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Muzeyyen Sangurlu Sezen

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Fixed Point Theorems for Fuzzy (γ, β)-Contractions in non-Archimedean Fuzzy Metric Spaces

Müzeyyen Sangurlu Sezen

Abstract. In this paper, we introduce new concepts of fuzzy (γ, β)-contraction and prove some fixed point results for fuzzy (γ, β)-contractions in complete non-Archimedean fuzzy metric spaces. Later, we define a fuzzy (γ, β)-weak contraction and establish some new fixed point results for fuzzy (γ, β)-weak contractions. Also, some examples are supplied in order to support the useability of our results.

1. Introduction

Zadeh introduced a definition of fuzzy set and found important applications in many fields of science [31]. So, on the fuzzy set also creating the metric structure has been one of the main problems of fuzzy mathematics. Later, some authors redefined this definition of fuzzy metric space by making some changes [4, 16]. On the other hand, Banach created a famous result called Banach contraction principle in the concept of the fixed point theory [2]. After this theorem, which is most important and fundamental for the fixed point theory topic in metric spaces, it has been also extensively studied in fuzzy metric spaces . Also, fixed point theorems have been studied in various other spaces [1, 5, 6, 11, 13–15, 18–20]. Most of authors proved some fixed point theorems for different contraction type mappings in fuzzy metric spaces, we refer the interested reader to (see[3, 21, 22, 25, 27, 29, 30]).In this

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work, we introduce new concepts of fuzzy (γ,β)-contraction and (γ,β)-weak contraction mappings. We prove some fixed point results in non-Archimedean fuzzy metric spaces. Later, we establish an example for our main theorem.

Definition 1.1 ([8])**.** Let X be a nonempty set, $*$ is a continuous t-norm and M is a fuzzy set on $X^2\times(0,\infty)$, satisfying the following conditions, for all $\zeta,\rho,\xi\in X$ and $s,t>0:$

(i) $M(\zeta,\rho,t)>0$,
(ii) $M(\zeta,\rho,t)=1$ iff $\zeta=\rho$,
(iii) $M(\zeta,\rho,t)=M(\rho,\zeta,t)$,
(iv) $M(\zeta,\xi,t+s)\geq M(\zeta,\rho,t)*M(\rho,\xi,s)$,
(v) $M(\zeta,\rho,\cdot):(0,\infty)\to[0,1]$ is continuous.

If the triangular inequality (iv) is replaced by

(NA) $M(\zeta,\xi,max\{t,s\})\geq M(\zeta,\rho,t)*M(\rho,\xi,s)$,

for all $\zeta,\rho,\xi\in X$ and $s,t>0$, or equivalently,

$$M(\zeta,\xi,t)\geq M(\zeta,\rho,t)*M(\rho,\xi,t),$$

then $(X,M,*)$ is called a non-Archimedean fuzzy metric space [12].

Throughout the paper, shortly we will use NAFMS instead of non-Archimedean fuzzy metric space.

Definition 1.2. Let $(X,M,*)$ be a fuzzy metric space (or NAFMS). Then

(i) A sequence $\{\zeta_n\}$ in X is said to converge to ζ in X, denoted by $\zeta_n\to\zeta$, if and only if $\lim\limits_{n\to\infty}M(\zeta_n,\zeta,s)=1$ for all $s>0$, i.e. for each $r\in(0,1)$ and $s>0$, there exists $n_0\in\mathbb{N}$ such that $M(\zeta_n,\zeta,s)>1-r$ for all $n\geq n_0$ [16, 24].
(ii) A sequence $\{\zeta_n\}$ is a M-Cauchy sequence if and only if for all $\varepsilon\in(0,1)$ and $s>0$, there exists $n_0\in\mathbb{N}$ such that $M(\zeta_n,\zeta_m,s)\geq 1-\varepsilon$ for all $m>n\geq n_0$ [8, 24]. A sequence $\{\zeta_n\}$ is a G-Cauchy sequence if and only if $\lim\limits_{n\to\infty}M(\zeta_n,\zeta_{n+p},s)=1$ for any $p>0$ and $s>0$ [9, 10, 28].
(iii) The fuzzy metric space $(X,M,*)$ is called M-complete (G-complete) if every M-Cauchy (G-Cauchy) sequence is convergent.

Definition 1.3 ([7])**.** Let $(X,M,*)$ be a fuzzy metric space and $h:X\to X$ be a given mapping. Then, h is said to be a triangular β-admissible mapping if there exists a mapping $\beta:X\times X\times(0,\infty)\to(0,\infty)$ such that

(T_{β_1}) $\beta(\zeta,\rho,s)\leq 1$ implies $\beta(h(\zeta),h(\rho),s)\leq 1$ for all $\zeta,\rho\in X$ and for all $s>0$,

(T_{β_2}) $\beta(\zeta,\xi,s) \leq 1$ and $\beta(\xi,\rho,s) \leq 1$ imply $\beta(\zeta,\rho,s) \leq 1$ for all $\zeta,\rho,\xi \in X$ and for all $s > 0$.

