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$\mathcal{I}$-convergence in Fuzzy Cone Normed Spaces

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Abstract. The aim of this paper is to define and study the concept of $\mathcal{I}$-convergence in fuzzy cone normed spaces which is a generalization of R. Saadati and S. M. Vaezpour type fuzzy normal space. We also obtained some basic properties of $\mathcal{I}$-convergence. In fuzzy cone normed space, $\mathcal{I}$-limit point and $\mathcal{I}$-cluster point were defined and studied.

1. Introduction

As an alternative to classical convergence, many types of convergence have been described to date and and these concepts were applied to various branches of mathematics. Among these concepts of convergence, statistical convergence is the most common field of application. This concept was first defined independently from each other by Fast [5] and Steinhaus [22] in 1951. Statistical convergence is related to natural density of positive integer sets. The family of integer sets , which have natural density of zero, forms an ideal. In this manner, the concept of ideal convergence is defined as a more general convergence type that includes statistical convergence using defined ideals on natural numbers. $\mathcal{I}$-convergence was introduced for sequences in classical metric spaces by Kostyrko [13] for the first time in 2000. Moreover,the concept of $\mathcal{I}$-limit points and $\mathcal{I}$-accumulation points, which is a generalization of statistical limit and statistical accumulation points of a sequences, were defined by Kostyrko [13]. Then they investigated basic properties of those concepts. In 2005, different topological properties of $\mathcal{I}$-accumulation points of a sequence was studied by Kostyrko et al. [14]. Also the concept of $\mathcal{I}$ -convergence of double sequences was introduced in a metric space and

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some properties of this convergence was studied by Das et al. [4]. This subject has been studied by various authors in different spaces ([10],[2]) until today.

The theory of fuzzy sets was initiated by Zadeh [26]. After the introduction of the notion of fuzzy set, many researcher have improved to utilize this concept in topology and analysis. Using fuzzy set with norm processors, the concept of fuzzy norm in fuzzy topological vector spaces was defined by Katsaras [11] and by this definition fuzzy set theory has become an important application area in terms of functions. Then, in 1992 Felbin [6] defined the concept of fuzzy normed space. Later, Cheng and Mordeson [3] (1994), Xiao and Zhu [25] (2002) and also Bag and Samanta [1] (2003) gave another fuzzy norm on a vector space related with the fuzzy metric type of Kramosil and Michalek [12]. Then in 2005 R. Saadati and S. M. Vaezpour [20] also introduced a fuzzy norm on a linear space whose associated fuzzy metric is George and Veeramani type in [8]. Moreover many researches studied given various definitions of fuzzy norms. Until now there are so many researches on this subject as convergence, continuity, summability methods, bounded sets... etc.

The concept of ideal convergence in fuzzy normed spaces was first studied by Kumar and Kumar [15]. Then, $\mathcal{I}$-convergence has been investigated in more general abstract spaces such as the fuzzy number spaces [16], 2-normed linear spaces [9], linear n-normed spaces [23] and intuitionistic fuzzy normed spaces [17].

In this paper, we aimed to generalize the notion of fuzzy norm by taking ordered Banach space instead of positive real numbers in the definition of fuzzy norm which is defined by R. Saadati. So we gave the definition of fuzzy cone normed space. Then we studied basic properties in fuzzy cone normed spaces. Also we introduced the notion of $\mathcal{I}$-convergence and $\mathcal{I}^*$-convergence in fuzzy cone normed space and obtained some properties related to these notion in this space. We also studied the concepts of $\mathcal{I}_{N_C}$-limit points and $\mathcal{I}_{N_C}$-cluster points of a sequences in fuzzy cone normed space.

2. Preliminaries

Definition 2.1 ([18])**.** Let X be a non empty set. Then a family of sets $\mathcal{I} \subset P(X)$ is said to be an ideal in X if

(a) $A, B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$,
(b) $A \in \mathcal{I}$ and $B \subset A$ imply $B \in \mathcal{I}$.

The ideal $\mathcal{I}$ is called nontrivial if $\mathcal{I} \neq \{\emptyset\}$ and $X \notin \mathcal{I}$. A nontrivial ideal $\mathcal{I}$ is called admisible if $\mathcal{I} \supset \{\{x\} : x \in X\}$

Definition 2.2 ([18])**.** Let X be a non empty set. Then a family of sets $\mathcal{F} \subset P(X)$ is said to be a filter in X if

(a) $\emptyset \notin \mathcal{F}$
(b) $A, B \in \mathcal{F}$ implies $A \cap B \in \mathcal{F}$,
(c) $B \in \mathcal{F}$ and $B \subset A$ imply $B \in \mathcal{F}$.

