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Introduction of Frame in Tensor Product of n-Hilbert Spaces

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Abstract. We study the concept of frame in tensor product of n-Hilbert spaces as tensor product of n-Hilbert spaces is again an n-Hilbert space. We generalize some of the known results about bases to frames in this new Hilbert space. A relationship between frame and bounded linear operator in tensor product of n-Hilbert spaces is studied. Finally, the dual frame in tensor product of n-Hilbert spaces is discussed.

1. Introduction

There are several techniques to rebuild signals by using a family of elementary signals. One of these techniques was given by D. Gabor in 1946 [10]. Thereafter frame in Hilbert space has been developed by Duffin and Schaeffer [7]. It is very useful to study nonharmonic Fourier series, i.e., sequences of the type $\left\{e^{i\lambda_n x}\right\}_{n\in\mathbb{Z}}$, where $\{\lambda_n\}_{n\in\mathbb{Z}}$ is a family of real or complex numbers. Daubechies et al, connected frames with Gabor system and Wavelet in 1986 [4]. To translate informations, frames are more flexible tools than bases and makes them very useful in signal processing [8], coding and communications [20], system modeling [6], filter bank theory [1], etc. In recent times, many generalizations of frames have been appeared. Some of them are g-frame [21], fusion frame [2] and g-fusion frame [19] etc. P. Ghosh and T. K. Samanta studied the stability of dual g-fusion frames and generalized atomic systems for operators in Hilbert spaces [11, 13].

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In 1970, Diminnie et al, introduced the concept of 2-inner product space [5]. A generalization of 2-inner product space for $n \geq 2$ was developed by A. Misiak in 1989 [16]. The basic concepts of tensor product of Hilbert spaces were presented by S. Rabinson [17].

In this paper, we study frame in the tensor product of n-Hilbert spaces and establish some of its properties. We note the result that in tensor product of n-Hilbert spaces, an image of a frame under a bounded linear operator is a frame if and only if the operator is invertible. Finally, dual of a frame in tensor product of n-Hilbert spaces is described.

Throughout this paper, X denotes the separable Hilbert space associated with the inner product $\langle \cdot, \cdot \rangle_1$ and $l^2(\mathbb{N})$ denotes the space of square summable scalar-valued sequences with index set of natural numbers $\mathbb{N}$.

2. Preliminaries

Definition 2.1 ([3])**.** A sequence $\{p_i\}_{i=1}^{\infty} \subseteq X$ is said to be a frame for X if there exist positive constants A, B such that

$$A\|p\|_1^2 \leq \sum_{i=1}^{\infty} | \langle p, p_i \rangle_1 |^2 \leq B\|p\|_1^2; \quad \forall p \in X.$$

The constants A and B are called frame bounds.

Definition 2.2 ([3])**.** Let $\{p_i\}_{i=1}^{\infty}$ be a frame for X. The synthesis operator, $T : l^2(\mathbb{N}) \to X$, defined by $T\{c_i\} = \sum_{i=1}^{\infty} c_i p_i$ is bounded operator and its adjoint, called the analysis operator, is given by $T^* p = \{\langle p, p_i \rangle_1\}_{i=1}^{\infty}$. The frame operator $S : X \to X$, is given by $Sp = TT^* p = \sum_{i=1}^{\infty} \langle p, p_i \rangle_1 p_i$, for all $p \in X$.

Definition 2.3 ([3])**.** A frame $\{q_i\}_{i=1}^{\infty}$ is said to be a dual of a frame $\{p_i\}_{i=1}^{\infty}$ for X if $p = \sum_{i=1}^{\infty} \langle p, q_i \rangle_1 p_i$, for all $p \in X$.

The tensor product of Hilbert spaces is introduced by several ways and it is a certain linear space of operators which was represented by Folland in [9], Kadison and Ringrose in [15].

Definition 2.4 ([18])**.** Let $(Y, \langle \cdot, \cdot \rangle_2)$ be a Hilbert space. Then the tensor product of Hilbert spaces X and Y is denoted by $X \otimes Y$ and is defined to be an inner product space with respect to the inner product:

$$\left\langle p \otimes q, p' \otimes q' \right\rangle = \left\langle p, p' \right\rangle_1 \left\langle q, q' \right\rangle_2, \quad \text{for all } p, p' \in X; q, q' \in Y.$$

The norm on $X \otimes Y$ is given by

$$\|p \otimes q\| = \|p\|_1 \|q\|_2, \quad \text{for all } p \in X \text{ and } q \in Y.$$

The space $X \otimes Y$ is complete with respect to the above inner product. Therefore the space $X \otimes Y$ is a Hilbert space.

Tensor product of operators $Q \in \mathcal{B}(X)$ and $T \in \mathcal{B}(Y)$, is denoted by $Q \otimes T$ and defined as $(Q \otimes T)\, A = Q A T^{*}$, for all $A \in X \otimes Y$. It can be easily verified that $Q \otimes T \in \mathcal{B}(X \otimes Y)$ [9].

Theorem 2.5 ([9])**.** *Suppose $Q, Q' \in \mathcal{B}(X)$ and $T, T' \in \mathcal{B}(Y)$. Then*

(i) $Q \otimes T \in \mathcal{B}(X \otimes Y)$ *and* $\|Q \otimes T\| = \|Q\| \|T\|$.
(ii) $(Q \otimes T)\,(f \otimes g) = Qf \otimes Tg$ *for all* $f \in X, g \in Y$.
(iii) $(Q \otimes T)\,(Q' \otimes T') = (QQ') \otimes (TT')$.
(iv) $Q \otimes T$ *is invertible if and only if* Q *and* T *are invertible, in which case* $(Q \otimes T)^{-1} = \left(Q^{-1} \otimes T^{-1}\right)$.
(v) $(Q \otimes T)^{*} = (Q^{*} \otimes T^{*})$.

Definition 2.6. [14] A real valued function $\|\cdot, \cdots, \cdot\| : H^{n} \to \mathbb{R}$ satisfying the following properties:

(i) $\|x_1, x_2, \ldots, x_n\| = 0$ if and only if $x_1, \ldots, x_n$ are linearly dependent,
(ii) $\|x_1, x_2, \ldots, x_n\|$ is invariant under permutations of $x_1, \ldots, x_n$,
(iii) $\|\alpha x_1, x_2, \ldots, x_n\| = |\alpha|\, \|x_1, x_2, \ldots, x_n\|$, $\alpha \in \mathbb{K}$,
(iv) $\|x + y, x_2, \ldots, x_n\| \leq \|x, x_2, \ldots, x_n\| + \|y, x_2, \ldots, x_n\|$,

for all $x_1, x_2, \ldots, x_n, x, y \in H$, is called n-norm on H. A linear space H, together with a n-norm $\|\cdot, \cdots, \cdot\|$, is called a linear n-normed space.

