Weighted Cebysev Type Inequalities for Double Integrals and Application

Asif R. Khan, Hira Nasir and S. Sikander Shirazi

Sahand Communications in Mathematical Analysis

Print ISSN: 2322-5807
Online ISSN: 2423-3900
Volume: 18
Number: 4
Pages: 59-72

Sahand Commun. Math. Anal.
DOI: 10.22130/scma.2021.129537.815

Sahand Communications in Mathematical Analysis (SCMA) Vol. 18 No. 4 (2021), 59-72
http://scma.maragheh.ac.ir
DOI: 10.22130/scma.2021.129537.815

Weighted Čebyšev Type Inequalities for Double Integrals and Application

Asif R. Khan[1], Hira Nasir[2] and S. Sikander Shirazi[3*]

Abstract. The purpose of this article is to generalize Čebyšev type inequalities for double integrals involving a weight function. By using an integral transform that is a weighted Montgomery identity, we obtained a generalized form of weighted Čebyšev type inequalities in L_m, $m \geq 1$ norm of differentiable functions. Also, we give some applications of the probability density function.

1. Introduction

A Čebyšev type inequality gives an estimation of bounded functionals which is based on Montgomery identity. For two absolutely continuous functions $f, g : [a_0, a_1] \to \mathbb{R}$ consider the functional,

$$T(f,g) = \frac{1}{a_1 - a_0}\int_{a_0}^{a_1} f(\xi)g(\xi)d\xi - \left(\frac{1}{a_1 - a_0}\int_{a_0}^{a_1} f(\xi)d\xi\right) \times \left(\frac{1}{a_1 - a_0}\int_{a_0}^{a_1} g(\xi)d\xi\right). \tag{1.1}$$

In 1882, Čebyšev [4] proved that if $f', g' \in L_\infty[a_0, a_1]$,($L_\infty$, space of essentially bounded measurable functions) then,

$$|T(f,g)| \leq \frac{1}{12}(a_1 - a_0)^2 \|f'\|_\infty \|g'\|_\infty, \tag{1.2}$$

where $T(f,g)$ is said to be Čebyšev functional which is defined in (1.1).

In [6], authors have proved the double integral Montgomery identity which is defined in the following manner,

$$f(\xi,\eta) = \frac{1}{a_1 - a_0}\int_{a_0}^{a_1} f(t,\eta)dt + \frac{1}{b_1 - b_0}\int_{b_0}^{b_1} f(\xi,s)ds \tag{1.3}$$

2020 *Mathematics Subject Classification.* 26D99,26D15, 26D20.

Key words and phrases. Čebyšev inequality, Weighted Čebyšev inequality, Montgomery identity, Probability density function.

Received: 20 June 2020, Accepted: 19 July 2021.

* Corresponding author.

$$- \frac{1}{(a_1 - a_0)(b_1 - b_0)} \int_{a_0}^{a_1} \int_{b_0}^{b_1} f(t,s)dsdt$$
$$+ \int_{a_0}^{a_1} \int_{b_0}^{b_1} U(\xi,s)V(\eta,t)\frac{\partial^2 f(s,t)}{\partial s \partial t}dtds,$$

where $U(\xi,s)$ and $V(\eta,t)$ are linear functionals which are named as Peano kernels defined by

$$U(\xi,s) = \begin{cases} \frac{s-a_0}{a_1-a_0}, & \text{if} \quad s \in [a_0,\xi], \\ \frac{s-a_1}{a_1-a_0}, & \text{if} \quad s \in (\xi, a_1], \end{cases} \tag{1.4}$$

and

$$V(\eta,t) = \begin{cases} \frac{t-b_0}{b_1-b_0}, & \text{if} \quad t \in [b_0,\eta], \\ \frac{t-b_1}{b_1-b_0}, & \text{if} \quad t \in (\eta, b_1]. \end{cases} \tag{1.5}$$

In [14], authors have proved the double integral weighted Montgomery identity which is defined in the following manner,

(1.6)

$$f(\xi,\eta)P(a_0,a_1) = -\int_{a_0}^{a_1}\int_{b_0}^{b_1} p_o(s,t)f(s,t)dtds + \int_{a_0}^{a_1}\int_{b_0}^{b_1} p_o(s,t)f(s,\eta)dtds$$
$$+ \int_{a_0}^{a_1}\int_{b_0}^{b_1} p_o(s,t)f(\xi,t)dtds$$
$$+ \int_{a_0}^{a_1}\int_{b_0}^{b_1} \bar{P}(\xi,s,\eta,t)f_{(1,1)}(s,t)dtds,$$

where P is defined as

$$P(a_0,a_1) = \int_{a_0}^{\xi}\int_{a_1}^{\eta} p_o(\xi,\eta)\, d\eta d\xi,$$

for some probability density function $p_o : [a_0,a_1]\times[b_0,b_1] \to \mathbb{R}_+ \quad and \quad \mathbb{R}_+ = [0,\infty)$. Furthermore, we recall an important concept which will be used throughout the paper, as follows.

Definition 1.1 ([16])**.** Let $L_p[s,t] \quad (1 \le p < \infty)$ be the space of integrable functions having $p - power$ on $[s,t]$ with norm

$$||f||_p = \left(\int_s^t |f(t)|^p dt\right)^{\frac{1}{p}},$$

and $L_\infty[s,t]$ be the space of all essential bounded functions on $[s,t]$ with norm

$$||f||_\infty = ess \sup_{x\in[s,t]} |f(x)|.$$

In the last few decades, many researchers have worked on the identity (1.1) and inequality (1.2). Researchers presented various generalizations and extensions of Čebyšev inequalities, we could mention the work [1, 3, 5, 8, 9, 15]. Our aim of this paper is to establish the generalization of Čebyšev inequality in terms of weight. In the first section, we generalize Čebyšev type inequalities and gives some special cases. In the second section, we give an application of the probability density function of our main result.