Lemma 1.4 ([7]). *Let $(X,M,*)$ be a NAFMS and h be a triangular (β)-admissible mapping. Assume that there exists $\zeta_0 \in X$ such that $\beta(\zeta_0,h(\zeta_0),s) \leq 1$. Define a sequence $\{\zeta_n\}$ by $h(\zeta_n) = \zeta_{n+1}$ for all $n \in \mathbb{N}$. Then, we have*

$$\beta(\zeta_m,\zeta_n,s) \leq 1, \quad \text{for all } m,n \in \mathbb{N} \text{ with } m < n.$$

Definition 1.5 ([26]). Let $\gamma : [0,1) \to \mathbb{R}$ be a strictly increasing, continuous mapping and for each sequence $\{\zeta_n\}_{n\in\mathbb{N}}$ of positive numbers $\lim_{n\to\infty} \zeta_n = 1$ if and only if $\lim_{n\to\infty} \gamma(\zeta_n) = +\infty$. Let Γ is the family of all γ functions. A mapping $h : X \to X$ is said to be a γ-contraction if there exists a $\delta \in (0,1)$ such that

$$M(h(\zeta),h(\rho),s) < 1 \quad \Rightarrow \quad \gamma(M(h(\zeta),h(\rho),s)) \geq \gamma(M(\zeta,\rho,s)) + \delta,$$

for all $\zeta, y \in X$, $s > 0$ and $\gamma \in \Gamma$.

Definition 1.6 ([26]). Let $(X,M,*)$ be a NAFMS. A mapping $h : X \to X$ is said to be a γ-weak contraction if there exists a $\delta \in (0,1)$ such that

$$M(h(\zeta),h(\rho),s) < 1,$$

then

$$\gamma(M(h(\zeta),h(\rho),s)) \geq \gamma(\min\{M(\zeta,\rho,s), M(\zeta,h(\zeta),s), M(\rho,h(\rho),s)\}) + \delta$$

for all $\zeta,\rho \in X$, $s > 0$ and $\gamma \in \Gamma$.

Remark 1.7 ([26]). Every γ-contraction h is a contractive mapping, that is,

$$M(hx,hy,t) > M(x,y,t),$$

for all $x, y \in X$, such that $hx \neq hy$. Thus every γ-contraction is a continuous mapping.

2. Main Results

In this section, we give some definitions and prove our main results.

Definition 2.1. Let $h : X \to X$ and $\beta : X \times X \times (0,\infty) \to (0,\infty)$ be two mappings. Then, h is said to be a fuzzy (γ,β)-contraction if there exist $\gamma \in \Gamma$ and $\delta \in (0,1)$ such that

$$\beta(\zeta,\rho,s) \leq 1 \quad \Rightarrow \quad \gamma(M(h(\zeta),h(\rho),s)) \geq \gamma(M(\zeta,\rho,s)) + \delta, \tag{2.1}$$

for all $\zeta,\rho \in X$ satisfying $M(h(\zeta),h(\rho),s) < 1$ and $s > 0$.

Theorem 2.2. *Let $(X,M,*)$ be a NAFMS and h be a fuzzy (γ,β)-contraction. Assume that the following conditions:*

(a) *h is a triangular β-admissible mapping,*

(b) *If there exists* $\zeta_0 \in X$ *such that* $\beta(\zeta_0, h(\zeta_0), s) \leq 1$.

Then h *has a fixed point in* X*. Also, if* $\beta(\varpi_1, \varpi_2, s) \leq 1$ *for all* ϖ_1, ϖ_2 *fixed points of* h*, then* h *has a unique fixed point.*

Proof. Let $\zeta_0 \in X$ such that $\beta(\zeta_0, h(\zeta_0), s) \leq 1$. Define sequence $\{\zeta_n\}$ by

$$\zeta_n = h(\zeta_{n-1}), \quad \text{for all } n \in \mathbb{N}. \tag{2.2}$$

If $\zeta_n = \zeta_{n-1}$ then $\zeta_n = \zeta$ is a fixed point of h. Suppose that $\zeta_n \neq \zeta_{n-1}$ for all $n \in \mathbb{N}$. By using condition (a) and (b), from Lemma (1.4), we have

$$\beta(\zeta_{n-1}, \zeta_n, s) \leq 1,$$

for all $n \in \mathbb{N}$ and $s > 0$. By applying the inequality (2.1) with $\zeta = \zeta_{n-1}$ and $y = \zeta_n$ and takes to (2.2), we obtain

$$\begin{aligned} \gamma(M(h(\zeta_{n-1}), h(\zeta_n), s)) &\geq \gamma(M(\zeta_{n-1}, \zeta_n, s)) + \delta \\ &= \gamma(M(h(\zeta_{n-2}), h(\zeta_{n-1}), s)) + \delta \\ &\geq \gamma(M(\zeta_{n-2}, \zeta_{n-1}, s)) + 2\delta \\ &\;\;\vdots \\ &\geq \gamma(M(\zeta_0, \zeta_1, s)) + n\delta. \end{aligned} \tag{2.3}$$

