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Investigation of the Boundary Layers of the Singular Perturbation Problem Including the Cauchy-Euler Differential Equation

Alireza Sarakhsi[1] and Siamak Ashrafi[2*]

Abstract. In this paper, for a singular perturbation problem consist of the Cauchy-Euler equation with local and non-local boundary conditions. We investigate the condition of the self-adjoint and the non-self-adjoint, also look for the formation or non-formation of boundary layers for local boundary conditions using the Frequent uniform limit method. Also, for the state of non-local conditions, we convert the non-local boundary conditions into local conditions by finding the fundamental solution and then obtaining the necessary conditions with the help of the 4-step method. Finally, we determine the formation or non-formation of a boundary layer for non-local conditions such as local conditions.

1. Introduction

One of the important subjects in applied mathematics is the theory of singular perturbation problems. The mathematical model for this kind of problem usually is in the form of either ordinary differential equations (ODE) or partial differential equations (PDE) in which the highest derivative is multiplied by some powers of ε as a positive small parameter [4] and[5]. The object theory of singular perturbation is to solve differential equation with some initial or boundary conditions with small parameter ε. These problems are essential in the heart problems of boundary value and initial value [5]. In book [1], which consists of three chapters, in the first chapter, there are some equations such as parameter. In chapter two, they studied boundary values that include

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these equations, and chapter three suggesting to solve those approximates using the software. Through these studies, we can find out that if the boundary conditions become local (Dirichlet) and the solution of the problem in the boundary layer verifies then in none of the boundary points was not any boundary layer. If the limitation of the problem solution in boundary condition chosen point does not adjust when $\varepsilon \to 0$ then there will be a boundary layer. If the limitation of problem solution when, $\varepsilon \to 0$ does not adjust in any of boundary points, so in both the boundary points will be boundary layer. In the book [1], after the first and second chapters, we see the non-solved problems of the boundary layer which shows that boundary layer problems with non-local boundary conditions have not been studied carefully. So, in this paper, for a singular perturbation problem consist of the Cauchy-Euler equation with local and non-local boundary conditions, we investigate the condition of the self-adjoint and the non-self-adjoint, also look for the existence or non-existence of boundary layers for local boundary conditions using the Frequent uniform limit method, Also, for the state of non-local conditions, we convert the non-local boundary conditions into local conditions by finding the fundamental solution and then obtaining the necessary conditions with the help of the 4-step method [6, 10–12] and finally we determine the existence or non-existence of a boundary layer for non-local conditions such as local conditions. Now we consider,

$$\varepsilon x^2 {y''}_{\varepsilon}(x) + ax{y'}_{\varepsilon}(x) + by_{\varepsilon}(x) = 0, \quad x \in (1, e) \tag{1.1}$$

$$ {y'}_{\varepsilon}(1) = \alpha, \qquad y_{\varepsilon}(e) = \beta \tag{1.2}$$

where ε is a small parameter and a, b, α and β are real constants.

2. The Investigating of Boundary Layers for Local Boundary Condition

In this section, first, the differential equation of Cauchy-Euler is considered with variance multiplication and with local boundary conditions as follows:

$$\varepsilon x^2 {y''}_{\varepsilon}(x) + ax{y'}_{\varepsilon}(x) + by_{\varepsilon}(x) = 0, \quad x \in (1, e) \tag{2.1}$$

$$ y'(1) = \alpha, \qquad y(e) = \beta \tag{2.2}$$

where ε is a small parameter and a, b, α and β are real constants, and the analytical solution of the equation is obtained as follows:

$$y_{\varepsilon}(x) = \sum_{j=1}^{2} c_j x^{k_j}, \quad j = 1, 2$$

Where

$$k_j = \frac{\varepsilon - a + (-1)^j \sqrt{(\varepsilon - a)^2 - 4b\varepsilon}}{2\varepsilon}, \quad j = 1, 2.$$