2. Weighted Čebyšev Type Inequality for Double Integrals

Here we state our first main result for the weighted Čebyšev inequality.

Theorem 2.1. *Let the functions $f, g : [a_0, a_1] \times [b_0, b_1] \to \mathbb{R}$ be differentiable mappings on $(a_0, a_1) \times (b_0, b_1)$ while $f_{(1,1)}(s,t)$ and $g_{(1,1)}(s,t)$, the second order partial derivatives, are integrable on $[a_0, a_1] \times [b_0, b_1]$. Then*

$$(2.1)\qquad |T(f,g,p,q)| \leq \frac{1}{\left(\int_{a_0}^{a_1} p(\xi)d\xi\right)^3 \left(\int_{b_0}^{b_1} q(\eta)d\eta\right)^3} \|f_{(1,1)}\|_m \|g_{(1,1)}\|_m \times \int_{a_0}^{a_1}\int_{b_0}^{b_1} p(\xi)q(\eta)H^2(\xi,\eta,n)\, d\eta d\xi,$$

where $p(\xi)$ and $q(\eta)$ are Riemann integrable functions provided that they are non-negative, moreover, $f_{(1,1)}(s,t) = \dfrac{\partial^2 f(s,t)}{\partial s \partial t}$ and $g_{(1,1)}(s,t) = \dfrac{\partial^2 g(s,t)}{\partial s \partial t}$, and

$$(2.2)\qquad H(\xi,\eta,n) = \left(\int_{a_0}^{a_1}\int_{b_0}^{b_1} \left|\bar{P}(\xi,s,\eta,t)\right|^n dtds\right)^{\frac{1}{n}},$$

while $\bar{P}$ is defined as

$$\bar{P}(\xi,s,\eta,t) = \begin{cases} \int_{a_0}^{s}\int_{b_0}^{t} p(r)q(v)drdv, & \text{if} \quad a_0 \leq s \leq \xi,\ b_0 \leq t \leq \eta, \\ -\int_{s}^{a_1}\int_{b_0}^{t} p(r)q(v)drdv, & \text{if} \quad \xi < s \leq a_1,\ b_0 \leq t \leq \eta, \\ -\int_{a_0}^{s}\int_{t}^{b_1} p(r)q(v)drdv, & \text{if} \quad a_0 \leq s \leq \xi,\ \eta < t \leq b_1, \\ \int_{s}^{a_1}\int_{t}^{b_1} p(r)q(v)drdv, & \text{if} \quad \xi < s \leq a_1,\ \eta < t \leq b_1, \end{cases}$$

where

$$(2.3)\qquad T(f,g,p,q) = \frac{1}{\left(\int_{a_0}^{a_1} p(\xi)d\xi\right)\left(\int_{b_0}^{b_1} q(\eta)d\eta\right)} \int_{a_0}^{a_1} p(\xi)\phi(\xi)\rho(\xi)d\xi$$

$$\times \int_{b_0}^{b_1} q(\eta)h(\eta)k(\eta)d\eta$$
$$- \frac{1}{\left(\int_{a_0}^{a_1} p(\xi)d\xi\right)^2 \left(\int_{b_0}^{b_1} q(\eta)d\eta\right)} \int_{a_0}^{a_1} p(\xi)\phi(\xi)d\xi$$
$$\times \int_{b_0}^{b_1} q(\eta)h(\eta)k(\eta)d\eta \int_{a_0}^{a_1} p(s)\rho(s)ds$$
$$- \frac{1}{\left(\int_{a_0}^{a_1} p(\xi)d\xi\right) \left(\int_{b_0}^{b_1} q(\eta)d\eta\right)^2} \int_{a_0}^{a_1} p(\xi)\phi(\xi)\rho(\xi)d\xi$$
$$\times \int_{b_0}^{b_1} q(\eta)h(\eta)d\eta \int_{b_0}^{b_1} q(t)k(t)dt$$
$$+ \frac{1}{\left(\int_{a_0}^{a_1} p(\xi)d\xi\right)^2 \left(\int_{b_0}^{b_1} q(\eta)d\eta\right)^2} \int_{a_0}^{a_1} p(\xi)\phi(\xi)d\xi$$
$$\times \int_{b_0}^{b_1} q(\eta)k(\eta)d\eta \int_{a_0}^{a_1} p(s)\rho(s)ds \int_{b_0}^{b_1} q(t)h(t)dt.$$

Proof. Let F,G, $\tilde{F}$ and $\tilde{G}$ be defined (see [14]), as follows.

$$F(\xi,\eta) = f(\xi,\eta) + \frac{1}{P(a_0,b_0)} \int_{a_0}^{a_1} \int_{b_0}^{b_1} p_o(s,t)f(s,t)\,dtds - \frac{1}{P(a_0,b_0)} \int_{a_0}^{a_1} \int_{b_0}^{b_1} p_o(s,t)f(s,\eta)\,dtds - \frac{1}{P(a_0,b_0)} \int_{a_0}^{a_1} \int_{b_0}^{b_1} p_o(s,t)f(\xi,t)\,dtds, \tag{2.4}$$

$$G(\xi,\eta) = g(\xi,\eta) + \frac{1}{P(a_0,b_0)} \int_{a_0}^{a_1} \int_{b_0}^{b_1} p_o(s,t)g(s,t)\,dtds - \frac{1}{P(a_0,b_0)} \int_{a_0}^{a_1} \int_{b_0}^{b_1} p_o(s,t)g(s,\eta)\,dtds - \frac{1}{P(a_0,b_0)} \int_{a_0}^{a_1} \int_{b_0}^{b_1} p_o(s,t)g(\xi,t)\,dtds, \tag{2.5}$$

$$\widetilde{F}(x,y) = \frac{1}{P(a_0,b_0)} \int_{a_0}^{a_1} \int_{b_0}^{b_1} \bar{P}(\xi,s,\eta,t)f_{(1,1)}(s,t)\,dtds, \tag{2.6}$$

and

$$\widetilde{G}(x,y) = \frac{1}{P(a_0,b_0)} \int_{a_0}^{a_1} \int_{b_0}^{b_1} \bar{P}(\xi,s,\eta,t)g_{(1,1)}(s,t)\,dtds. \tag{2.7}$$

