f$, and

$$\begin{aligned} f &= Pf \\ &= P\left(\sum_{j\in J} \Lambda_j^* \Gamma_j \gamma f\right) \\ &= \sum_{j\in J} (\Lambda_j P)^* (\Gamma_j P) \gamma f, \quad \forall f \in \mathcal{H}_0, \end{aligned}$$

so $(\{\Gamma_j P\}_{j\in J}, \gamma|_{\mathcal{H}_0})$ is an operator dual for $\{\Lambda_j P\}_{j\in J}$.

(iii) Assume that $P S^{-1} \Lambda_j^* = S_P^{-1} P \Lambda_j$, for all $j \in J$. For $f \in \mathcal{H}$,

$$\begin{aligned} S_P^{-1} P f &= S_P^{-1} P \left(\sum_{j\in J} \Lambda_j^* \Lambda_j S^{-1} f\right) \\ &= \sum_{j\in J} S_P^{-1} P \Lambda_j^* \Lambda_j S^{-1} f \\ &= \sum_{j\in J} P S^{-1} \Lambda_j^* \Lambda_j S^{-1} f \\ &= P\left(\sum_{j\in J} (\Lambda_j S^{-1})^* (\Lambda_j S^{-1}) f\right) \\ &= P S^{-1} f. \end{aligned}$$

The last equality is valid because S^{-1} is the frame operator of g-frame $\{\Lambda_j S^{-1}\}_{j\in J}$. □

In the ordinary dual case, if $\{\Gamma_j\}_{j\in J}$ is also a dual of $\{\Lambda_j\}_{j\in J}$, then $\{\Gamma_j P\}_{j\in J}$ is a dual of $\{\Lambda_j P\}_{j\in J}$. It is clear by $\gamma = id_{\mathcal{H}}$.

It is known that the pre-frame operator of a given g-frame in a Hilbert $\mathcal{A}$-module has closed range, this range is orthogonal complementable and the orthogonal projection onto this set is well-defined [10]. Considering this fact, the relation between pre-frame operators of a g-frame in a Hilbert $\mathcal{A}$-module and its duals is obtained in the following theorem.

Theorem 3.3. *If $\{(\Lambda_j, \mathcal{K}_j)\}_{j\in J}$ and $\{(\Gamma_j, \mathcal{K}_j)\}_{j\in J}$ are g-frames for $\mathcal{H}$ with pre-frame operator Θ_Λ and Θ_Γ, respectively, then $(\{\Gamma_j\}_{j\in J}, \gamma)$ is an operator dual for $\{\Lambda_j\}_{j\in J}$ if and only if $P_{\Theta_\Lambda}\Theta_\Gamma\gamma$ is a right inverse of Θ_Λ^*, where P_{Θ_Λ} is the orthogonal projection on the range of Θ_Λ. Moreover, if $\{\Lambda_j\}_{j\in J}$ is a Parseval g-frame, then $\Theta_\Lambda = P_{\Theta_\Lambda}\Theta_\Gamma\gamma$.*

Proof. Suppose that $(\{\Gamma_j\}_{j\in J}, \gamma)$ is an operator dual for $\{\Lambda_j\}_{j\in J}$. Then

$$\begin{aligned} id_{\mathcal{H}} &= \Theta_\Lambda^*\Theta_\Gamma\gamma \\ &= (P_{\Theta_\Lambda}\Theta_\Lambda)^*\Theta_\Gamma\gamma \\ &= \Theta_\Lambda^* P_{\Theta_\Lambda}\Theta_\Gamma\gamma. \end{aligned}$$

For converse, assume that $P_{\Theta_\Lambda}\Theta_\Gamma\gamma$ is a right inverse of Θ_Λ^*. Then

$$\begin{aligned} \Theta_\Lambda^* P_{\Theta_\Lambda}\Theta_\Gamma\gamma = id \quad &\Rightarrow \quad (P_{\Theta_\Lambda}\Theta_\Lambda)^*\Theta_\Gamma\gamma = id \\ &\Rightarrow \quad \Theta_\Lambda^*\Theta_\Gamma\gamma = id. \end{aligned}$$