And c_j for $j = 1, 2$ is achieved by using of Cramer method:

$$\begin{aligned} c_1 &= \frac{1}{\Delta} \begin{vmatrix} \alpha & k_2 \\ \beta & e^{k_2} \end{vmatrix} \\ &= \frac{\alpha e^{k_2} - \beta k_2}{\Delta} \end{aligned}$$

and

$$\begin{aligned} c_2 &= \frac{1}{\Delta} \begin{vmatrix} k_1 & \alpha \\ e^{k_1} & \beta \end{vmatrix} \\ &= \frac{k_1 \beta - \alpha e^{k_1}}{\Delta} \end{aligned}$$

where

$$\Delta = \begin{vmatrix} k_1 & k_2 \\ e^{k_1} & e^{k_2} \end{vmatrix}$$

And finally, the analytical solution is achieved as follows:

$$y_\varepsilon(x) = \frac{\alpha e^{k_2} - k_2 \beta}{k_1 e^{k_2} - k_2 e^{k_1}} \cdot x^{k_1} + \frac{k_1 \beta - e^{k_1} \alpha}{k_1 e^{k_2} - k_2 e^{k_1}} \cdot x^{k_2}. \tag{2.3}$$

Definition 2.1 (Frequent uniform limit method)**.** Suppose that $p_\varepsilon(y_\varepsilon) = 0$ is the representative of the perturbation problem with the differential operator p_ε and $y_\varepsilon(x)$ is the solution of the perturbation problem and $y_0(x)$ is the limit solution of the problem $y_\varepsilon(x)$ in which small parameter ε is conducted to zero. If the following uniform limit is defined, the perturbation problem is called non-singular:

$$\lim_{\varepsilon \to 0} \lim_{x \to x_0} y_\varepsilon(x) = \lim_{x \to x_0} \lim_{\varepsilon \to 0} y_\varepsilon(x) \tag{2.4}$$

And the problem is called singular perturbation if the uniform limit (2.4) in the boundary point $x = x_0$ is not established. In this problem, the boundary point $x = x_0$ is optional.

Definition 2.2 (The Boundary Layer)**.** In the singular state in which the relation (2.4) is not established, a non-harmony is created according to the differential equation order and the number of boundary conditions of problem in which the solution of $y_0(x)$ in boundary condition of problem in point $x = x_0$ is not applicable. This state in a singular perturbation problem is called the boundary layer phenomenon.

First, the limit behavior of $k_j(\varepsilon)$ for $j = 1, 2$ is studied when ε is conducted to zero:

$$\lim_{\varepsilon\to 0, j=1,2} k_j(\varepsilon) = \begin{cases} \lim\limits_{\varepsilon\to 0, j=1} k_1(\varepsilon) = \begin{cases} -\infty, & a > 0 \\ -\frac{b}{a}, & a < 0 \end{cases} = \begin{cases} -\frac{b}{a} > 0, & a < 0, \quad b < 0 \\ -\frac{b}{a} < 0, & a < 0, \quad b > 0 \end{cases} \\ \lim\limits_{\varepsilon\to 0, j=2} k_2(\varepsilon) = \begin{cases} -\frac{b}{a}, & a > 0 = \begin{cases} -\frac{b}{a} > 0, & a > 0, \ b < 0 \\ -\frac{b}{a} < 0, & a > 0, \ b > 0 \end{cases} \\ +\infty, & a < 0 \end{cases} \end{cases}$$

Or in another way:

$$\lim_{\varepsilon\to 0} k_1(\varepsilon) = \begin{cases} -\infty, & a > 0 \\ -\frac{b}{a}, & a < 0 \end{cases}$$

$$\lim_{\varepsilon\to 0} k_2(\varepsilon) = \begin{cases} +\infty, & a < 0 \\ -\frac{b}{a}, & a > 0 \end{cases}$$

There are two states for achieving the formation or non-formation of the boundary layer near boundary point:

The First State: When $a > 0$:

(a) The investigating of formation or non-formation of the boundary layer near boundary point when $x = e$: In this state of equation(2.3), we have:

$$\begin{aligned} y_0(x) &= \lim_{\varepsilon\to 0} y_\varepsilon(x) \\ &= \beta\left(\frac{x}{e}\right)^{-\frac{b}{a}} \end{aligned} \tag{2.5}$$

From direct solving of the Cauchy-Euler equation, when ε is conducted to zero, we will have:

$$\frac{dy_0}{y_0} = -\frac{b}{a}\frac{dx}{x} \quad \Rightarrow \quad y_0 = cx^{-\frac{b}{a}} \tag{2.6}$$

Now with the helping of the boundary condition of $y(e) = \beta$, we have:

$$a = B \cdot e^{\frac{b}{a}} \quad \Rightarrow \quad y_0(x) = B \cdot \left(\frac{x}{e}\right)^{-\frac{b}{a}} \tag{2.7}$$

Comparing the relations (2.5), (2.7) and with regards to the definition of Frequent uniform limit method, this result is achieved that there is no boundary layer in boundary point $x = e$.