Taking the limit as $n \to \infty$ and from (2.3), we have

$$\lim_{n\to\infty} \gamma(M(h(\zeta_{n-1}), h(\zeta_n), s)) = +\infty. \tag{2.4}$$

Then, we have

$$\lim_{n\to\infty} M(h(\zeta_{n-1}), h(\zeta_n), s) = 1. \tag{2.5}$$

To show that $\{\zeta_n\}$ is a Cauchy sequence, let us accept the opposite. So, $\{\zeta_n\}$ is not a Cauchy sequence. Then there are $\varepsilon \in (0,1)$ and $s_0 > 0$ such that for all $p \in \mathbb{N}$ there exist $n(p), m(p) \in \mathbb{N}$ with $n(p) > m(p) > p$ and

$$M(\zeta_{n(p)}, \zeta_{m(p)}, s_0) \leq 1 - \varepsilon. \tag{2.6}$$

Suppose that $m(p)$ is the least integer exceeding $n(p)$ satisfying (2.6). Then, we have

$$M(\zeta_{m(p)-1}, \zeta_{n(p)}, s_0) > 1 - \varepsilon, \tag{2.7}$$

and so, for all $p \in \mathbb{N}$, from (2.7), we get

$$\begin{aligned} 1 - \varepsilon &\geq M(\zeta_{n(p)}, \zeta_{m(p)}, s_0) \\ &\geq M(\zeta_{m(p)-1}, \zeta_{m(p)}, s_0) * M(\zeta_{m(p)-1}, \zeta_{n(p)}, s_0) \\ &\geq M(\zeta_{m(p)-1}, \zeta_{m(p)}, s_0) * (1 - \varepsilon\ ). \end{aligned} \tag{2.8}$$

Letting $p \to \infty$ in (2.8) and from (2.5), we obtain

$$\lim_{p\to\infty} M(\zeta_{n(p)}, \zeta_{m(p)}, s_0) = 1 - \varepsilon. \tag{2.9}$$

From the condition (iv) of definition of NAFMS, we get

$$\begin{aligned} M(\zeta_{m(p)+1}, \zeta_{n(p)+1}, s_0) \geq M(\zeta_{m(p)+1}, \zeta_{m(p)}, s_0) * M(\zeta_{m(p)}, \zeta_{n(p)}, s_0) \\ * M(\zeta_{n(p)}, \zeta_{n(p)+1}, s_0). \end{aligned} \tag{2.10}$$

Taking the limit as $p \to \infty$ in (2.10), we obtain

$$\lim_{p\to\infty} M(\zeta_{n(p)+1}, \zeta_{m(p)+1}, s_0) = 1 - \varepsilon\ . \tag{2.11}$$

From Lemma (1.4), we have $\beta(\zeta_{n(p)}, \zeta_{m(p)}, s) \leq 1$. By applying the inequality (2.1), we get

$$\begin{aligned} \gamma(M(\zeta_{n(p)+1}, \zeta_{m(k)+1}, s)) &= \gamma(M(h(\zeta_{n(p)}), h(\zeta_{m(p)}), s)) \\ &\geq \gamma(M(\zeta_{n(p)}, \zeta_{m(p)}, s)) + \delta. \end{aligned} \tag{2.12}$$

Taking the limit as $p \to \infty$ in (2.12), from (2.1), (2.9), (2.11) and continuity of γ, we obtain $\gamma(1-\varepsilon) \geq \gamma(1-\varepsilon) + \delta$. Thus a contradiction is obtained. Then $\{\zeta_n\}$ is a Cauchy sequence. Since $(X, M, *)$ is complete, there exists $\varpi \in X$ such that

$$\lim_{n\to\infty} \zeta_n = \varpi.$$

By using Remark (1.7), we have

$$\begin{aligned} M(h(\varpi), \varpi, s) &= \lim_{n\to\infty} M(h(\zeta_n), \zeta_n, s) \\ &= \lim_{n\to\infty} M(\zeta_{n+1}, \zeta_n, s) \\ &= 1. \end{aligned}$$