Since

$$FG = \tilde{F}\tilde{G},$$

now we will break $p_o(\xi,\eta)$ as the product, that is $p_o(\xi,\eta) = p(\xi)q(\eta)$ where $p(\xi)$ and $q(\eta)$ are Riemann integrable functions provided that they are non-negative. Applying integration $\displaystyle\int_{a_0}^{a_1}\int_{b_0}^{b_1} p(\xi)q(\eta)d\eta d\xi$ on FG over $[a_0,a_1]\times[b_0,b_1]$, then multiplying resultant equality by $\dfrac{1}{P(a_0,b_0)}$ and then break each functions

$$f(\xi,\eta) = \phi(\xi)h(\eta), \qquad g(\xi,\eta) = \rho(\xi)k(\eta),$$

provided that each functions are integrable in the given interval, which yields

(2.8)

$$\begin{aligned}
&\frac{1}{\left(\int_{a_0}^{a_1} p(\xi)d\xi\right)\left(\int_{b_0}^{b_1} q(\eta)d\eta\right)}\int_{a_0}^{a_1} p(\xi)\phi(\xi)\rho(\xi)d\xi\int_{b_0}^{b_1} q(\eta)h(\eta)k(\eta)d\eta\\
&\quad-\frac{1}{\left(\int_{a_0}^{a_1} p(\xi)d\xi\right)^2\left(\int_{b_0}^{b_1} q(\eta)d\eta\right)}\int_{a_0}^{a_1} p(\xi)\phi(\xi)d\xi\int_{b_0}^{b_1} q(\eta)h(\eta)k(\eta)d\eta\\
&\quad\times\int_{a_0}^{a_1} p(s)\rho(s)ds\\
&\quad-\frac{1}{\left(\int_{a_0}^{a_1} p(\xi)d\xi\right)\left(\int_{b_0}^{b_1} q(\eta)d\eta\right)^2}\int_{a_0}^{a_1} p(\xi)\phi(\xi)\rho(\xi)d\xi\int_{b_0}^{b_1} q(\eta)h(\eta)d\eta\\
&\quad\times\int_{b_0}^{b_1} q(t)k(t)dt\\
&\quad+\frac{1}{\left(\int_{a_0}^{a_1} p(\xi)d\xi\right)^2\left(\int_{b_0}^{b_1} q(\eta)d\eta\right)^2}\int_{a_0}^{a_1} p(\xi)\phi(\xi)d\xi\int_{b_0}^{b_1} q(\eta)k(\eta)d\eta\\
&\quad\times\int_{a_0}^{a_1} p(s)\rho(s)ds\int_{b_0}^{b_1} q(t)h(t)dt\\
&=\frac{1}{\left(\int_{a_0}^{a_1} p(\xi)d\xi\right)\left(\int_{b_0}^{b_1} q(\eta)d\eta\right)}\int_{a_0}^{a_1}\int_{b_0}^{b_1} p(\xi)q(\eta)
\end{aligned}$$

$$\times\left(\frac{1}{\left(\int_{a_0}^{a_1} p(\xi)d\xi\right)\left(\int_{b_0}^{b_1} q(\eta)d\eta\right)}\int_{a_0}^{a_1}\int_{b_0}^{b_1}\bar{P}(\xi,s,\eta,t)f_{(1,1)}(s,t)\,\mathrm{dtds}\right)$$
$$\times\left(\frac{1}{\left(\int_{a_0}^{a_1} p(\xi)d\xi\right)\left(\int_{b_0}^{b_1} q(\eta)d\eta\right)}\int_{a_0}^{a_1}\int_{b_0}^{b_1}\bar{P}(\xi,s,\eta,t)g_{(1,1)}(s,t)\,\mathrm{dtds}\right)\mathrm{d}\eta\mathrm{d}\xi,$$

consequently from (2.8), taking absolute value on it and then using Hölder's inequality for m and n such that $\frac{1}{m}+\frac{1}{n}=1$, we get

(2.9)

$$\begin{aligned}|T(f,g,p,q)| &\le \frac{1}{\left(\int_{a_0}^{a_1} p(\xi)d\xi\right)^3\left(\int_{b_0}^{b_1} q(\eta)d\eta\right)^3}\|f_{(1,1)}(s,t)\|_m\|g_{(1,1)}(s,t)\|_m \\ &\quad\times\int_{a_0}^{a_1}\int_{b_0}^{b_1}\left[|p(\xi)q(\eta)|\left(\int_{a_0}^{a_1}\int_{b_0}^{b_1}|\bar{P}(\xi,s,\eta,t)|^n\,dtds\right)^{\frac{2}{n}}\right]d\eta d\xi \\ &= \frac{1}{\left(\int_{a_0}^{a_1} p(\xi)d\xi\right)^3\left(\int_{b_0}^{b_1} q(\eta)d\eta\right)^3}\|f_{(1,1)}\|_m\|g_{(1,1)}\|_m \\ &\quad\times\int_{a_0}^{a_1}\int_{b_0}^{b_1}(p(\xi)q(\eta))H^2(\xi,\eta,n)d\eta d\xi,\end{aligned}$$

where

$$H(\xi,\eta,n)=\left(\int_{a_0}^{a_1}\int_{b_0}^{b_1}|\bar{P}(\xi,s,\eta,t)|^n\,dt\,ds\right)^{\frac{1}{n}},$$

and $T(f,g,p,q)$ defined above in (2.3). □

Remark 2.2. If $m\to\infty$ and $n=1$, such that $\frac{1}{m}+\frac{1}{n}=1$, in (2.9), then we get a special case with Hölder's inequality for infinite norm.