It concludes that $(\{\Gamma_j\}_{j\in J}, \gamma)$ is an operator dual for $\{\Lambda_j\}_{j\in J}$. Now, assume that $\{\Lambda_j\}_{j\in J}$ is a Parseval g-frame. Then Θ_Λ is an isometry because

$$\langle \Theta_\Lambda f, \Theta_\Lambda f\rangle = \sum_{j\in J} \langle \Lambda_j f, \Lambda_j f\rangle = \langle f, f\rangle, \quad \forall f \in \mathcal{H},$$

and by the polarization identity, Θ_Λ is unitary. Now, for $f \in \mathcal{H}$

$$\begin{aligned} \langle \Theta_\Lambda f, P_{\Theta_\Lambda}\Theta_\Gamma\gamma f\rangle &= \langle P_{\Theta_\Lambda}\Theta_\Lambda f, \Theta_\Gamma\gamma f\rangle \\ &= \langle \Theta_\Lambda f, \Theta_\Gamma\gamma f\rangle \\ &= \langle f, (\Theta_\Lambda)^*\Theta_\Gamma\gamma f\rangle \\ &= \langle f, f\rangle. \end{aligned} \tag{3.1}$$

On the other hand, $P_{\Theta_\Lambda}\Theta_\Gamma\gamma f \in R_{\Theta_\Lambda}$, so there exists $g \in \mathcal{H}$ such that $P_{\Theta_\Lambda}\Theta_\Gamma\gamma f = \Theta_\Lambda g$. Since Θ_Λ is unitary, the equality (3.1) concludes that $f = g$ and $\Theta_\Lambda = P_{\Theta_\Lambda}\Theta_\Gamma\gamma$. □

Example 3.4. It is well-known that $\mathcal{A}$ is a self-dual Hilbert $\mathcal{A}$-module. Alijani and Dehghan [2] showed that $\{f_j\}_{j\in J}$ is a frame for $\mathcal{A}$ if and only if $\{\Lambda_{f_j}\}_{j\in J}$ is a g-frame for $\mathcal{A}$, where $\Lambda_{f_j} f = \langle f, f_j\rangle$ on $\mathcal{A}$, for $j \in J$. Now, we can see that $(\{g_j\}_{j\in J}, \gamma)$ is an operator dual of $\{f_j\}_{j\in J}$ if and only if $(\{\Lambda_{g_j}\}_{j\in J}, \gamma)$ is an operator dual for $\{\Lambda_{f_j}\}_{j\in J}$. Since for $f \in \mathcal{A}$,

$$\begin{aligned}\sum_{j\in J} \langle \gamma f, g_j\rangle f_j &= \sum_{j\in J} \langle \gamma f, g_j\rangle (f_j^*)^* \\ &= \sum_{j\in J} \langle \langle \gamma f, g_j\rangle , f_j^*\rangle \\ &= \sum_{j\in J} \Lambda_{f_j}^*(\langle \gamma f, g_j\rangle) \\ &= \sum_{j\in J} \Lambda_{f_j}^*\Lambda_{g_j}\gamma f.\end{aligned}$$

By self-duality of $\mathcal{A}$, the set of (original duals) operator duals of $\{f_j\}_{j\in J}$ is one-to-one corresponding to the set of (original duals) operator duals of $\{\Lambda_{f_j}\}_{j\in J}$.

More precisely, a g-frame $\{\Gamma_j\}_{j\in J}$ is similar to a given g-frame $\{\Lambda_{f_j}\}_{j\in J}$ if and only if there exists a frame $\{g_j\}_{j\in J}$ such that $\{g_j\}_{j\in J}$ is equivalent to $\{f_j\}_{j\in J}$. To see this, assume that a g-frame $\{\Gamma_j\}_{j\in J}$ is similar to a given g-frame $\{\Lambda_{f_j}\}_{j\in J}$. Then there is an invertible operator $T \in B_*(\mathcal{A})$ such that $\Lambda_{f_j}T = \Gamma_j$, for every $j \in J$. Since $\mathcal{A}$ is self-dual, there exists a frame $\{g_j\}_{j\in J}$ such that $\Gamma_j = \Lambda_{g_j}$, for $j \in J$, and also for $j \in J$,

$$\begin{aligned}\Gamma_j f = \Lambda_{f_j} T f \quad &\Rightarrow \quad \langle f, g_j\rangle = \langle Tf, f_j\rangle \\ &\Rightarrow \quad \langle f, g_j\rangle = \langle f, T^* f_j\rangle .\end{aligned}$$

Now, set $\xi = T^*$, then the frames $\{f_j\}_{j\in J}$ and $\{g_j\}_{j\in J}$ are equivalent by the invertible and adjointable operator ξ. Similarly, the reverse of the relation is obtained.