Let $\mathcal{I}$ be nontrivial ideal of nonempty set X. Then

$$\mathcal{F}(\mathcal{I}) = \{A \subset X : X - A \in \mathcal{I}\}$$

is a filter on X, called the filter associated with the ideal $\mathcal{I}$.

Lemma 2.3 ([13])**.** *Let $\{F_i : i \in \mathbf{N}\}$ be a countable collection of subsets $\mathbf{N}$ such that $\{F_i\} \in \mathcal{F}(\mathcal{I})$ for each i where $\mathcal{I}$ is admissible ideal with the property (AP). Then there exists a set $F \subset \mathbf{N}$ such that $F \in \mathcal{F}(\mathcal{I})$ and the set $F \setminus F_i$ is finite for all i.*

Definition 2.4 ([21])**.** A binary operation $* : [0,1] \times [0,1] \to [0,1]$ is a continuous t-norm if $*$ satisfies the following conditions;

(a) $*$ is associative and commutative,
(b) $*$ is continuous,
(c) $a * 1 = a$ for all $a \in [0,1]$,
(d) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d, a, b, c, d \in [0,1]$.

The basic two continuous t-norms are (see [21]) the followings

(i) the minimum t-norm defined by $a * b = \min\{a, b\}$.
(ii) the product t-norm defined by $a * b = ab$

Definition 2.5 ([20])**.** A three-tuple $(X, \mathcal{N}, *)$ is said to be fuzzy normed space if X is arbitrary nonempty set, $*$ is a continuous t-norm, and $\mathcal{N}$ is fuzzy set on $X\times$ $(0, \infty)$ satisfying the following conditions for each $x, y \in X$ and $t, s > 0$,

(FN1) $\mathcal{N}(x, t) > 0$,
(FN2) $\mathcal{N}(x, t) = 1 \Leftrightarrow x = 0$,
(FN3) $\mathcal{N}(\alpha x, t) = \mathcal{N}(x, \frac{t}{|\alpha|})$ where $\alpha \neq 0$,
(FN4) $\mathcal{N}(x, t) * \mathcal{N}(y, s) \leq \mathcal{N}(x + y, t + s)$,
(FN5) $\lim_{t\to\infty} \mathcal{N}_C(x, t) = 1$.

Throughout this paper, E denotes a real Banach space and θ denotes the zero of E.

Definition 2.6 ([7])**.** A subset P of E is called a cone if

(a) P is closed, non-empty and $P \neq \{\theta\}$,
(b) If $a, b \in R$, $a, b \geq 0$ and $x, y \in P$ then $ax + by \in P$,
(c) If both $x \in P$ and $-x \in P$ then $x = \theta$.

For a given cone, a partial ordering $\preceq$ on E via P is defined by $x \preceq$ y if and only if $y - x \in P$. $x \prec y$ will stand for $x \preceq y$ and $x \neq y$ while $x \ll y$ will stand for $y - x \in int(P)$. Throughout the paper, we assume that all cones has nonempty interior.

Definition 2.7. A three-tuple $(X, M, *)$ is said to be fuzzy cone metric space if P is cone of E. X is arbitrary nonempty set, $*$ is a continuous t-norm, and M is fuzzy set on $X^2 \times P$ satisfying the following conditions for each $x, y, z, a \in X$ and $t, s \in int(P)$,

(FCM1) $M(x, y, t) > 0$ if $x \neq y$,
(FCM2) $M(x, y, t) = 1 \Leftrightarrow x = y$,
(GFM3) $M(x, y, t) = M(y, x, t)$,
(GFM4) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$ for all $x, y, z \in X$,
(GFM5) $M(x, y, .) : int(P) \to [0, 1]$ is continuous.

For $t \gg \theta$, the open ball $B(x, r, t)$ with center x, radius $r \in (0, 1)$ is defined by $B(x, r, t) = \{y \in X : M(x, y, t) > 1 - r\}$.

Definition 2.8 ([19])**.** Let $(X, M, *)$ be a fuzzy cone metric space and (x_n) be a sequence in X. Then the sequences (x_n) is said to be convergent at $x \in X$ if for any $t \gg \theta$ and any $0 < r < 1$, there exists a natural number N such that $G(x_n, x, t) > 1 - r$ whenever $n \geq N$ i.e $n \geq N \Rightarrow x_n \in B_G(x, r, t)$. We denote this by $x_n \to x$ as $n \to \infty$ or by $\lim_{n\to\infty} x_n = x$.

3. Fuzzy Cone Normed Spaces

In this section, we give the definition of fuzzy cone normed space which is a generalization of fuzzy normed space by R. Saadati and S. M. Vaezpour [20]. Also we give the relation between the concepts of fuzzy cone normed space and fuzzy cone metric space.