Definition 2.7 ([16])**.** Let $n \in \mathbb{N}$ and H be a linear space of dimension greater than or equal to n over the field $\mathbb{K}$, where $\mathbb{K}$ is the real or complex numbers field. A function $\langle \cdot, \cdot | \cdot, \cdots, \cdot \rangle : H^{n+1} \to \mathbb{K}$ satisfying the following five properties:

(i) $\langle x_1, x_1 | x_2, \ldots, x_n \rangle \geq 0$ and $\langle x_1, x_1 | x_2, \ldots, x_n \rangle = 0$ if and only if $x_1, x_2, \ldots, x_n$ are linearly dependent,
(ii) $\langle x, y | x_2, \ldots, x_n \rangle = \langle x, y | x_{i_2}, \ldots, x_{i_n} \rangle$ for every permutations $(i_2, \ldots, i_n)$ of $(2, \ldots, n)$,
(iii) $\langle x, y | x_2, \ldots, x_n \rangle = \overline{\langle y, x | x_2, \ldots, x_n \rangle}$,
(iv) $\langle \alpha x, y | x_2, \ldots, x_n \rangle = \alpha \langle x, y | x_2, \ldots, x_n \rangle, \alpha \in \mathbb{K}$
(v) $\langle x + y, z | x_2, \ldots, x_n \rangle = \langle x, z | x_2, \ldots, x_n \rangle + \langle y, z | x_2, \ldots, x_n \rangle$, for all $x, y, x_1, x_2, \ldots, x_n \in H$, is called an n-inner product on H, and the pair $(H, \langle \cdot, \cdot | \cdot, \cdots, \cdot \rangle)$ is called an n-inner product space.

Theorem 2.8 ([16])**.** *Let H be an n-inner product space. Then*

$$|\langle x, y | x_2, \ldots, x_n \rangle| \leq \|x, x_2, \ldots, x_n\| \, \|y, x_2, \ldots, x_n\|,$$

for all $x, y, x_2, \ldots, x_n \in H$, where

$$\|x_1, x_2, \ldots, x_n\| = \sqrt{\langle x_1, x_1 | x_2, \ldots, x_n \rangle},$$

is called Cauchy-Schwartz inequality.

Theorem 2.9 ([16])**.** *Let H be an n-inner product space. Then*

$$\|x_1, x_2, \ldots, x_n\| = \sqrt{\langle x_1, x_1 | x_2, \ldots, x_n \rangle}$$

defines an n-norm for which

$$\langle x, y | x_2, \ldots, x_n \rangle = \frac{1}{4}\left(\|x + y, x_2, \ldots, x_n\|^2 - \|x - y, x_2, \ldots, x_n\|^2 \right),$$

and

$$\|x + y, x_2, \ldots, x_n\|^2 + \|x - y, x_2, \ldots, x_n\|^2 = 2\left(\|x, x_2, \ldots, x_n\|^2 + \|y, x_2, \ldots, x_n\|^2 \right)$$

holds for all $x, y, x_1, x_2, \ldots, x_n \in H$.

Definition 2.10 ([14])**.** A sequence $\{x_k\}$ in linear n-normed space H is said to be convergent to $x \in H$ if

$$\lim_{k \to \infty} \|x_k - x, e_2, \ldots, e_n\| = 0$$

for every $e_2, \ldots, e_n \in H$ and it is called a Cauchy sequence if

$$\lim_{l, k \to \infty} \|x_l - x_k, e_2, \ldots, e_n\| = 0$$

for every $e_2, \ldots, e_n \in H$. The space H is said to be complete if every Cauchy sequence in this space is convergent in H. An n-inner product space is called n-Hilbert space if it is complete with respect to its induce norm.

3. Frame in Tensor Product of n-Hilbert Spaces

Definition 3.1. Let H be an n-Hilbert space and $a_2, \ldots, a_n \in H$. A sequence $\{p_i\}_{i=1}^{\infty}$ in H is said to be a frame associated to $(a_2, \ldots, a_n)$ for H if there exist constants $0 < A \leq B < \infty$ such that

$$A\,\|p, a_2, \ldots, a_n\|^2 \leq \sum_{i=1}^{\infty} |\langle p, p_i | a_2, \ldots, a_n \rangle|^2 \leq B\,\|p, a_2, \ldots, a_n\|^2 \tag{3.1}$$

for all $p \in H$. The constants A and B are called frame bounds. If $\{p_i\}_{i=1}^{\infty}$ satisfies the right hand inequality of (3.1), it is called a Bessel sequence associated to $(a_2, \ldots, a_n)$ in H with bound B.

By Cauchy-Schwartz inequality and the condition (i) of the definition 2.7, we may assume that every p_i and $a_2, \ldots, a_n$ are linearly independent.

Consider $F = \{a_2, a_3, \ldots, a_n\}$, where $a_2, a_3, \ldots, a_n$ are fixed elements in H and L_F denotes the linear subspace of H spanned by F. The quotient space H / L_F is a normed linear space with respect to the norm,

$\|p + L_F\|_F = \|p, a_2, \ldots, a_n\|$ for every $p \in H$. Let M_F be the algebraic complement of L_F, then $H = L_F \oplus M_F$. Define

$$\langle p, q \rangle_F = \langle p, q | a_2, \ldots, a_n \rangle \text{ on } H.$$

Then $\langle \cdot, \cdot \rangle_F$ is a semi-inner product on H and this induces an inner product on the quotient space H / L_F which is given by

$$\begin{aligned}\langle p + L_F, q + L_F \rangle_F &= \langle p, q \rangle_F \\ &= \langle p, q | a_2, \ldots, a_n \rangle ; \quad \forall p, q \in H.\end{aligned}$$

By identifying H / L_F with M_F in an obvious way, we obtain an inner product on M_F. Now, for every $p \in M_F$, we define $\|p\|_F = \sqrt{\langle p, p \rangle_F}$ and it can be easily verified that $(M_F, \|\cdot\|_F)$ is a norm space. Consider H_F as the completion of the inner product space M_F.

Theorem 3.2. *Let H be an n-Hilbert space. Then $\{p_i\}_{i=1}^{\infty} \subseteq H$ is a frame associated to $(a_2, \ldots, a_n)$ with bounds A, B if and only if it is a frame for the Hilbert space H_F with bounds A, B.*

Proof. Let $\{p_i\}_{i=1}^{\infty} \subseteq H$ be a frame associated to $(a_2, \ldots, a_n)$ for H with bounds A, B. Then the inequality (3.1) can be written as

$$A\|p\|_F^2 \leq \sum_{i=1}^{\infty} | \ \langle p, p_i \rangle_F |^2 \leq B\|p\|_F^2; \quad \forall p \in M_F.$$

This shows that $\{p_i\}_{i=1}^{\infty}$ is a frame for M_F. By the Lemma 5.1.2 of [3], the sequence $\{p_i\}_{i=1}^{\infty}$ is also a frame for H_F with the same bounds.

The converse is obvious. □

Definition 3.3. Let $\{p_i\}_{i=1}^{\infty}$ be a frame associated to $(a_2, \ldots, a_n)$ for H. Then the bounded linear operator $T_F : l^2(\mathbb{N}) \to H_F$, defined by $T_F\{c_i\} = \sum\limits_{i=1}^{\infty} c_i p_i$, is called pre-frame operator and its adjoint operator described by

$$T_F^* : H_F \to l^2(\mathbb{N}), T_F^* p = \{\langle p, p_i | a_2, \ldots, a_n \rangle\}_{i=1}^{\infty}$$

is called the analysis operator. The operator $S_F : H_F \to H_F$ given by

$$\begin{aligned}S_F p &= T_F T_F^* p \\ &= \sum_{i=1}^{\infty} \langle p, p_i | a_2, \ldots, a_n \rangle \, p_i, \quad \text{ for all } p \in H_F,\end{aligned}$$

is called the frame operator.