(b) The investigating of formation or non-formation of the boundary layer near boundary point $x = 1$:

At first, we have by the derivative of relation (2.7):

$$y'_0(x) = -\frac{Bb}{ea}\left(\frac{x}{e}\right)^{-\frac{b}{a}-1}$$

Now with applying the boundary condition $y'(1) = \alpha$, we will have:

$$y'_0(1) = \alpha \quad \Rightarrow \quad a\alpha e^{-\frac{b}{a}} + Bb = 0$$

So, the equation of the Cauchy-Euler is proposed in form of the theorem for the first state and first and second parts as follow to identify the formation or non-formation of the boundary layer near boundary points of $x = 1$ and $x = e$:

Theorem 2.3. *If $a > 0$ is in the Cauchy-Euler equation with local boundary condition of* (2.1)-(2.2)*, then we will have:*

(i) *The problem has no the boundary layer near boundary point $x = e$.*

(ii) *The problem with the condition of $a\alpha e^{-\frac{b}{a}} + Bb = 0$ has no the boundary layer near boundary point $x = 1$.*

The Second State: When $a < 0$:

(c) The investigating of formation or non-formation of the boundary layer near boundary point $x = e$:

In this state from equation (2.3), we will have:

$$\begin{aligned} y_0(x) &= \lim_{\varepsilon \to 0} y_\varepsilon(x) \\ &= -\frac{a\alpha}{b} x^{-\frac{b}{a}} \end{aligned}$$

Here, with applying boundary condition of $y(e) = \beta$ we have:

$$\begin{aligned} y_0(e) &= -\frac{a\alpha}{b} e^{-\frac{b}{a}} \\ &= B \end{aligned}$$

According to relation (2.7) we will have:

$$a\alpha e^{-\frac{b}{a}} + Bb = 0$$

The above relation shows the condition of non-formation of the boundary layer near boundary point $x = e$.

(d) The investigating of formation or non-formation of the boundary layer near boundary point when $x = 1$:

Here, by the derivative of relation (2.7) and applying the boundary condition of $y'(1) = \alpha$, we will have:

$$\begin{aligned} y_0'(1) &= -\frac{Bb}{a} e^{\frac{b}{a}} \\ &= \alpha \quad \Rightarrow \quad a\alpha e^{-\frac{b}{a}} + Bb = 0 \end{aligned}$$

So, the above relation shows the condition of non formation of the boundary layer near boundary point $x = 1$.

Now we are stated theorem as follows:

Theorem 2.4. *If it is $a < 0$ in the Cauchy-Euler equation with local boundary condition of* (2.1)*, and also the condition of $a\alpha e^{-\frac{b}{a}} + Bb = 0$ is established, then the problem* (2.1)-(2.2) *in this state has no boundary layer near boundary points of $x = 1$ and $x = e$.*

3. The Studying of the Self-Adjoint and Non-Self-Adjoint of Perturbation Problem

For to identify the conditions of the self-adjoint and non-self-adjoint of perturbation of problem, the problem is considered with the boundary conditions as follows:

$$\begin{aligned} &L_\varepsilon y_\varepsilon \equiv \varepsilon x^2 y''_\varepsilon(x) + axy'_\varepsilon(x) + by_\varepsilon(x) = f(x), \quad x \in (1, e) \\ &y'_\varepsilon(1) = \alpha, \qquad y_\varepsilon(e) = \beta \end{aligned}$$

where ε, and a, b are real and given constants and $f(x)$ is given real function. At first, the equation of the Cauchy-Euler is changed with the change of variation $t = Lnx$ as follows:

$$\begin{aligned} &L_\varepsilon Y_\varepsilon \equiv \varepsilon D(D-1) Y_\varepsilon + aY'_\varepsilon + bY_\varepsilon = f(t), \quad t \in (0, 1) \\ &Y'_\varepsilon(0) = \alpha, \qquad Y_\varepsilon(1) = \beta \end{aligned}$$