Corollary 2.3. *Let the assumptions of Theorem* 2.1 *be valid, then*

$$|T(f,g,p,q)| \le \frac{1}{\left(\int_{a_0}^{a_1} p(\xi)d\xi\right)^3\left(\int_{b_0}^{b_1} q(\eta)d\eta\right)^3}\|f_{(1,1)}\|_\infty\|g_{(1,1)}\|_\infty \times\int_{a_0}^{a_1}\int_{b_0}^{b_1}(p(\xi)q(\eta))\,H^2(\xi,\eta,1)\,d\eta d\xi, \tag{2.10}$$

where

$$H(\xi,\eta,1)=\int_{a_0}^{a_1}\int_{b_0}^{b_1}\left|\bar{P}(\xi,s,\eta,t)\right|dtds,$$

and $T(f,g,p,q)$ is defined as in (2.3).

Remark 2.4. If we simply put $p=q=1$ in (2.10), then we acquire the special case of [9], as follows.

Corollary 2.5. *Let the assumptions of Theorem* 2.1 *be valid, then*

$$|T_1(f,g)| \leq \frac{49}{3600}(a_1-a_0)^2(b_1-b_0)^2\|f_{(1,1)}(s,t)\|_\infty\|g_{(1,1)}(s,t)\|_\infty,$$

and $T_1(f,g)$ could acquire by just putting $p=q=1$ in (2.3)*, where the explicit mathematical form of Čebyšev functional $T_1(f,g)$ is defined in* (2.2) *of* [9].

Remark 2.6. If we simply put $p=q=1$ and $m=n=2$ such that $\frac{1}{m}+\frac{1}{n}=1$ in (2.9), then acquire result as follows.

Corollary 2.7. *Let the assumptions of Theorem* 2.1 *be valid, then*

$$\begin{aligned} |T(f,g,1,1)| \leq{} & \frac{1}{9\,(a_1-a_0)^2(b_1-b_0)^2}\|f_{(1,1)}\|_2\|g_{(1,1)}\|_2 \\ & \times\left\{3\xi^2-3(a_0+a_1)\xi+a_0^2+a_0a_1+a_1^2\right\} \\ & \times\left\{3\eta^2-3(b_0+b_1)\eta+b_0^2+b_0b_1+b_1^2\right\}. \end{aligned} \tag{2.11}$$

Here we state our second theorem.

Theorem 2.8. *Let the functions $\phi,\rho:[a_0,a_1]\times[b_0,b_1]\to\mathbb{R}$ be differentiable functions on $(a_0,a_1)\times(b_0,b_1)$ while $\phi_{(1,1)}(s,t)$ and $\rho_{(1,1)}(s,t)$ are the second order partial derivatives, are integrable on $[a_0,a_1]\times[b_0,b_1]$. Then*

$$\begin{aligned} |T_{p_o}(\phi,\rho)| \leq{} & \frac{1}{2P^2(a_0,b_0)}\int_{a_0}^{a_1}\int_{b_0}^{b_1}p_o(\xi,\eta) \\ & \times\left(|\rho(\xi,\eta)|\,\|\phi_{(1,1)}\|_m+|\phi(\xi,\eta)|\,\|\rho_{(1,1)}\|_m\right)H(\xi,\eta,n)\,d\xi d\eta, \end{aligned} \tag{2.12}$$

where

$$\begin{aligned} T_{p_o}(\phi,\rho) ={} & \frac{1}{P(a_0,b_0)}\int_{a_0}^{a_1}\int_{b_0}^{b_1}p_o(\xi,\eta)\phi(\xi,\eta)\rho(\xi,\eta)\,d\xi d\eta \\ & +\frac{1}{P^2(a_0,b_0)}\int_{a_0}^{a_1}\int_{b_0}^{b_1}p_o(\xi,\eta)\left(\rho(\xi,\eta)\int_{a_0}^{a_1}\int_{b_0}^{b_1}p_o(s,t)\phi(s,t)dtds\right)d\xi d\eta \\ & -\frac{1}{P^2(a_0,b_0)}\int_{a_0}^{a_1}\int_{b_0}^{b_1}p_o(\xi,\eta)\left(\rho(\xi,\eta)\int_{a_0}^{a_1}\int_{b_0}^{b_1}p_o(s,t)\phi(s,\eta)dtds\right)d\xi d\eta \\ & -\frac{1}{P^2(a_0,b_0)}\int_{a_0}^{a_1}\int_{b_0}^{b_1}p_o(\xi,\eta)\left(\rho(\xi,\eta)\int_{a_0}^{a_1}\int_{b_0}^{b_1}p_o(s,t)\phi(\xi,t)dtds\right)d\xi d\eta. \end{aligned} \tag{2.13}$$