Theorem 3.4. *Let $\{p_i\}_{i=1}^{\infty}$ be a frame associated to $(a_2, \ldots, a_n)$ for H with bounds A, B. Then the corresponding frame operator S_F is bounded, invertible, self-adjoint and positive.*

Proof. For each $p \in H_F$, we have

$$\begin{aligned}
\|S_F p\|_F^2 &= \|S_F p, a_2, \ldots, a_n\|^2 \\
&= \sup \left\{ |\langle S_F p, q | a_2, \ldots, a_n \rangle|^2 : \|q, a_2, \ldots, a_n\| = 1 \right\} \\
&= \sup_{\|q, a_2, \ldots, a_n\| = 1} \left| \left\langle \sum_{i=1}^{\infty} \langle p, p_i | a_2, \ldots, a_n \rangle \, p_i, q | a_2, \ldots, a_n \right\rangle \right|^2 \\
&\leq \sup_{\|q, a_2, \ldots, a_n\| = 1} \sum_{i=1}^{\infty} |\langle p, p_i | a_2, \ldots, a_n \rangle|^2 \sum_{i=1}^{\infty} |\langle q, p_i | a_2, \ldots, a_n \rangle|^2 \\
&\quad [\text{using Cauchy-Schwartz iequality}] \\
&\leq B^2 \|p, a_2, \ldots, a_n\|^2 = B^2 \|p\|_F^2 \\
&\quad [\text{since } \{p_i\}_{i=1}^{\infty} \text{ is a frame associated to } (a_2, \ldots, a_n)]
\end{aligned}$$

This shows that S_F is bounded. Since $S_F = T_F T_F^*$, it is easy to verify that S_F is self-adjoint. The inequality (3.1), can be written as

$$A \langle p, p | a_2, \ldots, a_n \rangle \leq \langle S_F p, p | a_2, \ldots, a_n \rangle \leq B \langle p, p | a_2, \ldots, a_n \rangle$$

and this gives $A I_F \leq S_F \leq B I_F$. Thus, S_F is positive and consequently it is invertible. □

Remark 3.5. In Theorem 3.4, it is proved that $A I_F \leq S_F \leq B I_F$. Since S_F^{-1} commutes with both S_F and I_F, multiplying both the sides of the inequality $A I_F \leq S_F \leq B I_F$ by S_F^{-1}, we get $B^{-1} I_F \leq S_F^{-1} \leq A^{-1} I_F$.

For more details on frames in n-Hilbert spaces one can go through the paper [12].

Let H and K be two n-Hilbert spaces associated with the n-inner products $\langle \cdot, \cdot | \cdot, \cdots, \cdot \rangle_1$ and $\langle \cdot, \cdot | \cdot, \cdots, \cdot \rangle_2$, respectively. The tensor product of H and K is denoted by $H \otimes K$ and it is defined to be an n-inner product space associated with the n-inner product given by

$$\begin{aligned}
&\langle f_1 \otimes g_1, f_2 \otimes g_2 | f_3 \otimes g_3, \ldots, f_n \otimes g_n \rangle \\
&\qquad = \langle f_1, f_2 | f_3, \ldots, f_n \rangle_1 \, \langle g_1, g_2 | g_3, \ldots, g_n \rangle_2 \,,
\end{aligned} \tag{3.2}$$

for all $f_1, f_2, f_3, \ldots, f_n \in H$ and $g_1, g_2, g_3, \ldots, g_n \in K$.
The n-norm on $H \otimes K$ is defined by

$$\begin{aligned}
&\|f_1 \otimes g_1, f_2 \otimes g_2, \ldots, f_n \otimes g_n\| \\
&\qquad = \|f_1, f_2, \ldots, f_n\|_1 \, \|g_1, g_2, \ldots, g_n\|_2 \,,
\end{aligned} \tag{3.3}$$

for all $f_1, f_2, \ldots, f_n \in H$ and $g_1, g_2, \ldots, g_n \in K$, where the n-norms $\|\cdot, \cdots, \cdot\|_1$ and $\|\cdot, \cdots, \cdot\|_2$ are generated by $\langle \cdot, \cdot | \cdot, \cdots, \cdot \rangle_1$ and $\langle \cdot, \cdot | \cdot, \cdots, \cdot \rangle_2$, respectively. The space $H \otimes K$ is complete with respect to the above n-inner product. Therefore the space $H \otimes K$ is an n-Hilbert space.

Consider $G = \{b_2, b_3, \ldots, b_n\}$, where $b_2, b_3, \ldots, b_n$ are fixed elements in K and L_G denotes the linear subspace of K spanned by G. Now, the Hilbert space K_G with respect to the inner product is given by

$$\begin{aligned}\langle p + L_G, q + L_G \rangle_G &= \langle p, q \rangle_G \\ &= \langle p, q | b_2, \ldots, b_n \rangle_2 ; \quad \forall p, q \in K.\end{aligned}$$

Remark 3.6. According to the definition 2.4, $H_F \otimes K_G$ is the Hilbert space with respect to the inner product:

$$\langle p \otimes q, p' \otimes q' \rangle = \langle p, p' \rangle_F \langle q, q' \rangle_G ,$$

for all $p, p' \in H_F$ and $q, q' \in K_G$.

Remark 3.7. From the definition of ordinary frames for separable Hilbert spaces, the sequence of vectors $\{p_i \otimes q_j\}_{i,j=1}^{\infty}$ in $H \otimes K$ can be consider as a frame associated to $(a_2 \otimes b_2, \ldots, a_n \otimes b_n)$ for $H \otimes K$ if there exist constants $0 < A \leq B < \infty$ such that

$$\begin{aligned}(3.4) \quad & A \|p \otimes q, a_2 \otimes b_2, \ldots, a_n \otimes b_n\|^2 \\ &\leq \sum_{i,j=1}^{\infty} |\langle p \otimes q, p_i \otimes q_j | a_2 \otimes b_2, \ldots, a_n \otimes b_n \rangle|^2 \\ &\leq B \|p \otimes q, a_2 \otimes b_2, \ldots, a_n \otimes b_n\|^2 ; \quad \forall p \otimes q \in H \otimes K,\end{aligned}$$

where $\{p_i\}_{i=1}^{\infty}$ and $\{q_j\}_{j=1}^{\infty}$ are the sequences of vectors in H and K, respectively and $a_2 \otimes b_2, a_3 \otimes b_3, \ldots, a_n \otimes b_n$ are fixed elements in $H \otimes K$. The constants A, B are called the frame bounds. If $A = B$ then it is called a tight frame associated to $(a_2 \otimes b_2, \ldots, a_n \otimes b_n)$. If the sequence $\{p_i \otimes q_j\}_{i,j=1}^{\infty}$ satisfies the right hand side inequality of (3.4), it is called a Bessel sequence associated to $(a_2 \otimes b_2, \ldots, a_n \otimes b_n)$ in $H \otimes K$.