So, we have with simplification:

$$\begin{aligned} L_\varepsilon Y_\varepsilon &\equiv \varepsilon Y''_\varepsilon + (a - \varepsilon) Y'_\varepsilon + bY_\varepsilon(x) \\ &= f(t), \quad t \in (0, 1) \end{aligned} \tag{3.1}$$

Definition 3.1 (Adjoint Operator)**.** Adjoint operator of

$$L[y] = p_n \frac{d^n y}{dx^n} + p_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + p_0 y$$

is defined on the interval (a, b) as follows:

$$L^*[y] = (-1)^n \frac{d^n}{dx^n}(\overline{p_n} y) + (-1)^{n-1} \frac{d^{n-1}}{dx^{n-1}}(\overline{p_{n-1}} y) + \cdots + \overline{p_0} y$$

Now if p_k is k times continuous derivative and u, v are n times continuous derivative on the interval (a, b), the Lagrange Identity on this interval is as follows:

$$\overline{v}L\left[u\right] - u\overline{L}^*\left[v\right] = \frac{d}{dx}B\left[u,v\right]$$

In which $B[u, v]$ is two-lined form as follows:

$$B\left[u,v\right] = \sum_{m=1}^{n}\sum_{j+k=m-1}(-1)^j u^k (p_m\overline{v})^j$$

in which $k > 0$ and $j > 0$ are established. Now using the Lagrange Identity and with the multiplication of $Z_\varepsilon(t)$ to two sides of equation (3.1) and with the integration in the interval $(0, 1)$ we will have:

$$\begin{aligned}(L_\varepsilon Y_\varepsilon, Z_\varepsilon) = \varepsilon\int_0^1 Y''_\varepsilon\left(t\right) Z_\varepsilon\left(t\right)dt + (a-\varepsilon)\int_0^1 Y'_\varepsilon\left(t\right) Z_\varepsilon\left(t\right)dt \\ + b\int_0^1 Y_\varepsilon\left(t\right) Z_\varepsilon\left(t\right)dt \\ = B(Y_\varepsilon, Z_\varepsilon) + (Y_\varepsilon, L^*_\varepsilon Z_\varepsilon)\end{aligned}$$

Now the two-lined form is as follows:

$$\begin{aligned}B(Y_\varepsilon, Z_\varepsilon) = \varepsilon Y'_\varepsilon(1)Z_\varepsilon(1) - \varepsilon Y'_\varepsilon(0)Z_\varepsilon(0) - \varepsilon Y_\varepsilon(1)Z'_\varepsilon(1) + \varepsilon Y_\varepsilon(0)Z'_\varepsilon(0) \\ + (a-\varepsilon)Y_\varepsilon(1)Z_\varepsilon(1) - (a-\varepsilon)Y_\varepsilon(0)Z_\varepsilon(0)\end{aligned}$$

And finally, adjoint equation is achieved as follows:

$$L^*_\varepsilon Z_\varepsilon = \varepsilon Z''_\varepsilon(t) - (a-\varepsilon)Z'_\varepsilon(t) + bZ_\varepsilon(t)$$

Now we can state the theorem as follows:

Theorem 3.2. *The line operator*(3.1) *with the condition of* $a = \varepsilon$ *is self-adjoint which means* $L_\varepsilon = L^*_\varepsilon$.

Proof. By Lagrange identity and the internal multiplication we have:

$$(L_\varepsilon Y_\varepsilon, Z_\varepsilon) - (Y_\varepsilon, L^*_\varepsilon Z_\varepsilon) = B(Y_\varepsilon, Z_\varepsilon)$$

And with supposing $B(Y_\varepsilon, Z_\varepsilon) = 0$, the self-adjoint of the linear operator of $L_\varepsilon Y_\varepsilon$ is established. □

4. The Fundamental Solution of Adjoint Equation

In this section, the following non-homogenous adjoint equation is considered for achieving the fundamental solution:

$$\begin{aligned}L^*_\varepsilon Z_\varepsilon &= \varepsilon Z''_\varepsilon(t) - (a-\varepsilon)Z'_\varepsilon(t) + bZ_\varepsilon(t) \\ &= g(t)\end{aligned} \tag{4.1}$$

where $g(t)$ is a uniform function.