Uniqueness: Let ϖ_1 and ϖ_2 are two fixed points of h. Indeed, if for ϖ_1, $\varpi_2 \in X$, then we get

$$\gamma(M(\varpi_1, \varpi_2, s)) \geq \gamma(M(\varpi_1, \varpi_2, s)) + \delta.$$

Thus a contradiction is obtained. Therefore, h has a unique fixed point. □

Theorem 2.3. *Let $(X, M, *)$ be a NAFMS and $h : X \to X$ be a mapping. If there exist two functions $\beta : X \times X \times (0,\infty) \to (0,\infty)$ and $\gamma \in \Gamma$ such that $M(h(\zeta), h(\rho), s) < 1$ implies that*

$$\beta(\zeta, \rho, s)\gamma(M(h(\zeta), h(\rho), s)) \geq \gamma(M(\zeta, \rho, s)) + \delta, \tag{2.13}$$

where $\gamma \in \Gamma$, $\delta \in (0,1)$, $s > 0$ and for all $\zeta, \rho \in X$. Suppose that the following conditions hold:

(a) *h is a triangular β-admissible mapping,*

(b) *If there exists* $\zeta_0 \in X$ *such that* $\beta(\zeta_0, h(\zeta_0), s) \leq 1$,

Then h *has a fixed point in* X*. Also, if* $\beta(\varpi_1, \varpi_2, s) \leq 1$ *for all* ϖ_1, ϖ_2 *fixed points of* h*, then* h *has a unique fixed point.*

Proof. Let $\zeta_0 \in X$ such that $\beta(\zeta_0, h(\zeta_0), s) \leq 1$. Define sequence $\{\zeta_n\}$ by

$$\zeta_n = h(\zeta_{n-1}), \quad \text{for all } n \in \mathbb{N}. \tag{2.14}$$

If $\zeta_n = \zeta_{n-1}$ then $\zeta_n = \zeta$ is a fixed point of h. Suppose that $\zeta_n \neq \zeta_{n-1}$ for all $n \in \mathbb{N}$. By using condition (a) and (b), from Lemma (1.4), we have

$$\beta(\zeta_{n-1}, \zeta_n, s) \leq 1,$$

for all $n \in \mathbb{N}$ and $s > 0$. By applying the inequality (2.13) with $\zeta = \zeta_{n-1}$ and $y = \zeta_n$ and takes to (2.14), we obtain

(2.15)

$$\begin{aligned}
\gamma(M(\zeta_{n-1}, \zeta_{n+1}, s)) &= \gamma(M(h(\zeta_{n-1}), h(\zeta_n), s)) \\
&\geq \beta(\zeta_{n-1}, \zeta_n, s)\gamma(M(h(\zeta_{n-1}), h(\zeta_n), s)) \\
&\geq \gamma(M(\zeta_{n-1}, \zeta_n, s)) + \delta \\
&= \gamma(M(h(\zeta_{n-2}), h(\zeta_{n-1}), s)) + \delta \\
&\geq \beta(\zeta_{n-2}, \zeta_{n-1}, s)\gamma(M(h(\zeta_{n-2}), h(\zeta_{n-1}), s)) + \delta \\
&\geq \gamma(M(\zeta_{n-2}, \zeta_{n-1}, s)) + 2\delta \\
&\ \ \vdots \\
&\geq \gamma(M(\zeta_0, \zeta_1, s)) + n\delta.
\end{aligned}$$

Taking the limit as $n \to \infty$, from (2.15) we have

$$\lim_{n\to\infty} \gamma(M(h(\zeta_{n-1}), h(\zeta_n), s)) = +\infty.$$

The proof of this theorem can obtained by using similar arguments as given in the proof of Theorem (2.2). □

Definition 2.4. Let $h : X \to X$ and $\beta : X \times X \times (0, \infty) \to (0, \infty)$ be two mappings. Then, h is said to be a fuzzy (γ, β)-contraction if there exist $\gamma \in \Gamma$ and $\delta \in (0, 1)$ such that $\beta(\zeta, \rho, s) \leq 1$ then

(2.16)

$$\gamma(M(h(\zeta), h(\rho), s)) \geq \gamma(\min\{M(\zeta, \rho, s), M(\zeta, h(\zeta), s), M(\rho, h(\rho), s)\}) + \delta$$

for all $\zeta, \rho \in X$ satisfying $M(h(\zeta), h(\rho), s) < 1$ and $s > 0$.