Proof. In [14], the double integral Montgomery identity is defined by

$$\begin{aligned}
\phi(\xi,\eta) = &-\frac{1}{P(a_0,b_0)}\int_{a_0}^{a_1}\int_{b_0}^{b_1} p_o(s,t)\phi(s,t)dtds \\
&+\frac{1}{P(a_0,b_0)}\int_{a_0}^{a_1}\int_{b_0}^{b_1} p_o(s,t)\phi(s,\eta)dtds \\
&+\frac{1}{P(a_0,b_0)}\int_{a_0}^{a_1}\int_{b_0}^{b_1} p_o(s,t)\phi(\xi,t)dtds \\
&+\frac{1}{P(a_0,b_0)}\int_{a_0}^{a_1}\int_{b_0}^{b_1} \bar{P}(\xi,s,\eta,t)\phi_{(1,1)}(s,t)dtds,
\end{aligned} \tag{2.14}$$

also applying (2.14) to the function "ρ", we obtain

$$\begin{aligned}
\rho(\xi,\eta) = &-\frac{1}{P(a_0,b_0)}\int_{a_0}^{a_1}\int_{b_0}^{b_1} p_o(s,t)\rho(s,t)dtds \\
&+\frac{1}{P(a_0,b_0)}\int_{a_0}^{a_1}\int_{b_0}^{b_1} p_o(s,t)\rho(s,\eta)dtds \\
&+\frac{1}{P(a_0,b_0)}\int_{a_0}^{a_1}\int_{b_0}^{b_1} p_o(s,t)\rho(\xi,t)dtds \\
&+\frac{1}{P(a_0,b_0)}\int_{a_0}^{a_1}\int_{b_0}^{b_1} \bar{P}(\xi,s,\eta,t)\rho_{(1,1)}(s,t)dtds.
\end{aligned} \tag{2.15}$$

Multiplying (2.14) by $\frac{1}{P(a_0,b_0)}\rho(\xi,\eta)$ and (2.16) by $\frac{1}{P(a_0,b_0)}\phi(\xi,\eta)$, and then summing the resultant equalities and then applying

$$\int_{a_0}^{a_1}\int_{b_0}^{b_1} p_o(\xi,\eta)\,d\xi d\eta \ on\ [a_0,a_1]\times[b_0,b_1],$$

we obtain

$$\begin{aligned}
&\frac{1}{P(a_0,b_0)}\int_{a_0}^{a_1}\int_{b_0}^{b_1} \left(p_o(\xi,\eta)\right)\phi(\xi,\eta)\rho(\xi,\eta)\,d\xi d\eta \\
&\quad= -\frac{1}{P^2(a_0,b_0)}\int_{a_0}^{a_1}\int_{b_0}^{b_1} p_o(\xi,\eta)\left(\rho(\xi,\eta)\int_{a_0}^{a_1}\int_{b_0}^{b_1} p_o(s,t)\phi(s,t)dtds\right)d\xi d\eta \\
&\qquad+\frac{1}{P^2(a_0,b_0)}\int_{a_0}^{a_1}\int_{b_0}^{b_1} p_o(\xi,\eta)\left(\rho(\xi,\eta)\int_{a_0}^{a_1}\int_{b_0}^{b_1} p_o(s,t)\phi(s,\eta)dtds\right)d\xi d\eta \\
&\qquad+\frac{1}{P^2(a_0,b_0)}\int_{a_0}^{a_1}\int_{b_0}^{b_1} p_o(\xi,\eta)\left(\rho(\xi,\eta)\int_{a_0}^{a_1}\int_{b_0}^{b_1} p_o(s,t)\phi(\xi,t)dtds\right)d\xi d\eta \\
&\qquad+\frac{1}{2P^2(a_0,b_0)}\int_{a_0}^{a_1}\int_{b_0}^{b_1} p_o(\xi,\eta)\left(\rho(\xi,\eta)\int_{a_0}^{a_1}\int_{b_0}^{b_1} \bar{P}(\xi,s,\eta,t)\,\phi_{(1,1)}(s,t)dtds\right.
\end{aligned} \tag{2.16}$$

$$+\, \phi(\xi,\eta)\int_{a_0}^{a_1}\int_{b_0}^{b_1}\bar{P}(\xi,s,\eta,t)\,\rho_{(1,1)}(s,t)dtds\Big)\,d\xi d\eta,$$

from that we deduce

$$T_{p_o}(\phi,\rho)=\frac{1}{2P^2(a_0,b_0)}\int_{a_0}^{a_1}\int_{b_0}^{b_1}p_o(\xi,\eta) \Big(\rho(\xi,\eta)\int_{a_0}^{a_1}\int_{b_0}^{b_1}\bar{P}(\xi,s,\eta,t)\,\phi_{(1,1)}(s,t)dtds + \phi(\xi,\eta)\int_{a_0}^{a_1}\int_{b_0}^{b_1}\bar{P}(\xi,s,\eta,t)\,\rho_{(1,1)}(s,t)dtds\Big)\,d\xi d\eta. \tag{2.17}$$

Consequently, from (2.17), taking absolute value on it and then applying Hölder's inequality for m and n such that $\frac{1}{m}+\frac{1}{n}=1$, we have

$$|T_{p_o}(\phi,\rho)|\le\frac{1}{2P^2(a_0,b_0)}\int_{a_0}^{a_1}\int_{b_0}^{b_1}p_o(\xi,\eta) \left(|\rho(\xi,\eta)|\,\|\phi_{(1,1)}\|_m+|\phi(\xi,\eta)|\,\|\rho_{(1,1)}\|_m\right) \times\left(\int_{a_0}^{a_1}\int_{b_0}^{b_1}|\bar{P}(\xi,s,\eta,t)|^n\,dtds\right)^{\frac{1}{n}}d\xi d\eta, \tag{2.18}$$

where

$$H(\xi,\eta,n)=\left(\int_{a_0}^{a_1}\int_{b_0}^{b_1}\left|\bar{P}(\xi,s,\eta,t)\right|^n dtds\right)^{\frac{1}{n}},$$