Theorem 3.8. *Let $\{p_i\}_{i=1}^{\infty}$ and $\{q_j\}_{j=1}^{\infty}$ be sequences of vectors in n-Hilbert spaces H and K. The sequence $\{p_i \otimes q_j\}_{i,j=1}^{\infty} \subseteq H \otimes K$ is a frame associated to $(a_2 \otimes b_2, \ldots, a_n \otimes b_n)$ for $H \otimes K$ if and only if $\{p_i\}_{i=1}^{\infty}$ is a frame associated to $(a_2, \ldots, a_n)$ for H and $\{q_j\}_{j=1}^{\infty}$ is a frame associated to $(b_2, \ldots, b_n)$ for K.*

Proof. Suppose that the sequence $\{p_i \otimes q_j\}_{i,j=1}^{\infty}$ is a frame associated to $(a_2 \otimes b_2, \ldots, a_n \otimes b_n)$ for $H \otimes K$. Then, for each $p \otimes q \in H \otimes K - \{\theta \otimes \theta\}$, there exist constants $A, B > 0$ such that

$$\begin{aligned}& A \|p \otimes q, a_2 \otimes b_2, \ldots, a_n \otimes b_n\|^2 \\ &\quad \leq \sum_{i,j=1}^{\infty} |\langle p \otimes q, p_i \otimes q_j | a_2 \otimes b_2, \ldots, a_n \otimes b_n \rangle|^2\end{aligned}$$

$$\leq B\,\|p\otimes q, a_2\otimes b_2, \ldots, a_n\otimes b_n\|^2\,.$$

Using the n-norm and n-inner product on $H\otimes K$, we get

$$\begin{aligned}
&A\,\|p, a_2, \ldots, a_n\|_1^2\,\|q, b_2, \ldots, b_n\|_2^2\\
&\quad\leq \left(\sum_{i=1}^{\infty}\left|\langle p, p_i|a_2, \ldots, a_n\rangle_1\right|^2\right)\times\left(\sum_{j=1}^{\infty}\left|\langle q, q_j|b_2, \ldots, b_n\rangle_2\right|^2\right)\\
&\quad\leq B\,\|p, a_2, \ldots, a_n\|_1^2\,\|q, b_2, \ldots, b_n\|_2^2\,.
\end{aligned}$$

Here, we may assume that each p_i and $a_2, \ldots, a_n$ are linearly independent and each q_j and $b_2, \ldots, b_n$ are linearly independent. Hence

$$\sum_{j=1}^{\infty}\left|\langle q, q_j|b_2, \ldots, b_n\rangle_2\right|^2, \qquad \sum_{i=1}^{\infty}\left|\langle p, p_i|a_2, \ldots, a_n\rangle_1\right|^2$$

are non-zero. Therefore, by the above inequality, we get

$$\begin{aligned}
&\frac{A\,\|q, b_2, \ldots, b_n\|_2^2\,\|p, a_2, \ldots, a_n\|_1^2}{\sum\limits_{j=1}^{\infty}\left|\langle q, q_j|b_2, \ldots, b_n\rangle_2\right|^2}\\
&\qquad\leq \sum_{i=1}^{\infty}\left|\langle p, p_i|a_2, \ldots, a_n\rangle_1\right|^2\\
&\qquad\leq \frac{B\,\|q, b_2, \ldots, b_n\|_2^2}{\sum\limits_{j=1}^{\infty}\left|\langle q, q_j|b_2, \ldots, b_n\rangle_2\right|^2}\,\|p, a_2, \ldots, a_n\|_1^2\,.
\end{aligned}$$

This implies that

$$A_1\,\|p, a_2, \ldots, a_n\|_1^2 \leq \sum_{i=1}^{\infty}\left|\langle p, p_i|a_2, \ldots, a_n\rangle_1\right|^2 \leq B_1\,\|p, a_2, \ldots, a_n\|_1^2\,,$$

for all $p\in H$, where

$$A_1 = \inf_{q\in K}\left\{\frac{A\,\|q, b_2, \ldots, b_n\|_2^2}{\sum\limits_{j=1}^{\infty}\left|\langle q, q_j|b_2, \ldots, b_n\rangle_2\right|^2}\right\}$$

and

$$B_1 = \sup_{q\in K}\left\{\frac{B\,\|q, b_2, \ldots, b_n\|_2^2}{\sum\limits_{j=1}^{\infty}\left|\langle q, q_j|b_2, \ldots, b_n\rangle_2\right|^2}\right\}.$$

This shows that $\{p_i\}_{i=1}^{\infty}$ is a frame associated to $(a_2, \ldots, a_n)$ for H. Similarly, it can be shown that $\{q_j\}_{j=1}^{\infty}$ is a frame associated to $(b_2, \ldots, b_n)$ for K.

Conversely, suppose that $\{p_i\}_{i=1}^{\infty}$ is a frame associated to $(a_2, \ldots, a_n)$ for H with bounds A, B and $\{q_j\}_{j=1}^{\infty}$ is a frame associated to $(b_2, \ldots, b_n)$ for K with bounds C, D. Then, for all $p \in H$ and $q \in K$, we have

$$A \left\|p, a_2, \ldots, a_n\right\|_1^2 \leq \sum_{i=1}^{\infty} \left|\langle p, p_i | a_2, \ldots, a_n\rangle_1\right|^2 \leq B \left\|p, a_2, \ldots, a_n\right\|_1^2,$$

$$C \left\|q, b_2, \ldots, b_n\right\|_2^2 \leq \sum_{j=1}^{\infty} \left|\langle q, q_j | b_2, \ldots, b_n\rangle_2\right|^2 \leq D \left\|q, b_2, \ldots, b_n\right\|_2^2.$$

Multiplying the above two inequalities with each and using (3.2) and (3.3), we get

$$\begin{aligned} &AC \left\|p \otimes q, a_2 \otimes b_2, \ldots, a_n \otimes b_n\right\|^2 \\ &\quad \leq \sum_{i,j=1}^{\infty} \left|\langle p \otimes q, p_i \otimes q_j | a_2 \otimes b_2, \ldots, a_n \otimes b_n\rangle\right|^2 \\ &\quad \leq BD \left\|p \otimes q, a_2 \otimes b_2, \ldots, a_n \otimes b_n\right\|^2. \end{aligned}$$

Hence, $\{p_i \otimes q_j\}_{i,j=1}^{\infty}$ is a frame associated to $(a_2 \otimes b_2, \ldots, a_n \otimes b_n)$ for $H \otimes K$. □

Theorem 3.9. *The sequence $\{p_i \otimes q_j\}_{i,j=1}^{\infty} \subseteq H \otimes K$ is a frame associated to $(a_2 \otimes b_2, \ldots, a_n \otimes b_n)$ for $H \otimes K$ if and only if $\{p_i \otimes q_j\}_{i,j=1}^{\infty}$ is a frame for $H_F \otimes K_G$.*

Proof. Suppose that the sequence $\{p_i \otimes q_j\}_{i,j=1}^{\infty}$ is a frame associated to $(a_2 \otimes b_2, \ldots, a_n \otimes b_n)$ for $H \otimes K$. By Theorem 3.8, $\{p_i\}_{i=1}^{\infty}$ is a frame associated to $(a_2, \ldots, a_n)$ for H and $\{q_j\}_{j=1}^{\infty}$ is a frame associated to $(b_2, \ldots, b_n)$ for K. Now, applying Theorem 3.2, $\{p_i\}_{i=1}^{\infty}$ and $\{q_j\}_{j=1}^{\infty}$ are frames for H_F and K_G, respectively. Hence, $\{p_i \otimes q_j\}_{i,j=1}^{\infty}$ is a frame for $H_F \otimes K_G$.