Example 2.5. Let $X = [0, 1]$ and M be the complete non-Archimedean fuzzy metric given by $M(\zeta, \rho, s) = \left(\frac{s}{s+1}\right)^{d(\zeta,\rho)}$ with $a * b = ab$ for all

$s > 0$ where d is a standart metric. Let $h : [0,1] \to [0,1]$ be defined by

$$h(\zeta) = \begin{cases} \frac{1}{2}, & \zeta \in [0,1), \\ \frac{1}{6}, & \zeta = 1, \end{cases}$$

and suppose that $\beta : X \times X \times [0,+\infty) \to \mathbb{R}^+$ is given by

$$\beta(\zeta,\rho,s) = \begin{cases} 1, & \zeta,\rho \in [0,1), \\ \frac{1}{8}, & \text{otherwise.} \end{cases}$$

Since h is not continuous, h is not fuzzy (γ,β)-contraction by Remark (1.7). If $\zeta \in [0,1)$ and $\rho = 1$, then $\beta(\zeta,\rho,s) \leq 1$ and also we have

$$\begin{aligned} M(h(\zeta),h(\rho),s) &= M\left(\frac{1}{2},\frac{1}{6},s\right) \\ &= \left(\frac{s}{s+1}\right)^{\left|\frac{1}{2}-\frac{1}{6}\right|} \\ &< 1. \end{aligned}$$

Then, we get

$$\begin{aligned} \min\left\{M(\zeta,1,s), M\left(\zeta,h(\zeta),s\right), M(1,h(1),s)\right\} &\leq \left(\frac{s}{s+1}\right)^{\left|1-\frac{1}{6}\right|} \\ &= \left(\frac{s}{s+1}\right)^{\frac{5}{6}}, \end{aligned}$$

and

$$M(h(\zeta),h(\rho),s) > \min\{M(\zeta,1,s), M(\zeta,h(\zeta),s), M(1,h(1),s)\}.$$

Consider a mapping $\gamma : [0,1) \to \mathbb{R}$ defined by $\gamma = \frac{1}{\sqrt{1-\zeta}}$ for all $\zeta \in [0,1)$. We see that $\gamma \in \Gamma$ since γ is a strictly increasing mapping and from the above inequality, we can write

$$\gamma(M(h(\zeta),h(\rho),s)) > \gamma\left(\min\{M(\zeta,\rho,s), M(\zeta,h(\zeta),s), M(\rho,h(\rho),s)\}\right)$$

so, there exist a $\delta \in (0,1)$ such that

$$\gamma(M(h(\zeta),h(\rho),s)) \geq \gamma(\min\{M(\zeta,\rho,s), M(\zeta,h(\zeta),s), M(\rho,h(\rho),s)\})+\delta$$

for all $\zeta,\rho \in X$ and $s > 0$. Therefore, h is a fuzzy (γ,β)-weak contraction.

Theorem 2.6. *Let $(X,M,*)$ be a NAFMS and h be a fuzzy (γ,β)-weak contraction. Suppose that the following conditions hold:*

(a) *h is a triangular β-admissible mapping,*
(b) *there exists $\zeta_0 \in X$ such that $\beta(\zeta_0,h(\zeta_0),s) \leq 1$,*

Then h has a fixed point in X. Also, if $\beta(\varpi_1,\varpi_2,s) \leq 1$ for all ϖ_1,ϖ_2 fixed points of h, then h has a unique fixed point.

Proof. Let $\zeta_0 \in X$ such that $\beta(\zeta_0, h\zeta_0, s) \leq 1$. Define sequence $\{\zeta_n\}$ by

$$\zeta_n = h(\zeta_{n-1}), \quad \text{for all } n \in \mathbb{N}. \tag{2.17}$$

If $\zeta_n = \zeta_{n-1}$ then $\zeta_n = \zeta$ is a fixed point of h. Assume that $\zeta_n \neq \zeta_{n-1}$ for all $n \in \mathbb{N}$. By using condition (a) and (b), from Lemma (1.4), we have

$$\beta(\zeta_{n-1}, \zeta_n, s) \leq 1,$$

for all $n \in \mathbb{N}$ and $s > 0$. By applying the inequality (2.16) with $\zeta = \zeta_{n-1}$ and $y = \zeta_n$ and takes to (2.17), we obtain,

$$\begin{aligned} \gamma(M(h(\zeta_{n-1}), h(\zeta_n), s)) &\geq \gamma(\min\{M(\zeta_{n-1}, \zeta_n, s), \\ &\quad M(\zeta_{n-1}, h(\zeta_{n-1}), s)), M(\zeta_n, h(\zeta_n), s)\}) + \delta \\ &= \gamma(\min\{M(\zeta_{n-1}, \zeta_n, s), \\ &\quad M(\zeta_{n-1}, \zeta_n, s), M(\zeta_n, \zeta_{n+1}, s)\}) + \delta \\ &= \gamma(\min\{M(\zeta_{n-1}, \zeta_n, s), M(\zeta_n, \zeta_{n+1}, s)\}) + \delta. \end{aligned} \tag{2.18}$$