Definition 3.1 ([24])**.** A three-tuble $(X, N_C, *)$ is said to be fuzzy cone normed space (FCNS in short), if X is a vector space, $\star$ is continuous t-norm, P is a cone of E and N_C is fuzzy set on $X \times int(P)$ satisfying the following conditions for every $x, y \in X$ and $t, s \in int(P)$;

(FCN1) $N_C(x, t) > 0$,
(FCN2) $N_C(x, t) = 1 \Leftrightarrow x = 0$,
(FCN3) $N_C(\alpha x, t) = N_C(x, \frac{t}{|\alpha|})$ for all $\alpha \neq 0$,
(FCN4) $N_C(x, t) * N_C(y, s) \leq N_C(x + y, t + s)$,
(FCN5) $N_C(x, .) : int(P) \to [0, 1]$ is continuous,
(FCN6) $\lim_{\|t\|\to\infty} N_C(x, t) = 1$.

Remark 3.2. If we take $E = \mathbf{R}$, $P = [0, \infty)$, $a * b = ab$ then every N-fuzzy normed spaces become a N_C-fuzzy cone normed spaces.

Lemma 3.3. *Let $(X, N_C, *)$ be a FCNS. Then $N_C(x, .) : int(P) \to [0, 1]$ is nondecreasing for all $x \in X$.*

Proof. Let $s, t \in int(P)$ and $t \ll s$. By (FCN4), we have $N_C(x, t) = N_C(x, t) * N_C(0, s - t) \leq N_C(x, s)$. We obtain $N_C(x, t) \leq N_C(x, s)$. □

Lemma 3.4. *Let $(X, N_C, *)$ be a FCNS. If we define*

$$M(x,y,t) = N_C(x-y,t)$$

then M is a fuzzy cone metric space which is called fuzzy cone metric induced by the fuzzy cone norm N_C.

Proof. (FCM 1-2-5) are obvious.
(FCM 3) $M(x,y,t) = N_C(x-y,t) = N_C(-(y-x),t) = N_C\left(y-x, \frac{t}{|-1|}\right) = M(y,x,t)$ by (FCN3).
(FCM 5) By (FCN5), we have $M(x,y,t) * M(y,z,s) = N_C(x-y,t) * N_C(y-z,s) \leq N_C(x-z,t) = M(x,z,t+s)$ □

Example 3.5. Let E=**R**. Then $P = [0,\infty) \subset E$ is a normal cone. Let $X = \mathbf{R}$, $a * b = ab$ for all $a, b \in [0,1]$ and $N_C : X \times Int(P) \to [0,1]$ defined by

$$N_C(x,t) = e^{\frac{-\sqrt{|x|}}{\sqrt{t}}}$$

for every $x \in X$ and $t \in int(P)$. Then $(X, N_C, *)$ is a fuzzy cone normed space.

(FCN1-2) are obvious.

(FCN3) For all $\alpha \neq 0$, $N_C(\alpha x, t) = e^{\frac{-\sqrt{|\alpha x|}}{\sqrt{t}}} = e^{\frac{-\sqrt{|x|}}{(\sqrt{t/\alpha})}} = N_C\left(x, \frac{t}{|\alpha|}\right)$.

(FCN4) For all $t, s \in int(P)$

$$\frac{\sqrt{x+y}}{\sqrt{t+s}} \leq \frac{\sqrt{x}}{\sqrt{t+s}} + \frac{\sqrt{y}}{\sqrt{t+s}}$$
$$\frac{\sqrt{x+y}}{\sqrt{t+s}} \leq \frac{\sqrt{x}}{\sqrt{t}} + \frac{\sqrt{y}}{\sqrt{s}}$$
$$e^{\frac{-\sqrt{|x+y|}}{\sqrt{t+s}}} \geq e^{\frac{-\sqrt{|x|}}{\sqrt{t}}} e^{\frac{-\sqrt{|y|}}{\sqrt{s}}}$$

Therefore

$$N_C(x,t) * N_C(y,s) \leq N_C(x+y, t+s).$$

Example 3.6. Suppose that $(X, \|\|)$ is a normed space, E is a Banach space and $P \subset E$ is a cone. Let $a * b = ab$ for all $a, b \in [0,1]$. For all $x \in X$ and every $t \in int(P)$, we take

$$N_C(x,t) = \frac{t}{t + \|x\|}.$$

Then $(X, N_C, *)$ is a fuzzy cone normed space.