4. Dilation of g-frames

The subject of dilation for frames has been considered in [7]. We are going to study this concept for g-frames in Hilbert C^*-modules. For this purpose, we need to use some results in [9] and [7]. Firstly, these results will be reviewed and finally, the conclusion about the dilation of g-frames will be investigated.

Theorem 4.1 ([9]). *Let $\Lambda = \{(\Lambda_j, \mathcal{K}_j)\}_{j\in J}$ and $\Gamma = \{(\Gamma_j, \mathcal{K}_j)\}_{j\in J}$ be two g-frames with the corresponding sequences $\{f_{jk} = \Lambda_j^* e_{jk}; k \in I_j, j \in J\}$ and $\{g_{jk} = \Lambda_j^* e_{jk}; k \in I_j, j \in J\}$ where $\{e_{jk}\}_{k\in I_j}$ is a Parseval frame for $\mathcal{K}_j$, for $j \in J$. Then:*

(i) *The sequence $\{\Lambda_j\}_{j\in J}$ is a g-frame (res. g-Bessel sequence, tight g-frame, modular g-Riesz basis) for $\mathcal{H}$ if and only if $\{f_{jk}\}_{k\in I_j, j\in J}$ is a frame (res. Bessel sequence, tight frame, modular Riesz basis) for $\mathcal{H}$.*

(ii) *The sequence* Γ *is a dual for* Λ *if and only if* $\{g_{jk}\}_{k\in I_j, j\in J}$ *is a dual for* $\{f_{jk}\}_{k\in I_j, j\in J}$.

Theorem 4.2 ([7]). *Let* $\{f_j\}_{j\in J}$ *and* $\{g_j\}_{j\in J}$ *be a dual frame pair for* $\mathcal{H}$. *Then there exists a Hilbert* $\mathcal{A}$*-module* $\mathcal{K} \supseteq \mathcal{H}$ *and a Riesz basis* $\{u_j\}_{j\in J}$ *of* $\mathcal{K}$ *which has a unique dual* $\{u_j^*\}_{j\in J}$ *and satisfies* $Pu_j = f_j$ *and* $Pu_j^* = g_j$, *for all* $j \in J$, *where* P *is the projection from* $\mathcal{K}$ *onto* $\mathcal{H}$.

Proposition 4.3 ([9]). *The sequence* $\{f_j\}_{j\in J}$ *is a modular Riesz basis for* $\mathcal{H}$ *if and only if* $\{f_j\}_{j\in J}$ *is a Riesz basis and has a unique dual frame.*

Theorem 4.4. *Let* $\Gamma = \{(\Gamma_j, \mathcal{K}_j)\}_{j\in J}$ *be a dual for a g-frame* $\Lambda = \{(\Lambda_j, \mathcal{K}_j)\}_{j\in J}$ *in* $\mathcal{H}$ *and let* $\mathcal{K}_j$*'s be closed submodules of a Hilbert* $\mathcal{A}$*-module* $\mathcal{K}$. *Then there exist a Hilbert* $\mathcal{A}$*-module* $\mathcal{X}$, *and a modular g-Riesz basis* $\{(\delta_j, \mathcal{K}_j)\}_{j\in J}$ *which has a unique dual* $\{(\sigma_j, \mathcal{K}_j)\}_{j\in J}$ *such that*

$$\delta_j|_{\mathcal{H}} = \Lambda_j, \qquad \sigma_j|_{\mathcal{H}} = \Gamma_j, \quad \forall j \in J.$$

Proof. For $j \in J$, assume that $\{e_{jk}\}_{k\in I_j}$ is a Parseval frame for $\mathcal{K}_j$. Set

$$f_{jk} = \Lambda_j^* e_{jk}, \qquad g_{jk} = \Gamma_j^* e_{jk}, \quad \forall k \in I_j.$$

Since Λ and Γ are g-frames, the sequences $\{f_{jk}\}_{k\in I_j, j\in J}$ and $\{g_{jk}\}_{k\in I_j, j\in J}$ are frames for $\mathcal{H}$ by Theorem 4.1. Also, Theorem 4.1 concludes that $\{g_{jk}\}_{k\in I_j, j\in J}$ is a dual for $\{f_{jk}\}_{k\in I_j, j\in J}$ because Λ is a dual of Γ. Now, Theorem 4.2 concludes that there exist a Hilbert $\mathcal{A}$-module $\mathcal{X}$ and a Riesz basis $\{x_{jk}\}_{k\in I_j, j\in J}$ of $\mathcal{X}$ with a unique dual $\{y_{jk}\}_{k\in I_j, j\in J}$ such that

$$\mathcal{H} \subseteq \mathcal{X}, \qquad Px_{jk} = f_{jk}, \qquad Py_{jk} = g_{jk}, \quad \forall k \in I_j, \quad \forall j \in J,$$

when P is the projection from $\mathcal{X}$ onto $\mathcal{H}$. Now, Proposition 4.3 concludes that the Riesz basis $\{x_{jk}\}_{k\in I_J, j\in J}$ is a modular Riesz basis. We define for $j \in J$

$$\delta_j^* : \mathcal{K}_j \longrightarrow \mathcal{X}, \qquad \delta_j^* e_{jk} = x_{jk}, \quad \forall k \in I_j.$$