The proof of the converse is obvious. □

Remark 3.10. Let $\{p_i \otimes q_j\}_{i,j=1}^{\infty}$ be a frame associated to $(a_2 \otimes b_2, \ldots, a_n \otimes b_n)$ for $H \otimes K$. According to the definition 3.3, the frame operator $S_{F \otimes G} : H_F \otimes K_G \to H_F \otimes K_G$ is described by

$$S_{F \otimes G}(p \otimes q) = \sum_{i,j=1}^{\infty} \langle p \otimes q, p_i \otimes q_j | a_2 \otimes b_2, \ldots, a_n \otimes b_n\rangle \left(p_i \otimes q_j\right)$$

for all $p \otimes q \in H_F \otimes K_G$.

Theorem 3.11. *If S_F, S_G and $S_{F\otimes G}$ are the corresponding frame operators for $\{p_i\}_{i=1}^{\infty}, \{q_j\}_{j=1}^{\infty}$ and $\{p_i \otimes q_j\}_{i,j=1}^{\infty}$, respectively, then $S_{F\otimes G} = S_F \otimes S_G$ and $S_{F\otimes G}^{-1} = S_F^{-1} \otimes S_G^{-1}$.*

Proof. Since $S_{F\otimes G}$ is the frame operator for $\{p_i \otimes q_j\}_{i,j=1}^{\infty}$, we have

$$\begin{aligned}
S_{F\otimes G}(p\otimes q) &= \sum_{i,j=1}^{\infty} \langle p\otimes q, p_i \otimes q_j | a_2 \otimes b_2, \ldots, a_n \otimes b_n\rangle (p_i \otimes q_j) \\
&= \sum_{i,j=1}^{\infty} \langle p, p_i | a_2, \ldots, a_n\rangle_1 \langle q, q_j | b_2, \ldots, b_n\rangle_2 (p_i \otimes q_j) \\
&= \left(\sum_{i=1}^{\infty} \langle p, p_i | a_2, \ldots, a_n\rangle_1 p_i\right) \otimes \left(\sum_{j=1}^{\infty} \langle q, q_j | b_2, \ldots, b_n\rangle_2 q_j\right) \\
&= S_F(p) \otimes S_G(q) \\
&= (S_F \otimes S_G)(p\otimes q), \quad \forall p\otimes q \in H_F \otimes K_G.
\end{aligned}$$

Thus $S_{F\otimes G} = S_F \otimes S_G$. Since S_F and S_G are invertible, by Theorem 2.5 (iv), $S_{F\otimes G}^{-1} = (S_F \otimes S_G)^{-1} = S_F^{-1} \otimes S_G^{-1}$. □

Theorem 3.12. *Let $\{p_i\}_{i=1}^{\infty}$ be a frame associated to $(a_2, \ldots, a_n)$ for H with bounds A, B and $\{q_j\}_{j=1}^{\infty}$ be a frame associated to $(b_2, \ldots, b_n)$ for K with bounds C, D with their corresponding frame operators S_F and S_G, respectively. Then $\Lambda = \left\{S_{F\otimes G}^{-1}(p_i \otimes q_j)\right\}_{i,j=1}^{\infty}$ is a frame associated to $(a_2 \otimes b_2, \ldots, a_n \otimes b_n)$ for $H \otimes K$ with the corresponding frame operator $S_{F\otimes G}^{-1}$.*

Proof. For each $p\otimes q \in H_F \otimes K_G$, we have

(3.5)

$$\begin{aligned}
&\sum_{i,j=1}^{\infty} \left|\left\langle p\otimes q, S_{F\otimes G}^{-1}(p_i \otimes q_j) | a_2 \otimes b_2, \ldots, a_n \otimes b_n\right\rangle\right|^2 \\
&= \sum_{i,j=1}^{\infty} \left|\left\langle p\otimes q, \left(S_F^{-1} \otimes S_G^{-1}\right)(p_i \otimes q_j) | a_2 \otimes b_2, \ldots, a_n \otimes b_n\right\rangle\right|^2 \\
&= \sum_{i,j=1}^{\infty} \left|\left\langle p\otimes q, S_F^{-1}(p_i) \otimes S_G^{-1}(q_j) | a_2 \otimes b_2, \ldots, a_n \otimes b_n\right\rangle\right|^2 \\
&= \sum_{i=1}^{\infty} \left|\left\langle p, S_F^{-1} p_i | a_2, \ldots, a_n\right\rangle_1\right|^2 \sum_{j=1}^{\infty} \left|\left\langle q, S_G^{-1} q_j | b_2, \ldots, b_n\right\rangle_2\right|^2 \\
&= \sum_{i=1}^{\infty} \left|\left\langle S_F^{-1} p, p_i | a_2, \ldots, a_n\right\rangle_1\right|^2 \sum_{j=1}^{\infty} \left|\left\langle S_G^{-1} q, q_j | b_2, \ldots, b_n\right\rangle_2\right|^2
\end{aligned}$$

$\leq B \left\|S_F^{-1}(p), a_2, \ldots, a_n\right\|_1^2 D \left\|S_G^{-1}(q), b_2, \ldots, b_n\right\|_2^2$

[since $\{p_i\}_{i=1}^{\infty}$ is a frame associated to $(a_2, \ldots, a_n)$, and $\{q_j\}_{j=1}^{\infty}$ is a frame associated to $(b_2, \ldots, b_n)$]

$\leq BD \left\|S_F^{-1}\right\|^2 \left\|S_G^{-1}\right\|^2 \|p, a_2, \ldots, a_n\|_1^2 \|q, b_2, \ldots, b_n\|_2^2$

$\left[\text{since } S_F^{-1}, S_G^{-1} \text{ are bounded operators on } H_F \text{ and } K_G\text{, respectively}\right]$

$\leq \dfrac{BD}{A^2C^2} \|p \otimes q, a_2 \otimes b_2 \ldots, a_n \otimes b_n\|^2$ [using (3.3)]

[since $B^{-1}I_F \leq S_F^{-1} \leq A^{-1}I_F$ and $D^{-1}I_G \leq S_G^{-1} \leq C^{-1}I_G$ where, I_G denotes the identity operator on K_G].