Definition 3.7. Let $(X, N_C, *)$ be a FCNS. For any $t \gg \theta$, $a \in X$ and $r \in (0,1)$,

$$B_{N_C}(a,r,t) = \{x \in X : N_C(x-a,t) > 1-r\}$$

is called open ball with center a and radius r with respect to t. A subset $G \subset X$ is called open if for each $a \in G$, there exist $t \gg \theta$ and $r \in (0,1)$ such that $B_{N_C}(a,r,t) \in G$. Let τ_{N_C} denote the family of all open subsets of X. τ_{N_C} is called the topology induced by the fuzzy cone norm. Note that this topology is same as the topology induced by the fuzzy cone metric which is Hausdorff (see, Theorem 2.6 and Theorem 2.7 of [19]).

Definition 3.8. Let $(X, N_C, *)$ be a FCNS. A sequences x=(x_n) of elements of X is said to be convergent to $a \in X$ with respect to fuzzy cone norm N_C if for each $\varepsilon > 0$ and $t \in int(P)$ there exists a positive integer n_0 such that $N_C(x_n - a, t) > 1 - \varepsilon$, $x_n \in B_G(a, \varepsilon, t)$ for each n> n_0. We denote this by $x_n \to$ a as $n \to \infty$ or by $N_C - \lim\limits_{n\to\infty} x_n = a$.

Definition 3.9. Let $\mathcal{I}$ be an ideal on $\mathbf{N}$ and $(X, N_C, *)$ be a FCNS. A sequences $x = (x_n)$ in X is said to be $\mathcal{I}$-convergent to $a \in X$ with respect to fuzzy cone norm on X if for each $\varepsilon > 0$ and $t \in int(P)$, the set

$$A(\varepsilon) = \{n \in \mathbf{N} : N_C(x_n - a, t) \leq 1 - \varepsilon\} \in \mathcal{I}$$

and we write $\mathcal{I}_{N_C} - \lim\limits_{n\to\infty} x_n = a$. It is obvious that $\mathcal{I}_{N_C} - \lim\limits_{n\to\infty} x_n = a$ iff for each $\varepsilon > 0$ and $t \in int(P)$, the set $A^c(\varepsilon) = \{n \in \mathbf{N} : N_C(x_n - a, t) > 1 - \varepsilon\} \in \mathcal{F}(I)$.

Example 3.10.
1) If $\mathcal{I}$ is an admissible ideal, then ordinary convergence implies $\mathcal{I}$-convergence.
2) If $\mathcal{I}$ is an admissible ideal which doesn't contain any infinite set, the $\mathcal{I}$-convergence coincides with the usual convergence with respect to fuzzy cone norm.

Example 3.11. Let E=$\mathbf{R}$. Then P=$[0, \infty) \subset E$ is a normal cone. Let $X = \mathbf{R}^2$, $a * b = ab$ and $N_C : X \times int(P) \to [0,1]$ defined by $N_C(x,t) = \frac{t}{t + \|x\|}$ for all $x \in X$ and $t \in int(P)$. Let $\mathcal{I} = \mathcal{P}(A)$ such that $A = \{2, 4, 6, \ldots, 2n, \ldots\}$. Define a sequence (x_n) in X

$$x_n = \begin{cases} (n,1), & n = 2k, \\ (1,1), & n = 2k+1, \end{cases}$$

Then (x_n) is $\mathcal{I}_{N_C}$-convergence to $(1,1)$.

Lemma 3.12. *Let (x_n) be a sequence in fuzzy cone normed space $(X, N_C, *)$.*

(i) *If $\mathcal{I}_{N_C} - \lim\limits_{n\to\infty} x_n = a$ and $\mathcal{I}_{N_C} - \lim\limits_{n\to\infty} x_n = b$ then $a = b$.*

(ii) *$\mathcal{I}_{N_C} - \lim\limits_{n\to\infty} x_n = a$ iff $\mathcal{I} - \lim\limits_{n\to\infty} N_C(x_n - a, t) = 1$*

Proof. (ii) ($\Rightarrow$) Suppose that $\mathcal{I}_{N_C} - \lim\limits_{n\to\infty} x_n = a$. Since for each $\varepsilon > 0$ and $t \in int(P)$, we have

$$\{n \in \mathbf{N} : |N_C(x_n - a, t) - 1| \geq \varepsilon\} = \{n \in \mathbf{N} : N_C(x_n - a, t) \geq 1 + \varepsilon\}$$

$$\cup \{n \in \mathbf{N} : N_C(x_n - a, t) \leq 1 - \varepsilon\}$$

and for each $\varepsilon > 0$ and $t \in int(P)$, the set $\{n \in \mathbf{N} : N_C(x_n - a, t) \geq 1 + \varepsilon\} = \emptyset \in \mathcal{I}$. By hypothesis, $\{n \in \mathbf{N} : N_C(x_n - a, t) \leq 1 - \varepsilon\} \in \mathcal{I}$. Hence we obtain $\mathcal{I} - \lim_{n\to\infty} N_C(x_n - a, t) = 1$.