In this case, let's assume that $\min\{M(\zeta_{n-1}, \zeta_n, s), M(\zeta_n, \zeta_{n+1}, s)\} = M(\zeta_n, \zeta_{n+1}, s)$. From (2.18) becomes

$$\begin{aligned} \gamma(M(h(\zeta_{n-1}), h(\zeta_n), s)) &= M(\zeta_n, \zeta_{n+1}, s) \\ &\geq \gamma(M(\zeta_n, \zeta_{n+1}, s)) + \delta \\ &> \gamma(M(\zeta_n, \zeta_{n+1}, s)), \end{aligned}$$

which is a contradiction. Therefore,

$$\min\{M(\zeta_{n-1}, \zeta_n, s), M(\zeta_n, \zeta_{n+1}, s)\} = M(\zeta_{n-1}, \zeta_n, s) \tag{2.19}$$

for all $n \in \mathbb{N}$. Since γ is strictly increasing, from (2.18) and (2.19), we have

$$M(\zeta_n, \zeta_{n+1}, s) > M(\zeta_{n-1}, \zeta_n, s).$$

Thus, from (2.18), we have

$$\gamma(M(\zeta_n, \zeta_{n+1}, s)) \geq \gamma(M(\zeta_{n-1}, \zeta_n, s)) + \delta,$$

for all $n \in \mathbb{N}$. It implies that

$$\gamma(M(\zeta_n, \zeta_{n+1}, s)) \geq \gamma(M(\zeta_{n-1}, \zeta_n, s)) + n\delta. \tag{2.20}$$

Letting $n \to \infty$ in (2.20), we obtain

$$\begin{aligned} \lim_{n\to\infty} \gamma(M(\zeta_n, \zeta_{n+1}, s)) &= \lim_{n\to\infty} \gamma(M(h(\zeta_{n-1}), h(\zeta_n), s)) \\ &= \infty, \end{aligned}$$

and then, we get

$$\lim_{n\to\infty} M(h(\zeta_{n-1}), h(\zeta_n), s) = 1. \tag{2.21}$$

The proof can be continued as in Theorem 2.2. Since $(X, M, *)$ is complete, there exists $\varpi \in X$ such that

$$\lim_{n\to\infty} \zeta_n = \varpi.$$

Now, we show that ϖ is a fixed point of h. Since γ is continuous, so we should examine both the following cases:

Case 1: For each $n \in \mathbb{N}$, there exists $i_n \in \mathbb{N}$ such that $\zeta_{i_n+1} = h(\varpi)$ and $i_n > i_{n-1}$ where $i_0 = 1$. Then, we get

$$\begin{aligned} \varpi &= \lim_{n\to\infty} \zeta_{i_n+1} \\ &= \lim_{n\to\infty} h\varpi \\ &= h\varpi. \end{aligned}$$

This proves that ϖ is a fixed point of h.

Case 2: There exists $n_0 \in \mathbb{N}$ such that $\zeta_{n+1} \neq h\varpi$ for all $n \geq n_0$. That is $M(h(\zeta_n), h(\varpi), s) < 1$ for all $n \geq n_0$. It follows from (2.16), property of γ, we have

(2.22)
$$\begin{aligned} \gamma(M(\zeta_{n+1}, h(\varpi), s)) &= \gamma(M(h(\zeta_n), h(\varpi), s)) \\ &\geq \gamma(\min\{M(\zeta_n, \varpi, s), M(\zeta_n, h(\zeta_n), s), M(\varpi, h(\varpi), s)\}) + \delta \\ &= \gamma(\min\{M(\zeta_n, \varpi, s), M(\zeta_n, \zeta_{n+1}, s), M(\varpi, h(\varpi), s)\}) + \delta. \end{aligned}$$

If $M(\varpi, h(\varpi), s) < 1$, then by the fact

$$\lim_{n\to\infty} M(\zeta_n, \varpi, s) = 1,$$

there exists $n_1 \in \mathbb{N}$ such that for all $n \geq n_1$, we get

$$\min\{M(\zeta_n, \varpi, s), M(\zeta_n, \zeta_{n+1}, s), M(\varpi, h(\varpi), s)\} = M(\varpi, h(\varpi), s).$$

From (2.22), we have

$$\gamma(M(\zeta_{n+1}, h(\varpi), s)) \geq \gamma(M(\varpi, h(\varpi), s)) + \delta, \tag{2.23}$$

for all $n \geq \max\{n_0, n_1\}$. Since γ is continuous, letting $n \to \infty$ in (2.23), we obtain

$$\gamma(M(\varpi, h(\varpi), s)) \geq \gamma(M(\varpi, h(\varpi), s)) + \delta.$$

Thus a contradiction is obtained. Therefore, $M(\varpi, h(\varpi), s) = 1$, that is ϖ is a fixed point of h.