Now, for $p \in H_F$, we have

$$\begin{aligned} \|p, a_2, \ldots, a_n\|_1 &\leq \|S_F\| \left\|S_F^{-1}(p), a_2, \ldots, a_n\right\|_1 \\ &\leq B \left\|S_F^{-1}(p), a_2, \ldots, a_n\right\|_1, \end{aligned} \tag{3.6}$$

and similarly for $q \in K_G$, we get

$$\|q, b_2, \ldots, b_n\|_2 \leq D \left\|S_G^{-1}(q), b_2, \ldots, b_n\right\|_2. \tag{3.7}$$

On the other hand, from (3.5),

$$\begin{aligned} &\sum_{i,j=1}^{\infty} \left|\left\langle p \otimes q, S_{F\otimes G}^{-1}\left(p_i \otimes q_j\right) | a_2 \otimes b_2, \ldots, a_n \otimes b_n\right\rangle\right|^2 \\ &\quad= \sum_{i=1}^{\infty} \left|\left\langle S_F^{-1}p, p_i | a_2, \ldots, a_n\right\rangle_1\right|^2 \sum_{j=1}^{\infty} \left|\left\langle S_G^{-1}q, q_j | b_2, \ldots, b_n\right\rangle_2\right|^2 \\ &\quad\geq A \left\|S_F^{-1}(p), a_2, \ldots, a_n\right\|_1^2 C \left\|S_G^{-1}(q), b_2, \ldots, b_n\right\|_2^2 \\ &\quad\geq \frac{AC}{B^2D^2} \|p, a_2, \ldots, a_n\|_1^2 \|q, b_2, \ldots, b_n\|_2^2 \text{ [by (3.6) and (3.7)]} \\ &\quad= \frac{AC}{B^2D^2} \|p \otimes q, a_2 \otimes b_2 \ldots, a_n \otimes b_n\|^2. \end{aligned}$$

Hence, Λ is a frame associated to $(a_2 \otimes b_2, \ldots, a_n \otimes b_n)$ for $H \otimes K$ with bounds $\dfrac{AC}{B^2D^2}$ and $\dfrac{BD}{A^2C^2}$.

Furthermore, for each $p \otimes q \in H_F \otimes K_G$, we have

$$\sum_{i,j=1}^{\infty} \left\langle p \otimes q, S_{F\otimes G}^{-1}\left(p_i \otimes q_j\right) | a_2 \otimes b_2, \ldots, a_n \otimes b_n\right\rangle S_{F\otimes G}^{-1}\left(p_i \otimes q_j\right)$$

$$= \sum_{i,j=1}^{\infty} \left\langle p \otimes q, S_F^{-1} p_i \otimes S_G^{-1} q_j | a_2 \otimes b_2, \ldots, a_n \otimes b_n \right\rangle S_F^{-1} p_i \otimes S_G^{-1} q_j$$
$$= \left(\sum_{i=1}^{\infty} \left\langle S_F^{-1} p, p_i | a_2, \ldots, a_n \right\rangle_1 S_F^{-1} p_i \right)$$
$$\otimes \left(\sum_{j=1}^{\infty} \left\langle S_G^{-1} q, q_j | b_2, \ldots, b_n \right\rangle_2 S_G^{-1} q_j \right)$$
$$= S_F^{-1} S_F \left(S_F^{-1} p \right) \otimes S_G^{-1} S_G \left(S_G^{-1} q \right)$$
$$= S_F^{-1} p \otimes S_G^{-1} q = S_{F \otimes G}^{-1} (p \otimes q).$$

Hence, the corresponding frame operator for Λ is $S_{F \otimes G}^{-1}$. □

Theorem 3.13. *Let $\{p_i\}_{i=1}^{\infty}$ be a frame associated to $(a_2, \ldots, a_n)$ for H with bounds A, B and $\{q_j\}_{j=1}^{\infty}$ be a frame associated to $(b_2, \cdots, b_n)$ for K with bounds C, D with their corresponding frame operators S_F and S_G, respectively. Then $\{\Delta_{ij} = (U_1 \otimes U_2)(p_i \otimes q_j)\}_{i,j=1}^{\infty}$ is a frame associated to $(a_2 \otimes b_2, \cdots, a_n \otimes b_n)$ for $H \otimes K$ if and only if $U_1 \otimes U_2$ is an invertible operator on $H_F \otimes K_G$.*

Proof. First we suppose that $U_1 \otimes U_2$ is an invertible on $H_F \otimes K_G$. Then by Theorem 2.5, U_1 and U_2 are invertible on H_F and K_G, respectively. For each $p \in H_F$ and $q \in K_G$, we have

$$\|p, a_2, \ldots, a_n\|_1 \leq \left\|U_1^{-1}\right\| \|U_1^*(p), a_2, \ldots, a_n\|_1 , \tag{3.8}$$

and

$$\|q, b_2, \ldots, b_n\|_2 \leq \left\|U_2^{-1}\right\| \|U_2^*(q), b_2, \ldots, b_n\|_2 . \tag{3.9}$$

Now, for each $p \otimes q \in H_F \otimes K_G$, we have

$$\sum_{i,j=1}^{\infty} \left|\left\langle p \otimes q, (U_1 \otimes U_2)(p_i \otimes q_j) | a_2 \otimes b_2, \ldots, a_n \otimes b_n \right\rangle\right|^2 \tag{3.10}$$
$$= \sum_{i,j=1}^{\infty} \left|\left\langle p \otimes q, U_1(p_i) \otimes U_2(q_j) | a_2 \otimes b_2, \ldots, a_n \otimes b_n \right\rangle\right|^2$$
$$= \sum_{i=1}^{\infty} \left|\left\langle U_1^* p, p_i | a_2, \ldots, a_n \right\rangle_1\right|^2 \sum_{j=1}^{\infty} \left|\left\langle U_2^* q, q_j | b_2, \ldots, b_n \right\rangle_2\right|^2$$
$$\leq BD \|U_1^*(p), a_2, \ldots, a_n\|_1^2 \|U_2^*(q), b_2, \ldots, b_n\|_2^2$$

[since $\{p_i\}_{i=1}^{\infty}$ is a frame associated to $(a_2, \ldots, a_n)$, and $\{q_j\}_{j=1}^{\infty}$ is a frame associated to $(b_2, \ldots, b_n)$]

$$\leq BD \|U_1^*\|^2 \|U_2^*\|^2 \|p, a_2, \ldots, a_n\|_1^2 \|q, b_2, \ldots, b_n\|_2^2$$

$$= BD \|U_1 \otimes U_2\|^2 \|p \otimes q, a_2 \otimes b_2 \ldots, a_n \otimes b_n\|^2 .$$

On the other hand, from (3.10)

$$\sum_{i,j=1}^{\infty} |\langle p \otimes q, (U_1 \otimes U_2)(p_i \otimes q_j) | a_2 \otimes b_2, \ldots, a_n \otimes b_n\rangle|^2$$
$$\geq AC \|U_1^*(p), a_2, \ldots, a_n\|_1^2 \|U_2^*(q), b_2, \ldots, b_n\|_2^2$$
$$\geq \frac{AC \|p, a_2, \ldots, a_n\|_1^2 \|q, b_2, \ldots, b_n\|_2^2}{\|U_1^{-1}\|^2 \|U_2^{-1}\|^2} [\text{by (3.8) and (3.9)}]$$
$$= \frac{AC}{\left\|(U_1 \otimes U_2)^{-1}\right\|^2} \|p \otimes q, a_2 \otimes b_2, \ldots, a_n \otimes b_n\|^2 .$$

Therefore, $\{\Delta_{ij}\}_{i,j=1}^{\infty}$ is a frame associated to $(a_2 \otimes b_2, \ldots, a_n \otimes b_n)$ for $H \otimes K$.