($\Leftarrow$) It is obvious. □

Theorem 3.13. *Let (x_n) and (y_n) be sequences in fuzzy cone normed space $(X, N_C, *)$.*

(i) *If $\mathcal{I}_{N_C} - \lim_{n\to\infty} x_n = a$ and $\mathcal{I}_{N_C} - \lim_{n\to\infty} y_n = b$, then $\mathcal{I}_{N_C} - \lim_{n\to\infty} (x_n + y_n) = a + b$.*

(ii) *If $\mathcal{I}_{N_C} - \lim_{n\to\infty} x_n = a$ and k be any real number, then $\mathcal{I} - \lim_{n\to\infty} kx_n = ka$.*

Proof. (i) Let $\varepsilon > 0$. By Remark 1.6 [8], we have $\varepsilon_0 \in (0, 1)$ such that

$$(1 - \varepsilon_0) * (1 - \varepsilon_0) > 1 - \varepsilon \tag{3.1}$$

For $t \in int(P)$, Put

$$A = \{n \in \mathbf{N} : N_C(x_n + y_n - (a + b), t) \leq 1 - \varepsilon\} =$$
$$K_1(\varepsilon_0) = \{n \in \mathbf{N} : N_C(x_n - a, t) \leq 1 - \varepsilon_0\},$$
$$K_2(\varepsilon_0) = \{n \in \mathbf{N} : N_C(y_n - b, t) \leq 1 - \varepsilon_0\},$$

By assumption $K_1(\varepsilon_0) \in \mathcal{I}$ and $K_2(\varepsilon_0) \in \mathcal{I}$. Since $\mathcal{I}$ is an ideal, $K = K_1(\varepsilon_0) \cup K_2(\varepsilon_0) \in \mathcal{I}$ and $K^c \in \mathcal{F}(I)$. We shall show that

$$K^c \subseteq A^c.$$

Let m $\in K^c$. Then we have $N_C(x_m - a, \frac{t}{2}) > 1 - \varepsilon_0$ and $N_C(y_m - b, \frac{t}{2}) > 1 - \varepsilon_0$. Since N_C is a fuzzy cone norm and by (3.1),

$$\begin{aligned} N_C(x_m + y_m - (a + b), t) &\geq N_C\left(x_m - a, \frac{t}{2}\right) * N_C\left(y_m - b, \frac{t}{2}\right) \\ &> (1 - \varepsilon_0) * (1 - \varepsilon_0) \\ &> 1 - \varepsilon. \end{aligned}$$

Then we obtain that $m \in A^c$.

Since $K^c \in \mathcal{F}(I)$, we have $A^c \in \mathcal{F}(I)$, So $\mathcal{I}_{N_C} - \lim_{n\to\infty} (x_n + y_n) = a + b$.

(ii) Case-(1) $k = 0$, is obvious.

Case-(2) $|k| > 1$: For $t \in int(P)$, Put

$$A(\varepsilon) = \{n \in \mathbf{N} : N_C(x_n - a, t) \leq 1 - \varepsilon\},$$
$$B(\varepsilon) = \{n \in \mathbf{N} : N_C(k(x_n - a), t) \leq 1 - \varepsilon\}.$$

Since N_C is a fuzzy cone norm,

$$N_C(k(x_n - a), t) = N_C\left(x_n - a, \frac{t}{|k|}\right) \tag{3.2}$$

Since N_C is an nondecreasing function and $\frac{t}{|k|} \leq t$ for $|k| > 1$,

$$N_C\left(x_n - a, \frac{t}{|k|}\right) \leq N_C(x_n - a, t) \tag{3.3}$$

As $\mathcal{I}_{N_C} - \lim_{n\to\infty} x_n = a$, $A(\varepsilon) \in \mathcal{I}$. By (3.2) and (3.3), it follows that

$$B(\varepsilon) \subseteq A(\varepsilon).$$

Therefore $B(\varepsilon) \in \mathcal{I}$.

Case-(3) $|k| < 1$ and $k \neq 0$: For each $\varepsilon > 0$ and $t \in int(P)$,

$$K(\varepsilon) = \{n \in \mathbf{N} : N_C(x_n - a, t) > 1 - \varepsilon\},$$
$$M(\varepsilon) = \{n \in \mathbf{N} : N_C(k(x_n - a), t) > 1 - \varepsilon\}.$$

Since N_C is a fuzzy cone norm, we obtain

$$\begin{aligned} N_C\left(x_n - a, \frac{t}{|k|}\right) &\geq N_C(x_n - a, t) * N_C\left(0, \frac{t}{|k|} - t\right) \\ &= N_C(x_n - a, t), \end{aligned} \tag{3.4}$$

As $\mathcal{I}_{N_C} - \lim_{n\to\infty} x_n = a$, $K(\varepsilon) \in \mathcal{F}(\mathcal{I})$. By (3.2) and (3.4), it follows that

$$K(\epsilon) \subseteq M(\epsilon).$$

Then, we obtain $M(\epsilon) \in \mathcal{F}(\mathcal{I})$.