and $\bar{P}(\xi,s,\eta,t)$ is defined in (2.2) and $p_o(\xi,\eta)$ is the probability density function. Therefore

$$|T_{p_o}(\phi,\rho)|\le\frac{1}{2P^2(a_0,b_0)}\int_{a_0}^{a_1}\int_{b_0}^{b_1}(p_o(\xi,\eta))\left(|\rho(\xi,\eta)|\,\|\phi_{(1,1)}\|_m +|\phi(\xi,\eta)|\,\|\rho_{(1,1)}\|_m\right)\,H(\xi,\eta,n)d\xi d\eta, \tag{2.19}$$

where $T_{p_o}(\phi,\rho)$ is defined in (2.13). □

Remark 2.9. If $m\to\infty$ and $n=1$, such that $\frac{1}{m}+\frac{1}{n}=1$, in (2.12), then we get a special case with Hölder's inequality for infinite norm as follows.

Corollary 2.10. *Let the assumptions of the Theorem* 2.8 *be valid. Then*

$$|T_{p_o}(\phi,\rho)|\le\frac{1}{2P^2(a_0,b_0)}\int_{a_0}^{a_1}\int_{b_0}^{b_1}(p_o(\xi,\eta))\left(|\rho(\xi,\eta)|\,\|\phi_{(1,1)}\|_\infty +|\phi(\xi,\eta)|\,\|\rho_{(1,1)}\|_\infty\right)\,H(\xi,\eta,1)d\xi d\eta. \tag{2.20}$$

Remark 2.11. If we simply put $p_0(\xi,\eta)=1$ in (2.20), then we acquire a special case of [9] as follows.

Corollary 2.12. *Let the assumptions of Theorem* 2.8 *be valid, then*

(2.21)
$$\begin{aligned}|T_{p_o}(\phi,\rho)| \le & \frac{1}{2(a_1-a_0)^2(b_1-b_0)^2}\int_{a_0}^{a_1}\int_{b_0}^{b_1}\big(|\rho(\xi,\eta)|\,\|\phi_{(1,1)}\|_\infty \\ & +|\phi(\xi,\eta)|\,\|\rho_{(1,1)}\|_\infty\big)\; H(\xi,\eta,1) \quad d\xi d\eta. \\ = & \frac{1}{8(a_1-a_0)^2(b_1-b_0)^2} \\ & \times \int_{a_0}^{a_1}\int_{b_0}^{b_1}\big(|\rho(\xi,\eta)|\,\|\phi_{(1,1)}\|_\infty + |\phi(\xi,\eta)|\,\|\rho_{(1,1)}\|_\infty\big) \\ & \times \left[\left((\xi-a_0)^2+(a_1-\xi)^2\right)\left((\eta-b_0)^2+(b_1-\eta)^2\right)\right] d\xi d\eta.\end{aligned}$$

Remark 2.13. If we simply put $p = q = 1$ and $m = n = 2$ such that $\frac{1}{m}+\frac{1}{n}=1$ in (2.12) then the acquired result will be as follows.

Corollary 2.14. *Let the assumptions of Theorem* 2.8 *be valid, then*

$$\begin{aligned}|T_{p_o}(\phi,\rho)| \le & \frac{1}{6\,(a_1-a_0)^2(b_1-b_0)^2}\int_{a_0}^{a_1}\int_{b_0}^{b_1}\big(|\rho(\xi,\eta)|\,\|\phi_{(1,1)}\|_2 \\ & +|\phi(\xi,\eta)|\,\|\rho_{(1,1)}\|_2\big)\,\sqrt{\{(\xi-a_0)^3-(\xi-a_1)^3\}} \\ & \times \sqrt{\{(\eta-b_0)^3-(\eta-b_1)^3\}}d\xi d\eta.\end{aligned} \tag{2.22}$$

Theorem 2.15. *Under the assumptions of Theorem* 2.1*, we have*

$$\begin{aligned}|F(\xi,\eta)| \le & \frac{1}{P(a_0,b_0)}\|f_{(1,1)}\|_\infty\left(\int_{a_0}^{a_1}\int_{b_0}^{b_1}|\bar{P}(\xi,s,\eta,t)|\,dtds\right) \\ \le & \frac{1}{P(a_0,b_0)}\|f_{(1,1)}\|_\infty H(\xi,\eta,1).\end{aligned}$$

Proof. From identities (2.4) and (2.6), we let

$$F = \widetilde{F}. \tag{2.23}$$

Applying absolute value on both sides of (2.23) result and then applying Hölder's inequality, we get

$$\begin{aligned}|F(\xi,\eta)| = & \left|\frac{1}{P(a_0,b_0)}\int_{a_0}^{a_1}\int_{b_0}^{b_1}\bar{P}(\xi,s,\eta,t)f_{(1,1)}(s,t)\,dtds\right| \\ \le & \frac{1}{P(a_0,b_0)}\int_{a_0}^{a_1}\int_{b_0}^{b_1}|\bar{P}(\xi,s,\eta,t)f_{(1,1)}(s,t)|\;\,dtds \\ \le & \frac{1}{P(a_0,b_0)}\|f_{(1,1}\|_\infty\left(\int_{a_0}^{a_1}\int_{b_0}^{b_1}|\bar{P}(\xi,s,\eta,t)|\,dtds\right) \\ = & \frac{1}{P(a_0,b_0)}\|f_{(1,1)}\|_\infty H(\xi,\eta,1).\end{aligned} \tag{2.24}$$

Remark 2.16. If we simply put $p = q = 1$ in (2.24), then we acquire a special case as follows.

$$|F(\xi,\eta)| \leq \frac{1}{4(a_1-a_0)^2(b_1-b_0)^2}\|f_{(1,1)}\|_\infty$$
$$\times\left[\left((\xi-a_0)^2+(a_1-\xi)^2\right)\left((\eta-b_0)^2+(b_1-\eta)^2\right)\right].$$

□

In the next section, we are going to discuss some applications of our main result for probability density functions. In [7] authors shown the application of the related probability density function for single integrals.