Conversely, suppose that $\{\Delta_{ij}\}_{i,j=1}^{\infty}$ is a frame associated to $(a_2 \otimes b_2, \ldots, a_n \otimes b_n)$ for $H \otimes K$. Now, for each $p \otimes q \in H_F \otimes K_G$,

$$\sum_{i,j=1}^{\infty} \langle p \otimes q, \Delta_{ij} | a_2 \otimes b_2, \ldots, a_n \otimes b_n\rangle \Delta_{ij}$$
$$= \sum_{i,j=1}^{\infty} \langle p \otimes q, U_1 p_i \otimes U_2 q_j | a_2 \otimes b_2, \ldots, a_n \otimes b_n\rangle (U_1 p_i \otimes U_2 q_j)$$
$$= \sum_{i=1}^{\infty} \langle U_1^* p, p_i | a_2, \ldots, a_n\rangle_1 U_1 p_i \otimes \sum_{j=1}^{\infty} \langle U_2^* q, q_j | b_2, \ldots, b_n\rangle_2 U_2 q_j$$
$$= U_1 S_F U_1^*(p) \otimes U_2 S_G U_2^*(q)$$
$$= (U_1 \otimes U_2)(S_F \otimes S_G)(U_1^* \otimes U_2^*)(p \otimes q)$$
$$= (U_1 \otimes U_2) S_{F \otimes G} (U_1 \otimes U_2)^* (p \otimes q).$$

Hence, the frame operator for $\{\Delta_{ij}\}_{i,j=1}^{\infty}$ is $(U_1 \otimes U_2) S_{F \otimes G} (U_1 \otimes U_2)^*$ and therefore it is invertible. Also, we know that $S_{F \otimes G}$ is invertible and hence $U_1 \otimes U_2$ is invertible on $H_F \otimes K_G$. □

4. Dual frame in tensor product of n-Hilbert spaces

In this section, dual frame in n-Hilbert spaces and their tensor product are discussed.

Definition 4.1. Let $\{p_i\}_{i=1}^{\infty}$ be a frame associated to $(a_2, \ldots, a_n)$ for H. Then a frame $\{q_i\}_{i=1}^{\infty}$ associated to $(a_2, \ldots, a_n)$ satisfying

$$p = \sum_{i=1}^{\infty} \langle p, q_i | a_2, \ldots, a_n \rangle_1 \, p_i, \quad \forall p \in H$$

is called a dual frame or alternative dual frame associated to $(a_2, \ldots, a_n)$ of $\{p_i\}_{i=1}^{\infty}$.

Theorem 4.2. *Let $\{p_i\}_{i=1}^{\infty}$ and $\{q_i\}_{i=1}^{\infty}$ be two Bessel sequences associated to $(a_2, \ldots, a_n)$ in H. Then the following are equivalent:*

(i) $p = \sum\limits_{i=1}^{\infty} \langle p, q_i | a_2, \ldots, a_n \rangle_1 \, p_i; \ \forall p \in H_F$.

(ii) $p = \sum\limits_{i=1}^{\infty} \langle p, p_i | a_2, \ldots, a_n \rangle_1 \, q_i; \ \forall p \in H_F$.

Proof. *(i)* $\Rightarrow$ *(ii)* Let T_F and T_G be the pre-frame operators of $\{p_i\}_{i=1}^{\infty}$ and $\{q_i\}_{i=1}^{\infty}$, respectively. Composing T_F with the adjoint of T_G, for all $p \in H_F$, we get

$$T_F T_G^{*} : H_F \to H_F, T_F T_G^{*}(p) = \sum_{i=1}^{\infty} \langle p, q_i | a_2, \ldots, a_n \rangle_1 \, p_i.$$

Now, in terms of pre-frame operators (i) can be written as $T_F T_G^{*} = I_F$ and this is equivalent to $T_G T_F^{*} = I_F$. Therefore, for each $p \in H_F$,

$$p = \sum_{i=1}^{\infty} \langle p, p_i | a_2, \ldots, a_n \rangle_1 \, q_i.$$

Similarly, (ii) $\Rightarrow$ (i) follows. □

Remark 4.3. Suppose that the equivalent conditions of Theorem 4.2 are satisfied. Then using Cauchy-Schwartz inequality, for every $p \in H_F$, we have

$$\begin{aligned}
\|p, a_2, \ldots, a_n\|_1^2 &= \langle p, p | a_2, \ldots, a_n \rangle_1 \\
&= \left\langle \sum_{i=1}^{\infty} \langle p, p_i | a_2, \ldots, a_n \rangle_1 \, q_i, p | a_2, \ldots, a_n \right\rangle_1 \\
&= \sum_{i=1}^{\infty} \langle p, p_i | a_2, \ldots, a_n \rangle_1 \, \langle q_i, p | a_2, \ldots, a_n \rangle_1 \\
&\leq \left(\sum_{i=1}^{\infty} \left| \langle p, p_i | a_2, \ldots, a_n \rangle_1 \right|^2 \right)^{1/2} \left(\sum_{i=1}^{\infty} \left| \langle p, q_i | a_2, \ldots, a_n \rangle_1 \right|^2 \right)^{1/2}
\end{aligned}$$

$$\leq \left(\sum_{i=1}^{\infty} |\langle p, p_i | a_2, \ldots, a_n \rangle_1|^2\right)^{1/2} B^{1/2} \|p, a_2, \ldots, a_n\|_1$$

[since $\{q_i\}_{i=1}^{\infty}$ is a Bessel sequences associated to $(a_2, \ldots, a_n)$]

$$\Rightarrow \frac{1}{B} \|p, a_2, \ldots, a_n\|_1^2 \leq \sum_{i=1}^{\infty} |\langle p, p_i | a_2, \ldots, a_n \rangle_1|^2.$$

This shows that $\{p_i\}_{i=1}^{\infty}$ is a frame associated to $(a_2, \ldots, a_n)$ for H. Similarly, it can be shown that $\{q_i\}_{i=1}^{\infty}$ is also a frame associated to $(a_2, \ldots, a_n)$ for H.

We now present the concept of a dual frame in $H \otimes K$.

Remark 4.4. Let $\{p_i \otimes q_j\}_{i,j=1}^{\infty}$ be a frame associated to $(a_2 \otimes b_2, \cdots, a_n \otimes b_n)$ for $H \otimes K$. Then according to the definition 4.1, a frame $\{e_i \otimes h_j\}_{i,j=1}^{\infty}$ associated to $(a_2 \otimes b_2, \ldots, a_n \otimes b_n)$ for $H \otimes K$ satisfying

$$p \otimes q = \sum_{i,j=1}^{\infty} \langle p \otimes q, e_i \otimes h_j | a_2 \otimes b_2, \ldots, a_n \otimes b_n \rangle (p_i \otimes q_j), \tag{4.1}$$

for all $p \otimes q \in H \otimes K$, can be consider as a dual frame associated to $(a_2 \otimes b_2, \cdots, a_n \otimes b_n)$ of $\{p_i \otimes q_j\}_{i,j=1}^{\infty}$.