Definition 4.1. The fundamental solution of a ordinary differential equation is a generalized function that is applicable in a differential equation and its right side is the Delta-Dirac function. In other words, if L is a differential operator with constant coefficients as

$$L(D) = \sum_{\alpha=0}^{m} a_\alpha D^\alpha$$

the generalized function of η which is applicable in the equation of $L(D)\eta = \delta$, it is called a fundamental operator of $L(D)$. Now we have from the homogeneous equation:

$$Z''_\varepsilon(t) - (a-\varepsilon)Z'_\varepsilon(t) + bZ_\varepsilon(t) = 0$$

in which the solution of the equation is $Z_\varepsilon(t) = e^{\rho(\varepsilon)t}$ and $\rho(\varepsilon)$ that are the roots of the characteristic or auxiliary equation of

$$\rho^2(\varepsilon) - (a-\varepsilon)\rho(\varepsilon) + b = 0$$

which is administrated as follows:

$$\rho_j(\varepsilon) = \frac{(a-\varepsilon) + (-1)^j\sqrt{(a-\varepsilon)^2 - 4b\varepsilon}}{2\varepsilon}, \quad j = 1, 2$$

The solution of the homogeneous adjoint equation for $\rho_1(\varepsilon) = \rho_2(\varepsilon)$ is as follows:

$$Z_{j\varepsilon}(t) = e^{\rho_j(\varepsilon)t}, \quad j = 1, 2$$

So, the general solution of the homogeneous adjoint equation is as follows:

$$Z_\varepsilon(t) = \sum_{j=1}^{2} C_j e^{\rho_j(\varepsilon)t} \tag{4.2}$$

where C_j is selected constants. Now we achieve the especial solution of the non-homogeneous adjoint equation of (4.1) by the method of changing parameter:

$$Z_\varepsilon(t) = \int_t^1 \frac{e^{\rho_1(\varepsilon)(t-\xi)} - e^{\rho_2(\varepsilon)(t-\xi)}}{\varepsilon(\rho_2(\varepsilon) - \rho_1(\varepsilon))} g(\xi)d\xi$$

We have from the general solution (4.2):

$$\begin{aligned} Z_\varepsilon(t) &= C_1 e^{\rho_1(\varepsilon)(t)} + C_2 e^{\rho_2(\varepsilon)(t)} \\ Z'_\varepsilon(t) &= C'_1(t) e^{\rho_1(\varepsilon)(t)} + C'_2(t) e^{\rho_2(\varepsilon)(t)} \\ &\quad + C_1(t)\rho_1(\varepsilon) e^{\rho_1(\varepsilon)(t)} + C_2(t)\rho_2(\varepsilon) e^{\rho_2(\varepsilon)(t)} \end{aligned}$$

Now, we have the following systems:

$$\begin{cases} C_1'(t)e^{\rho_1(\varepsilon)(t)} + C_2'(t)e^{\rho_2(\varepsilon)(t)} = 0 \\ C_1'(t)\rho_1(\varepsilon)e^{\rho_1(\varepsilon)(t)} + C_2'(t)\rho_2(\varepsilon)e^{\rho_2(\varepsilon)(t)} = \frac{1}{\varepsilon}g(t) \end{cases}$$

Now we achieve $C_1'(t), C_2'(t)$:

$$\begin{aligned} C_1'(t) =& \frac{1}{w(t)} \begin{vmatrix} 0 & e^{\rho_2(\varepsilon)(t)} \\ \frac{g(t)}{\varepsilon} & \rho_2(\varepsilon)e^{\rho_2(\varepsilon)(t)} \end{vmatrix} \\ =& \frac{-g(t)}{\varepsilon(\rho_2 - \rho_1)} e^{-\rho_1(\varepsilon)(t)} \end{aligned}$$

$$\begin{aligned} C_2'(t) =& \frac{1}{w(t)} \begin{vmatrix} e^{\rho_1(\varepsilon)(t)} & 0 \\ \rho_1(\varepsilon)e^{\rho_1(\varepsilon)(t)} & \frac{g(t)}{\varepsilon} \end{vmatrix} \\ =& \frac{g(t)}{\varepsilon(\rho_2 - \rho_1)} e^{-\rho_2(\varepsilon)(t)} \end{aligned}$$

in which $w(t)$ is Wronskian and is achieved as follows:

$$w(t) = (\rho_2(\varepsilon) - \rho_1(\varepsilon))e^{(\rho_1(\varepsilon)+\rho_2(\varepsilon))t}$$