Uniqueness: Let ϖ_1, ϖ_2 be two fixed points of h. Suppose that $\varpi_1 \neq \varpi_2$. Then $h(\varpi_1) \neq h(\varpi_2)$. It follows from (2.13), we have

$$\begin{aligned} \gamma(M(\varpi_1, \varpi_2, s)) &= \gamma(M(h(\varpi_1), h(\varpi_2), s)) \\ &\geq \gamma(\min\{M(\varpi_1, \varpi_2, s), M(\varpi_1, h(\varpi_1), s), M(\varpi_2, h(\varpi_2), s)\}) + \delta \\ &= \gamma(\min\{M(\varpi_1, \varpi_2, s), M(\varpi_1, \varpi_1, s), M(\varpi_2, \varpi_2, s)\}) + \delta \\ &= \gamma(M(\varpi_1, \varpi_2, s)) + \delta. \end{aligned}$$

Thus a contradiction is obtained. Then, $M(\varpi_1, \varpi_2, s) = 1$, that is $\varpi_1 = \varpi_2$. □

Theorem 2.7. *Let $(X, M, *)$ be a NAFMS and $h : X \to X$ be a mapping. If there exist two functions $\beta : X \times X \times (0, \infty) \to (0, \infty)$ and $\gamma \in \Gamma$ such that $M(h(\zeta), h(\rho), s) < 1$ implies that*

$$\beta(\zeta, \rho, t)\gamma(M(h(\zeta), h(\rho), s)) \geq \gamma(\min\{M(\zeta, \rho, s), M(\zeta, h(\zeta), s), M(\rho, h(\rho), s)\}) + \delta. \tag{2.24}$$

where $\gamma \in \Gamma$, $\delta \in (0, 1)$, $s > 0$ and for all $\zeta, \rho \in X$. Suppose that the following conditions hold:

(a) *h is a triangular β-admissible mapping,*
(b) *there exists $\zeta_0 \in X$ such that $\beta(\zeta_0, h(\zeta_0), s) \leq 1$,*
(c) *if $\{\zeta_n\} \in X$ such that $\beta(\zeta_n, \zeta_{n+1}, s) \leq 1$ for all $n \in \mathbb{N}$ and $\zeta_n \to \varpi$ as $n \to \infty$, then $\beta(\zeta_n, \varpi, s) \leq 1$.*

Then h has a fixed point in X. Also, if $\beta(\varpi_1, \varpi_2, s) \leq 1$ for all ϖ_1, ϖ_2 fixed points of h, then h has a unique fixed point.

Proof. Let $\zeta_0 \in X$ such that $\beta(\zeta_0, h\zeta_0, s) \leq 1$. Define sequence $\{\zeta_n\}$ by

$$\zeta_n = h(\zeta_{n-1}), \quad \text{for all } n \in \mathbb{N}.$$

Following the proof of Theorem 2.6, we see that $\{\zeta_n\}$ is a Cauchy sequence. Since $(X, M, *)$ is complete, there exists $\varpi \in X$ such that

$$\lim_{n\to\infty} \zeta_n = \varpi.$$

By using Lemma (1.4) with the condition (c), we get

$$\beta(\zeta_n, \varpi, s) \leq 1,$$

for all $n \in \mathbb{N}$ and $s > 0$. Now, we show that ϖ is a fixed point of h. Since γ is continuous, so we should examine both the following cases:

Case 1: For each $n \in \mathbb{N}$, there exists $i_n \in \mathbb{N}$ such that $\zeta_{i_n+1} = h(\varpi)$ and $i_n > i_{n-1}$ where $i_0 = 1$. Then, we get

$$\begin{aligned} \varpi &= \lim_{n\to\infty} \zeta_{i_n+1} \\ &= \lim_{n\to\infty} h\varpi \\ &= h\varpi. \end{aligned}$$

This proves that ϖ is a fixed point of h.

Case 2: There exists $n_0 \in \mathbb{N}$ such that $\zeta_{n+1} \neq h\varpi$ for all $n \geq n_0$. That is $M(h(\zeta_n), h(\varpi), s) < 1$ for all $n \geq n_0$. It follows from (2.24), property of γ, we have

$$\begin{aligned} \gamma(M(\zeta_{n+1}, h(\varpi), s)) &= \gamma(M(h(\zeta_n), h(\varpi), s)) \\ &\geq \beta(\zeta_n, \varpi, s)\gamma(M(h(\zeta_n), h(\varpi), s)) \\ &\geq \gamma(\min\{M(\zeta_n, \varpi, s), M(\zeta_n, h(\zeta_n), s), M(\varpi, h(\varpi), s)\}) + \delta \end{aligned} \tag{2.25}$$

$$= \gamma(\min\{M(\zeta_n, \varpi, s), M(\zeta_n, \zeta_{n+1}, s), M(\varpi, h(\varpi), s)\}) + \delta.$$