□

Theorem 3.14. *Let $\mathcal{I}$ be an admissible ideal and (x_n) be a sequence in fuzzy cone normed space $(X, N_C, *)$. If each subsequence of (x_n) is $\mathcal{I}_{N_C}$-convergence to a, then (x_n) is also $\mathcal{I}_{N_C}$-convergence to a.*

Proof. Suppose that (x_n) is not $\mathcal{I}_{N_C}$-convergence to a. Then there exists $\varepsilon > 0$ and t$\in int(P)$ such that

$$A = \{n \in \mathbf{N} : N_C(x_n - a, t) \geq 1 - \varepsilon\} \notin \mathcal{I}.$$

Since $\mathcal{I}$ is an admissible ideal, A must be an infinite set. Let A=$\{n_1 < n_2 < n_3 < \ldots < n_m < \ldots\}$. Let $y_m = x_{n_m}$ for $m \in \mathbf{N}$ which is not $\mathcal{I}_{N_C}$-convergence to a. This is a contradiction. □

The following example shows that the converse of Theorem 3.14 may not be true, in general.

Example 3.15. Let $E = \mathbf{R}^2$. Then $P = \{(a_1, a_2) : a_1, a_2 \geq 0\} \subset E$ is a normal cone. Let $X = \mathbf{R}$, $a * b = a.b$ and $N : X \times int(P) \to [0, 1]$ defined by $N(x, t) = e^{-\frac{|x|}{\|t\|}}$ for all $x \in X$ and $t \in int(P)$. Let $\mathcal{I} = \{A \subset \mathbf{N} : \delta(A) = 0\}$. Define a sequence (x_n) in X

$$x_n = \begin{cases} 1, & k = n^2, \\ 0, & k \neq n^2, \end{cases}$$

Then (x_n) is $\mathcal{I}_{N_C}$-convergence to 0. But $(x_{k_m}) = (1)$, subsequence of (x_n), is not $\mathcal{I}_{N_C}$-convergence to 0.

Theorem 3.16. *Let $\mathcal{I}$ be an admissible ideal with the property (AP) and (x_n) be a sequence in fuzzy cone normed space $(X, N_C, *)$. (x_n) is $\mathcal{I}_{N_C}$-convergence in X if and only if there is a N_C-convergence sequence (y_n) such that $\{i \in \mathbf{N} : x_i \neq y_i\} \in \mathcal{I}$.*

Proof. Let $\mathcal{I}_{N_C} - \lim_{n\to\infty} x_n = a$. For each t $\in int(P)$, $k \in \mathbf{N}$. Let

$$F_k = \left\{ n \in \mathbf{N} : N_C(x_n - a, t) \geq \frac{1}{k} \right\}.$$

Then, $F_k \in \mathcal{F}(\mathcal{I})$ for $k \in \mathbf{N}$. Since $\mathcal{I}$ is an admissible ideal with the property (AP), by Lemma 2.3, there exists $F \subset \mathbf{N}$ such that $F \in \mathcal{F}(I)$ and $F \setminus F_k$ is finite for all $k \in \mathbf{N}$. We can see that $x_n \to_F a$ i.e for each $\varepsilon > 0$, $t \in int(P)$, so there exists a positive integer n_0 such that $N_C(x_n - a, t) > 1 - \varepsilon$ for all $n \geq n_0$ and $n \in F$.

Define a sequence (y_n) in X

$$y_n = \begin{cases} x_n, & n \in F \\ a, & n \in \mathbf{N} \backslash F \end{cases}$$

The sequence (y_n) is N_C-convergence to a. Thus we have $\{i \in \mathbf{N} : x_i \neq y_i\} \in \mathcal{I}$.