3. Application to Probability Density Function

Let X be a continuous random variable with the probability density functions $\phi, \rho : [a_0, a_1] \to \mathbb{R}_+$ and the expected value of X is given by

$$E_\phi(X) = \int_{a_0}^{a_1} s\phi(s)ds,$$

and its weighted expectation will be

$$E_{p,\phi}(X) = \int_{a_0}^{a_1} p(s)s\phi(s)ds,$$

the distribution function I_ϕ is given as:

$$I_\phi(\xi) = \int_a^\xi \phi(s)ds,$$

for $\xi \in [a_0, a_1]$, such that $I_\phi(a_0) = 0$ and $I_\phi(a_1) = 1$.
Moreover, let Y be another continuous random variable with the probability density functions $h, k : [b_0, b_1] \to \mathbb{R}_+$ and the expected value of Y is given by

$$E_h(Y) = \int_{b_0}^{b_1} th(t)dt,$$

and its weighted expectation will be

$$E_{q,h}(Y) = \int_{b_0}^{b_1} q(t)th(t)dt,$$

the distribution function M_h is given as:

$$M_h(\eta) = \int_{b_0}^{\eta} h(t)dt,$$

for $\eta \in [b_0, b_1]$, such that $M_h(b_0) = 0$ and $M_h(b_1) = 1$.

Then we have the following theorem.

Theorem 3.1. *Let X, Y, I_ϕ, M_h be defined as above. Then the following inequality holds*

(3.1)
$$\Bigg|\frac{1}{(\int_{a_0}^{a_1} p(\xi)d\xi)(\int_{b_0}^{b_1} q(\eta)d\eta)}\int_{a_0}^{a_1} p(\xi)I_\phi(\xi)I_\rho(\xi)d\xi\int_{b_0}^{b_1} q(\eta)M_h(\eta)M_k(\eta)d\eta$$
$$-\frac{1}{(\int_{a_0}^{a_1} p(\xi)d\xi)^2(\int_{b_0}^{b_1} q(\eta)d\eta)}\left(a_1p(a_1)-E_{p,\phi}(X)-\int_{a_0}^{a_1}\xi p'(\xi)I_\phi(\xi)d\xi\right)$$
$$\times\int_{b_0}^{b_1} q(\eta)M_h(\eta)M_k(\eta)d\eta\times\left(a_1p(a_1)-E_{p,\rho}(X)-\int_{a_0}^{a_1} sp'(s)I_\rho(s)ds\right)$$
$$-\frac{1}{(\int_{a_0}^{a_1} p(\xi)d\xi)(\int_{b_0}^{b_1} q(\eta)d\eta)^2}\left(\int_{a_0}^{a_1} p(\xi)I_\phi(\xi)I_\rho(\xi)d\xi\right)$$
$$\times\left(b_1q(b_1)-E_{q,h}(Y)-\int_{b_0}^{b_1} M_h(\eta)\eta q'(\eta)d\eta\right)$$
$$\times\left(b_1q(b_1)-E_{q,k}(Y)-\int_{b_0}^{b_1} M_k(t)tq'(t)dt\right)$$
$$+\frac{1}{(\int_{a_0}^{a_1} p(\xi)d\xi)^2(\int_{b_0}^{b_1} q(\eta)d\eta)^2}\left(b_1q(b_1)-E_{q,k}(Y)-\int_{b_0}^{b_1} M_k(t)tq'(t)dt\right)$$
$$\times\left(a_1p(a_1)-E_{p,\phi}(X)-\int_{a_0}^{a_1} I_\phi(s)sp'(s)ds\right)$$
$$\times\left(b_1q(b_1)-E_{q,h}(Y)-\int_{b_0}^{b_1} M_h(t)tq'(t)dt\right)$$
$$\times\left(a_1p(a_1)-E_{p,\rho}(X)-\int_{a_0}^{a_1} I_\rho(s)sp'(s)ds\right)\Bigg|$$
$$\leq\frac{1}{(\int_{a_0}^{a_1} p(\xi)d\xi)^3(\int_{b_0}^{b_1} q(\eta)d\eta)^3}\|f\|_m\|g\|_m\times\int_{a_0}^{a_1}\int_{b_0}^{b_1} p(\xi)q(\eta)H^2(\xi,\eta,n)d\eta d\xi.$$

Proof. By substituting $\phi = I_\phi, \rho = I_\rho, h = M_h, k = M_k$ in equation (2.1), and the identities mentioned below we get the required inequality (3.1),

$$\int_{a_0}^{a_1} p(s)I_\phi(s)ds = a_1p(a_1)-E_{p,\phi}(X)-\int_{a_0}^{a_1} sp'(s)I_\phi(s)ds$$
$$\int_{a_0}^{a_1} p(s)I_\rho(s)ds = a_1p(a_1)-E_{p,\rho}(X)-\int_{a_0}^{a_1} sp'(s)I_\rho(s)ds$$
$$\int_{b_0}^{b_1} q(t)M_h(t)dt = b_1q(b_1)-E_{q,h}(Y)-\int_{b_0}^{b_1} tq'(t)M_h(t)dt$$
$$\int_{b_0}^{b_1} q(t)M_k(t)dt = b_1q(b_1)-E_{q,k}(Y)-\int_{b_0}^{b_1} tq'(t)M_k(t)dt.$$