Every δ_j^* is an $\mathcal{A}$-module map and adjointable by the adjointable operator $\delta_j x = \sum_{k\in I_j} \langle x, x_{jk}\rangle e_{jk}$, because for $j \in J$, $x \in \mathcal{X}$, and $g_j \in \mathcal{K}_j$, we have

$$\begin{aligned}\langle \delta_j x, g_j \rangle &= \left\langle \sum_{k\in I_j} \langle x, x_{jk}\rangle e_{jk}, g_j \right\rangle \\ &= \sum_{k\in I_j} \langle x, x_{jk}\rangle \langle e_{jk}, g_j\rangle \\ &= \sum_{k\in I_j} \left\langle x, \delta_j^* e_{jk}\right\rangle \langle e_{jk}, g_j\rangle \\ &= \sum_{k\in I_j} \left\langle x, \langle g_j, e_{jk}\rangle \delta_j^* e_{jk}\right\rangle \\ &= \left\langle x, \delta_j^* \left(\sum_{k\in I_j} \langle g_j, e_{jk}\rangle e_{jk} \right) \right\rangle \\ &= \left\langle x, \delta_j^* g_j \right\rangle .\end{aligned}$$

The last equality is given by the fact that $\{e_{jk}\}_{k\in I_j}$ is a Parseval frame, for all $j \in J$. Similarly, the operator $\sigma_j^* e_{jk} = y_{jk}$ from $\mathcal{K}_j$ into $\mathcal{X}$ is an adjointable $\mathcal{A}$-module map with the adjoint

$$\sigma_j x = \sum_{k\in I_j} \langle x, y_{jk}\rangle e_{jk},$$

for $j \in J$. Again by Theorem 4.1, the g-frame $\{(\sigma_j, \mathcal{K}_j)\}_{j\in J}$ is a dual for modular g-Riesz basis $\{(\delta_j, \mathcal{K}_j)\}_{j\in J}$. We must show that $\delta_j|_{\mathcal{H}} = \Lambda_j$ and $\sigma_j|_{\mathcal{H}} = \Gamma_j$, for $j \in J$. To see this, let $f \in \mathcal{H}$ and $j \in J$, and since the sequence $\{e_{jk}\}_{k\in I_j}$ is a Parseval frame for $\mathcal{K}_j$,

$$\begin{aligned}\Lambda_j f &= \sum_{k\in I_j} \langle \Lambda_j f, e_{jk}\rangle e_{jk} \\ &= \sum_{k\in I_j} \left\langle f, \Lambda_j^* e_{jk}\right\rangle e_{jk} \\ &= \sum_{k\in I_j} \langle f, f_{jk}\rangle e_{jk}.\end{aligned} \tag{4.1}$$

Now, suppose that P_{R_j} is the orthogonal projection onto the range of the pre-frame operator $\{g_{jk}\}_{k\in I_j}$, and $\{t_{jk}\}_{k\in I_j}$ is an orthonomal basis of $l^2(\mathcal{A})$ such that $x_{jk} = f_{jk} \oplus P_{R_j}^{\perp} t_{jk}$, for $k \in I_j$ and $j \in J$. By (4.1) and the structure of x_{jk}'s in the proof of Theorem 4.2, we obtain for $f \in \mathcal{H}$,

$$\begin{aligned}\delta_j f = \delta_j(f \oplus 0) &= \sum_{k \in I_j} \langle f \oplus 0, x_{jk} \rangle e_{jk} \\ &= \sum_{k \in I_j} \left\langle f \oplus 0, f_{jk} \oplus P_{R_j}^{\perp} t_{jk} \right\rangle e_{jk} \\ &= \sum_{k \in I_j} \langle f, f_{jk} \rangle e_{jk} \\ &= \Lambda_j f.\end{aligned}$$

Similarly, $\sigma_j|_{\mathcal{H}} = \Lambda_j$, for $j \in J$. □

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