Conversely, we assume that $\{n \in \mathbf{N} : x_n \neq y_n\} \in \mathcal{I}$ and $N_C - \lim_{n\to\infty} y_n = a$. Then for every $\varepsilon > 0$, $t \in int(P)$, we have

$\{n \in \mathbf{N} : N_C(x_n - a, t) \leq 1 - \varepsilon\} \subseteq \{n \in \mathbf{N} : x_n \neq y_n\} \cup \{n \in \mathbf{N} : N_C(y_n - a, t) \leq 1 - \varepsilon\}$ Since $\mathcal{I}$ is an admissible ideal and by hypothesis, we obtain $\{n \in \mathbf{N} : N_C(y_n - a, t) \leq 1 - \varepsilon\} \in \mathcal{I}$. By the definition of ideal, $\{n \in \mathbf{N} : N_C(x_n - a, t) \leq 1 - \varepsilon\} \in \mathcal{I}$. So we obtain $\mathcal{I}_{N_C} - \lim_{n\to\infty} x_n = a$. ☐

Definition 3.17. Let $(X, N_C, *)$ be a fuzzy cone normed space and (x_n) be a sequence in X. Then (x_n) is said to be $\mathcal{I}^*_{N_C}$-convergent to a in X if there exists a subset

$$M = \{k_m : k_1 < k_2 < \cdots\} \subset \mathbf{N}$$

such that $M \in \mathcal{F}(\mathcal{I})$ and N_C-$\lim_{n\to\infty} x_n = a$ for each $t \in int(P)$. We denote $\mathcal{I}^*_{N_C}$-$\lim_{n\to\infty} x_n = a$.

Theorem 3.18. *Let $(X, N_C, *)$ be a fuzzy cone normed space, $\mathcal{I}$ be an admissible ideal and (x_n) be a sequence in X,*

$$\mathcal{I}^*_{N_C} - \lim_{n\to\infty} x_n = a \quad \Rightarrow \quad \mathcal{I}_{N_C} - \lim_{n\to\infty} x_n = a$$

Proof. Assume that (x_n) is an $\mathcal{I}^*_{N_C}$-convergent to a in X if there exists a subset

$$M = \{k_m : k_1 < k_2 < \cdots\} \subset \mathbf{N}$$

such that $M \in \mathcal{F}(\mathcal{I})$ and N_C-$\lim_{n\to\infty} x_n = a$ for each $t \in int(P)$. Then, there exists a natural number n_0 such that $N_C(x_{k_m} - a, t) > 1 - \varepsilon$ for every $k_m \geq n_0$. Let $A = \{k_1, k_2, \ldots, k_{n_0}\}$. I is an admissible ideal and $A \in \mathcal{I}$. Since $M \in \mathcal{F}(I)$, there exists a set $B \in \mathcal{I}$ such that $M = \mathbf{N} \setminus B$.

$$A(\varepsilon) = \{n \in \mathbf{N} : N_C(x_n - a, t) \leq 1 - \varepsilon\} \subset A \cup B.$$

By the definition of ideal, $A \cup B \in \mathcal{I}$ and $A(\varepsilon) \in \mathcal{I}$. □

Definition 3.19. Let $(X, N_C, *)$ be a fuzzy cone normed space and (x_n) be a sequence in X. Then

(1) An element $a \in X$ is said to be a $\mathcal{I}_{N_C}$-limit point of (x_n) if there is a set J=$\{n_1 < n_2 < \cdots < n_k < \cdots\} \subset \mathbf{N}$ such that the set $J \notin \mathcal{I}$ and $N_C - \lim_{k\to\infty} x_{m_k} = a$ for every $t \in int(P)$. The set of $\mathcal{I}_{N_C}$-limit points of (x_n) is denoted by $\mathcal{L}^{\mathcal{I}}_{N_C}(x_n)$ in $(X, N_C, *)$.

(2) An element $a \in X$ is said to be a $\mathcal{I}_{N_C}$-cluster point of (x_n) if for every $\varepsilon > 0$ and $t \in int(P)$, the set $\{n \in \mathbf{N} : N_C(x_n - a, t) > 1 - \varepsilon\} \notin \mathcal{I}$. The set of $\mathcal{I}_{N_C}$-cluster points of (x_n) is denoted by $\mathcal{C}l^{\mathcal{I}}_{N_C}(x_n)$ in $(X, N_C, *)$.

Theorem 3.20. *Let $(X, N_C, *)$ be a fuzzy cone normed space and $\mathcal{I}$ be an admissible ideal. Let (x_n) be a sequence in X. Then the set $\mathcal{C}l^{\mathcal{I}}_{N_C}(x_n)$ is closed with respect to topology induced by the fuzzy cone norm N_C.*

Proof. Let $y \in \overline{\mathcal{C}l^{\mathcal{I}}_{N_C}(x_n)}$. Then we obtain $B(y, r, t) \cap \mathcal{C}l^{\mathcal{I}}_{N_C}(x_n) \neq \emptyset$ where $t \in int(P)$ and $0 < r < 1$. Let $z \in B(y, r, t) \cap \mathcal{C}l^{\mathcal{I}}_{N_C}(x_n)$. Choose $0 < r_0 < 1$ such that $B(z, r_0, t) \subset B(y, r, t)$. We have

$$\begin{aligned} G &= \{n \in \mathbf{N} : N_C(x_n - y, t) > 1 - r\} \\ &\supseteq \{n \in \mathbf{N} : N_C(x_n - z, t) > 1 - r_0\} \\ &= H. \end{aligned}$$