Remark 4.5. According to the Theorem 3.2, $\{p_i \otimes q_j\}_{i,j=1}^{\infty}$ and $\{e_i \otimes h_j\}_{i,j=1}^{\infty}$ are pair of dual frames associated to $(a_2 \otimes b_2, \cdots, a_n \otimes b_n)$ for $H \otimes K$ if and only if $\{p_i \otimes q_j\}_{i,j=1}^{\infty}$ and $\{e_i \otimes h_j\}_{i,j=1}^{\infty}$ are pair of dual frames for $H_F \otimes K_G$.

Theorem 4.6. *Let $\{p_i\}_{i=1}^{\infty}, \{e_i\}_{i=1}^{\infty}$ be a pair of dual frames associated to $(a_2, \ldots, a_n)$ for H and $\{q_j\}_{j=1}^{\infty}, \{h_j\}_{j=1}^{\infty}$ be a pair of dual frames associated to $(b_2, \ldots, b_n)$ for K. Then $\{e_i \otimes h_j\}_{i,j=1}^{\infty}$ is a dual frame associated to $(a_2 \otimes b_2, \ldots, a_n \otimes b_n)$ of $\{p_i \otimes q_j\}_{i,j=1}^{\infty}$.*

Proof. By Theorem 3.8, $\{p_i \otimes q_j\}_{i,j=1}^{\infty}$, $\{e_i \otimes h_j\}_{i,j=1}^{\infty}$ are frames associated to $(a_2 \otimes b_2, \ldots, a_n \otimes b_n)$ for $H \otimes K$. Since $\{e_i\}_{i=1}^{\infty}$ and $\{h_j\}_{j=1}^{\infty}$ are dual frames associated to $(a_2, \ldots, a_n)$ and $(b_2, \ldots, b_n)$ of $\{p_i\}_{i=1}^{\infty}$ and $\{q_j\}_{j=1}^{\infty}$, respectively, for all $p \in H$, $q \in K$,

$$p = \sum_{i=1}^{\infty} \langle p, e_i | a_2, \ldots, a_n \rangle_1 p_i, , \qquad q = \sum_{j=1}^{\infty} \langle q, h_j | b_2, \ldots, b_n \rangle_2 q_j.$$

Then, for all $p \otimes q \in H \otimes K$, we have

$$p \otimes q = \left(\sum_{i=1}^{\infty} \langle p, e_i | a_2, \ldots, a_n \rangle_1 p_i\right) \otimes \left(\sum_{j=1}^{\infty} \langle q, h_j | b_2, \ldots, b_n \rangle_2 q_j\right)$$

$$= \sum_{i,j=1}^{\infty} \langle p, e_i | a_2, \ldots, a_n \rangle_1 \langle q, h_j | b_2, \ldots, b_n \rangle_2 (p_i \otimes q_j)$$

$$= \sum_{i,j=1}^{\infty} \langle p \otimes q, e_i \otimes h_j | a_2 \otimes b_2, \ldots, a_n \otimes b_n \rangle (p_i \otimes q_j).$$

This completes the proof. □

Theorem 4.7. *Let $\{p_i\}_{i=1}^{\infty}, \{e_i\}_{i=1}^{\infty}$ be a pair of dual frames associated to $(a_2, \ldots, a_n)$ for H and $\{q_j\}_{j=1}^{\infty}, \{h_j\}_{j=1}^{\infty}$ be a pair of dual frames associated to $(b_2, \ldots, b_n)$ for K. Suppose $U \in \mathcal{B}(H_F)$ and $V \in \mathcal{B}(K_G)$ are unitary operators. Then $\Lambda = \{(U \otimes V)(p_i \otimes q_j)\}_{i,j=1}^{\infty}$ and $\Gamma = \{(U \otimes V)(e_i \otimes h_j)\}_{i,j=1}^{\infty}$ also form a pair of dual frames associated to $(a_2 \otimes b_2, \ldots, a_n \otimes b_n)$ for $H \otimes K$.*

Proof. By Theorem 4.6, $\{p_i \otimes q_j\}_{i,j=1}^{\infty}$ and $\{e_i \otimes h_j\}_{i,j=1}^{\infty}$ form a pair of dual frames associated to $(a_2 \otimes b_2, \ldots, a_n \otimes b_n)$ for $H \otimes K$. Now, for each $p \otimes q \in H_F \otimes K_G$, we have

$$\sum_{i,j=1}^{\infty} |\langle p \otimes q, (U \otimes V)(e_i \otimes h_j) | a_2 \otimes b_2, \ldots, a_n \otimes b_n \rangle|^2$$

$$= \sum_{i,j=1}^{\infty} |\langle p \otimes q, (Ue_i \otimes Vh_j) | a_2 \otimes b_2, \ldots, a_n \otimes b_n \rangle|^2$$

$$= \sum_{i=1}^{\infty} |\langle U^*p, e_i | a_2, \ldots, a_n \rangle_1|^2 \otimes \sum_{j=1}^{\infty} \left|\langle V^*q, h_j | b_2, \ldots, b_n \rangle_2 \, q_j\right|^2.$$

Since $\{e_i\}_{i=1}^{\infty}$ is a frame associated to $(a_2, \ldots, a_n)$ for H and $\{h_j\}_{j=1}^{\infty}$ is a frame associated to $(b_2, \ldots, b_n)$ for K, the above calculation shows that Γ is a frame associated to $(a_2 \otimes b_2, \ldots, a_n \otimes b_n)$ for $H \otimes K$. Similarly, it can be shown that Λ is a frame associated to $(a_2 \otimes b_2, \ldots, a_n \otimes b_n)$ for $H \otimes K$. Furthermore, for each $p \otimes q \in H_F \otimes K_G$, we have

$$\sum_{i,j=1}^{\infty} \langle p \otimes q, (U \otimes V)(e_i \otimes h_j) | a_2 \otimes b_2, \ldots, a_n \otimes b_n \rangle (U \otimes V)(p_i \otimes q_j)$$

$$= \sum_{i,j=1}^{\infty} \langle p \otimes q, (Ue_i \otimes Vh_j) | a_2 \otimes b_2, \ldots, a_n \otimes b_n \rangle (Up_i \otimes Vq_j)$$

$$= U \sum_{i=1}^{\infty} \langle U^*p, e_i | a_2, \ldots, a_n \rangle_1 \, p_i \otimes V \sum_{j=1}^{\infty} \langle V^*q, h_j | b_2, \ldots, b_n \rangle_2 \, q_j$$

$$= UU^*(p) \otimes VV^*(q) = p \otimes q [\text{since } U, V \text{ are unitary operators}].$$

[Since $\{p_i\}_{i=1}^{\infty}$, $\{e_i\}_{i=1}^{\infty}$ are dual frames associated to $(a_2, \ldots, a_n)$, and $\{q_j\}_{j=1}^{\infty}, \{h_j\}_{j=1}^{\infty}$ are dual frames associated to $(b_2, \ldots, b_n)$].

Hence, according to the remark 4.5, Λ and Γ form a pair of dual frames associated to $(a_2 \otimes b_2, \ldots, a_n \otimes b_n)$ for $H \otimes K$. □

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