□

Remark 3.2. If we substitute $p = q = 1$, we achieve a special case of the above theorem.

$$
\begin{aligned}
(3.2)\qquad &\Bigg| \frac{1}{(a_1-a_0)(b_1-b_0)} \int_{a_0}^{a_1} I_\phi(\xi) I_\rho(\xi) d\xi \int_{b_0}^{b_1} M_h(\eta) M_k(\eta) d\eta \\
&- \frac{1}{(a_1-a_0)^2(b_1-b_0)} \left(a_1 - E_\phi(X)\right) \left(\int_{b_0}^{b_1} M_h(\eta) M_k(\eta) d\eta \right) \\
&\times \left(a_1 - E_\rho(X)\right) \\
&- \frac{1}{(a_1-a_0)(b_1-b_0)^2} \left(\int_{a_0}^{a_1} I_\phi(\xi) I_\rho(\xi) d\xi \right) \left(b_1 - E_h(Y)\right) \\
&\times \left(b_1 - E_k(Y)\right) \\
&+ \frac{1}{(a_1-a_0)^2(b_1-b_0)^2} \left(b_1 - E_k(Y)\right) \left(a_1 - E_\phi(X)\right) \\
&\times \left(b_1 - E_h(Y)\right) \left(a_1 - E_\rho(X)\right) \Bigg| \\
\leq\ & \frac{49}{3600} (a_1-a_0)^2 (b_1-b_0)^2 \|f\|_m \|g\|_m .
\end{aligned}
$$

4. Conclusion

In this article, we have generalized results of [9], which gives an estimation of bounded functionals. By using the Montgomery identity as defined in [6, 14], we have obtained weighted Čebyšev type inequality for double integrals. At last, we have presented an application of our main result which is achieved by applying the probability density function.

Acknowledgment. The research of the first and the third authors is supported by the Higher Education Commission of Pakistan under Indigenous PhD Fellowship for 5000 Scholars, HEC, (Phase-II) Batch-VI.

References

1. F. Ahmad, N.S. Barnett and S.S. Dragomir, *New weighted Ostrowski and Čebyšev type inequalities*, Nonlinear Analysis, 71 (2009), pp. 1408-1412.
2. N.S. Barnett and S.S. Dragomir, *An Ostrowski type inequality for double integrals and applications for cubature formulae*, Soochow J. Math., 27(1), (2001), pp. 1-10.
3. K. Boukerrioua and A. Guezane-Lakoud, *On generalization of Čebyšev type inequalities*, J. Inequal. Pure Appl. Math., 8(2), (2007), Art 55.
4. P.L. Čebyšev, *Sur les expressions approximatives des intégrales définies par les autres prises entre les mêmes limites*, Proc. Math. Soc. Charkov, 2(1882), pp. 93-98.
5. S. Sever Dragomir, N. Irshad and A.R. Khan, *Generalization of weighted Ostrowski-Gruss type inequality by using Korkine's identity*, Stud. Univ. Babes-Bolyai, Math., 65(2), (2020), pp. 183-198.

6. S.S. Dragomir, P. Cerone, N.S. Barnett and J. Roumeliotis, *An inequality of the Ostrowski type for double integrals and applications for cubature formulae*, Tamsui Oxf. J. Math. Sci., 16(1), (2000), pp. 1-16.
7. N. Irshad and A.R. Khan, *On Weighted Ostrowski Grüss Inequality with Applications*, Transylv. J. Math. Mech., 10(1), (2018), pp. 15-22.
8. A.R. Khan, J. Pečarić and M. Praljak, *Weighted Montgomery inequalities for higher order differentiable functions of two variables*, Rev. Anal. Numér. Théor. Approx., 42(1), (2013), pp. 49-71.
9. A. Guezane-Lakoud and F. Aissaoui, *New Čebyšev Type Inequalities For Double Integrals*, J. Math. Inequal., 5(4), (2011), pp. 453-462.
10. D.S. Mitrinović, J.E. Pečarić and A.M. Fink, *Inequalities Involving Functions and their Integrals and Derivatives*, Kluwer Academic Publishers, Dordrecht, 1991.
11. D.S. Mitrinović, J.E. Pečarić and A.M. Fink, *Classical and New Inequalities in Analysis*, Kluwer Academic Publishers, Dordrecht, 1993.
12. A.M. Ostrowski, *Über die absolutabweichung einer differentiebaren funktion von ihren integralmittelwert*, Comment. Math. Helv., 10 (1938), pp. 226-227.
13. B.G. Pachpatte, *On Čebyšev-Gruss type inequalities via Pecarics extention of the Montgomery identity*, J. Inequal. Pure Appl. Math., 7(1), (2006), Art 11.
14. J. Pečarić and A. Vukelić, *Montgomery's identities for function of two variables*, J. Math. Anal. Appl., 332 (2007), pp. 617-630.
15. G. Rahman, K. Sooppy Nisar, B. Ghanbari and T. Abdeljawad, *On generalized fractional integral inequalities for the monotone weighted Chebyshev functionals*, Adv. Difference Equ., (2020), 2020:368.
16. R.L. Wheeden and A. Zygmund, *Measure and Integral*, Marcel Dekker, The New York, 1977.

---

[1] Department of Mathematics, University of Karachi, University Road, Karachi-75270 Pakistan.

E-mail address: asifrk@uok.edu.pk

[2] Department of Mathematics, Federal Urdu University of Arts, Science and Technology , University Road, Karachi-75270 Pakistan.

E-mail address: hira.nasir@hotmail.com

[3] Department of Basic Sciences, Muhammad Ali Jinnah University, P.E.C.H.S. Main Shahrah-e-Faisal, Karachi-75400, Pakistan.

E-mail address: sikander.shirazi@jinnah.edu