Since $z \in \mathcal{C}l^{\mathcal{I}}_{N_C}(x_n)$, $H \notin \mathcal{I}$. So $G \notin \mathcal{I}$. Hence, we have $y \in \mathcal{C}l^{\mathcal{I}}_{N_C}(x_n)$. □

Theorem 3.21. *Let $(X, N_C, *)$ be a fuzzy cone normed space and $\mathcal{I}$ be admissible ideal. Then for every sequence is (x_n) in X. we obtain*

$$\mathcal{L}^{\mathcal{I}}_{N_C}(x_n) \subset \mathcal{C}l^{\mathcal{I}}_{N_C}(x_n)$$

Proof. Assume that a $\in \mathcal{L}^{\mathcal{I}}_{N_C}(x_n)$. Then there exits a set $J = \{n_1 < n_2 < \cdots < n_k < \cdots\} \subset \mathbf{N}$ such that the set $J \notin \mathcal{I}$ and N_C-$\lim_{n\to\infty} x_{n_k} = a$ for every $t \in int(P)$.
Let $\varepsilon > 0$ and $t \in int(P)$. By hypothesis, there exists an integer $k_0 \in \mathbf{N}$ such that $N_C(x_{n_k} - a, t) > 1 - \varepsilon$ for each $k > k_0$. Thus, we have

$$\{n \in \mathbf{N} : N_C(x_n - a, t) > 1 - \varepsilon\} \supseteq J \setminus \{n_1 < n_2 < \cdots < n_{k_0}\}.$$

Therefore,

$$\{n \in \mathbf{N} : N_C(x_n - a, t) > 1 - \varepsilon\} \notin \mathcal{I}$$

and $a \in \mathcal{C}l^{\mathcal{I}}_{N_C}(x_n)$. □

Theorem 3.22. *Let $(X, N_C, *)$ be a fuzzy cone normed space and $\mathcal{I}$ be an ideal. If (x_n) be $\mathcal{I}_{N_C}$-convergent to a in X, then*

$$\mathcal{L}^{\mathcal{I}}_{N_C}(x_n) = \mathcal{C}l^{\mathcal{I}}_{N_C}(x_n) \\ = \{a\}.$$

Proof. Assume that (x_n) is $\mathcal{I}_{N_C}$-convergent to a Then, for each $\varepsilon > 0$ and $t \in int(P)$, the set

$$A = \{n \in \mathbf{N} : N_C(x_n - b, t) \leq 1 - \varepsilon\} \in \mathcal{I}$$

and so

$$\{n \in \mathbf{N} : N_C(x_n - b, t) > 1 - \varepsilon\} \notin \mathcal{I}$$

and a $\in \mathcal{C}l^{\mathcal{I}}_{N_C}(x_n)$. We assume that $\mathcal{C}l^{\mathcal{I}}_{N_C}(x_n) = \{b\}$ where $a \neq b$. By Definition 3.19 , for every $\varepsilon > 0$ and $t \in int(P)$, the sets

$$K = \{n \in \mathbf{N} : N_C(x_n - a, t) > 1 - \varepsilon\} \notin \mathcal{I}.$$
$$L = \{n \in \mathbf{N} : N_C(x_n - b, t) > 1 - \varepsilon\} \notin \mathcal{I}.$$

For $a \neq b$, we have $K \cap L = \emptyset$. By hypothesis,

$$K^c = \{n \in \mathbf{N} : N_C(x_n - a, t) \leq 1 - \varepsilon\} \in \mathcal{I}$$

So we have L $\in \mathcal{I}$, which contradictions to K $\notin \mathcal{I}$. Therefore, $\mathcal{C}l^{\mathcal{I}}_{N_C}(x_n) = \{a\}$.
On the other hand, by the hypothesis, Theorem 3.21 and Definition 3.19, we obtain $a \in \mathcal{L}^{\mathcal{I}}_{N_C}(x_n)$. By previous theorem, we obtain $\mathcal{L}^{\mathcal{I}}_{N_C}(x_n) = \mathcal{C}l^{\mathcal{I}}_{N_C}(x_n) = \{a\}$. □

4. Conculusion

In this paper, we have generalized the notion of fuzzy norm by taking ordered Banach space instead of positive real numbers in the definition of fuzzy norm which is defined by R. Saadati. Then we gave the basic properties of fuzzy cone normed spaces and also introduced the notion of $n\mathcal{I}$ -convergence and $n\mathcal{I}^\star$-convergence in fuzzy cone normed space.

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