If $M(\varpi, h(\varpi), s) < 1$, then by the fact

$$\lim_{n\to\infty} M(\zeta_n, \varpi, s) = 1,$$

there exists $n_1 \in \mathbb{N}$ such that for all $n \geq n_1$, we get

$$\min\{M(\zeta_n, \varpi, s), M(\zeta_n, \zeta_{n+1}, s), M(\varpi, h(\varpi), s)\} = M(\varpi, h(\varpi), s).$$

From (2.25), we have

$$\gamma(M(\zeta_{n+1}, h(\varpi), s)) \geq \gamma(M(\varpi, h(\varpi), s)) + \delta, \tag{2.26}$$

for all $n \geq \max\{n_0, n_1\}$. Since γ is continuous, letting $n \to \infty$ in (2.26), we obtain

$$\gamma(M(\varpi, h(\varpi), s)) \geq \gamma(M(\varpi, h(\varpi), s)) + \delta.$$

Thus a contradiction is obtained. Therefore, $M(\varpi, h(\varpi), s) = 1$, that is ϖ is a fixed point of h. The uniqueness of the fixed point h is done by the proof of Theorem 2.6. □

Example 2.8. Let $X = [0, \frac{1}{2}]$, $a * b = \min\{a, b\}$. Let the complete NAFMS be $M(\zeta, \rho, s) = \frac{\min\{\zeta,\rho\}}{\max\{\zeta,\rho\}}$ for all $s > 0$. Let $\gamma : [0, 1) \to \mathbb{R}$ be defined by $\gamma = \frac{1}{1-\zeta}$ for all $\zeta \in [0, 1)$ and $h : X \to X$ be defined by $h(\zeta) = \sqrt{\zeta}$ for all $\zeta \in X$. Assume that $\beta : X \times X \times (0, \infty) \to \mathbb{R}^+$ is given by

$$\beta(\zeta, \rho, s) = \begin{cases} e^{-(\zeta+\rho)}, & \zeta, \rho \in [0, \frac{1}{2}), \\ 1, & \text{otherwise}. \end{cases}$$

h is a triangular (β)-admissible mapping. Indeed, if $\beta(\zeta, \rho, s) \leq 1$, then $e^{-(\zeta+\rho)} \leq 1$. Then $\zeta, \rho \in [0, \frac{1}{2})$. Thus $\beta(h(\zeta), h(\rho), s) = e^{-(h(\zeta)+h(\rho))} \leq 1$. Now assume that $\beta(\zeta, \xi, s) \leq 1$ and $\beta(\xi, \rho, s) \leq 1$, so $\zeta, \xi \in [0, \frac{1}{2})$ and $\xi, \rho \in [0, \frac{1}{2})$. Then, $\zeta, \rho \in [0, \frac{1}{2})$ and so $\beta(\zeta, \rho, s) \leq 1$. Hence, h is a triangular β-admissible mapping.

Let $M(h(\zeta), h(\rho), s) < 1$.

Case 1: If ζ or ρ is equal to $\frac{1}{2}$, then $\beta(\zeta, \rho, s) = 1$ and hence we get

$$\begin{aligned} \gamma(M(h(\zeta), h(\rho), s)) &= \gamma\left(\sqrt{\zeta}\right) \\ &\geq \gamma(\zeta) + \delta \\ &= \gamma(M(\zeta, \rho, s)) + \delta, \end{aligned}$$

such that there exists a $\delta \in (0, 1)$.

Case 2: If both ζ and ρ are in $[0, \frac{1}{2})$, then we have $\beta(\zeta, \rho, s) \leq 1$. Hence for $\zeta < \rho$ (or $\zeta > \rho$), we get

$$\gamma(M(h(\zeta), h(\rho), s)) = \gamma\left(\frac{\sqrt{\zeta}}{\sqrt{\rho}}\right)$$

$$\geq \gamma\left(\frac{\zeta}{\rho}\right)+\delta$$
$$=\gamma(M(\zeta,\rho,s))+\delta,$$

such that there exists a $\delta \in (0,1)$. So, (2.1) is provided for both cases above. Hence h is a fuzzy (γ,β) contractive mapping. Further, there exists $\zeta_0 \in X$ such that $\beta(\zeta_0, h(\zeta_0), s) \leq 1$. Indeed for $\zeta_0 = 0$, we have $\beta(0, h(0), s) = 1 \leq 1$. Thus, all the required hypotheses of Theorem 2.2 are satisfied and h has a unique fixed point $\varpi = 0$.

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Department of Mathematics, Faculty of Science, University of Gazi, 06500, Ankara, Turkey.
E-mail address: muzeyyensangurlu@gazi.edu.tr