According to the integration of $C_1'(t)$ and $C_2'(t)$ accordingly on the intervals (t_1, t) and (t_2, t), we have:

$$C_1(t) = C_1 - \int_{t_1}^{t} \frac{-g(\xi)}{\varepsilon(\rho_2(\varepsilon) - \rho_1(\varepsilon))} e^{-\rho_1(\varepsilon)\xi} d\xi$$

$$C_2(t) = C_2 - \int_{t_2}^{t} \frac{g(\xi)}{\varepsilon(\rho_2(\varepsilon) - \rho_1(\varepsilon))} e^{-\rho_2(\varepsilon)\xi} d\xi$$

Hence, we have from the general solution of homogenous adjoint equation:

$$\begin{aligned} Z_\varepsilon(t) &= \sum_{j=1}^{2} C_j e^{\rho_j(\varepsilon)t} \\ &= C_1 e^{\rho_1(\varepsilon)(t)} + C_2 e^{\rho_2(\varepsilon)(t)} + \int_{t_1}^{t} \frac{g(\xi)e^{\rho_1(\varepsilon)(t-\xi)}}{\varepsilon(\rho_2(\varepsilon) - \rho_1(\varepsilon))} d\xi \\ &\quad - \int_{t_2}^{t} \frac{g(\xi)e^{\rho_2(\varepsilon)(t-\xi)}}{\varepsilon(\rho_2(\varepsilon) - \rho_1(\varepsilon))} d\xi \end{aligned}$$

Now with supposing $t_1 = 0$ and $t_2 = 1$ and with changing the symptoms of $\rho_1(\varepsilon)$ and $\rho_2(\varepsilon)$, we give four states for the fundamental solution of adjoint equation:

The First State: Both roots of $\rho_1(\varepsilon)$ and $\rho_2(\varepsilon)$ are positive:

In this state, the fundamental solution of integration on the interval $(t,1)$ is as follows:

$$Z_\varepsilon(t-\xi)=\begin{cases}\frac{e^{\rho_1(\varepsilon)(t-\xi)}-e^{\rho_2(\varepsilon)(t-\xi)}}{\varepsilon[\rho_2(\varepsilon)-\rho_1(\varepsilon)]}, & 0<t<\xi<1\\ 0, & 0<\xi<t<1\end{cases} \tag{4.3}$$
$$=\frac{\theta(\xi-t)}{\varepsilon\left[\rho_2(\varepsilon)-\rho_1(\varepsilon)\right]}\left(e^{\rho_1(\varepsilon)(t-\xi)}-e^{\rho_2(\varepsilon)(t-\xi)}\right)$$

in which $\theta(\xi-t)$ is the Heaviside function:

The Second State:$\rho_1(\varepsilon)<0$ and $\rho_2(\varepsilon)<0$:

In this state, the fundamental solution of integration on the interval $(0,t)$is as follows:

$$Z_\varepsilon(t-\xi)=\begin{cases}0, & 0<t<\xi<1\\ \frac{e^{\rho_1(\varepsilon)(t-\xi)}-e^{\rho_2(\varepsilon)(t-\xi)}}{\varepsilon[\rho_2(\varepsilon)-\rho_1(\varepsilon)]}, & 0<\xi<t<1\end{cases} \tag{4.4}$$

The Third State:$\rho_1(\varepsilon)<0$ and $\rho_2(\varepsilon)>0$:

In this case, the fundamental solution is as follows:

$$Z_\varepsilon(t-\xi)=\begin{cases}\frac{e^{\rho_2(\varepsilon)(t-\xi)}}{\varepsilon(\rho_2(\varepsilon)-\rho_1(\varepsilon))}, & 0<t<\xi<1\\ \frac{e^{\rho_1(\varepsilon)(t-\xi)}}{\varepsilon(\rho_2(\varepsilon)-\rho_1(\varepsilon))}, & 0<\xi<t<1\end{cases} \tag{4.5}$$

The Fourth State: $\rho_1(\varepsilon)>0$ and $\rho_2(\varepsilon)<0$:

In this state, the fundamental solution is as follows:

$$Z_\varepsilon(t-\xi)=\begin{cases}\frac{-e^{\rho_1(\varepsilon)(t-\xi)}}{\varepsilon(\rho_2(\varepsilon)-\rho_1(\varepsilon))}, & 0<t<\xi<1\\ \frac{-e^{\rho_2(\varepsilon)(t-\xi)}}{\varepsilon(\rho_2(\varepsilon)-\rho_1(\varepsilon))}, & 0<\xi<t<1\end{cases} \tag{4.6}$$

Now we can propose the following theorems:

Theorem 4.2. *If the non-homogeneous adjoint equation of* (4.1) *with the conditions of*

$$(a-\varepsilon)>2\sqrt{\varepsilon b}$$

and $a>\varepsilon$ *is established in which* $0<\varepsilon\ll 1$*, we have the fundamental solution of* (4.3)*.*

Theorem 4.3. *If the non-homogeneous adjoint equation of* (4.1) *with the conditions of*

$$a>\varepsilon+2\sqrt{\varepsilon b}$$

and $a < \varepsilon$ is established in which $0 < \varepsilon \ll 1$, we have the fundamental solution of (4.4)*.*

Theorem 4.4. *If the non-homogeneous adjoint equation of* (4.1) *with the conditions of*

$$a > \varepsilon + 2\sqrt{\varepsilon b}$$

and $a = \varepsilon$ is established in which $0 < \varepsilon \ll 1$, we have the fundamental solution of (4.5)*.*

Theorem 4.5. *If the non-homogeneous adjoint equation of* (4.1) *with the condition of*

$$a > \varepsilon + 2\sqrt{\varepsilon b}$$

is established in which $0 < \varepsilon \ll 1$, we have the fundamental solution of (4.6)*.*

Theorem 4.6. *If $a < 0$ is in the Cauchy-Euler equation with local boundary condition of* (2.1)*, and also the condition of $a\alpha e^{-\frac{b}{a}} + Bb = 0$ is established, then the problem* (2.1)-(2.2) *in this state has no boundary layer near boundary points of $x = 1$ and $x = e$.*

5. Obtaining the Necessary Conditions

In this section, we investigate the boundary layers for the perturbation problem the including Cauchy Euler equation when it is in the following form with non-local boundary conditions:

$$\begin{aligned} L_\varepsilon y_\varepsilon &\equiv \varepsilon x^2 y''_\varepsilon(x) + axy'_\varepsilon + by_\varepsilon(x) \\ &= f(x), \quad x \in (1, e) \end{aligned} \tag{5.1}$$

$$\begin{aligned} L_\varepsilon y_\varepsilon &\equiv \sum_{j=0}^{1} \left\{ \alpha_{kj} y_\varepsilon^{(1-j)}(1) + \beta_{kj} y_\varepsilon^{(1-j)}(e) \right\} \\ &= \gamma_k, \quad k = 1, 2 \end{aligned} \tag{5.2}$$

where ε is a small parameter and $a, b, \alpha_{kj}, \beta_{kj}$, and γ_k are real constants and also the $f(x)$ is a real function. Now, by changing the variable $x = e^t$, we get the above perturbation problem into the following form:

$$\begin{aligned} L_\varepsilon Y_\varepsilon &\equiv \varepsilon Y''_\varepsilon + (a - \varepsilon) Y'_\varepsilon + bY_\varepsilon(x) \\ &= f(t), \quad t \in (0, 1) \end{aligned} \tag{5.3}$$

$$\begin{aligned} L_\varepsilon Y_\varepsilon &\equiv \sum_{j=0}^{1} \left\{ \alpha_{kj} Y_\varepsilon^{(1-j)}(0) + \beta_{kj} Y_\varepsilon^{(1-j)}(1) \right\} \\ &= \gamma_k, \quad k = 1, 2 \end{aligned} \tag{5.4}$$

In this case, using the 4-step method [6, 10–12] of localization, we seek to find the necessary conditions. According to this, in this paper we analyze the fundamental solution to the first case. At first, we have the following definition:

Definition 5.1. Suppose $f(x)$ is a function such that for $x \leq x_0$ and $x \geq x_0$ we have $f \in C$, in which case the derivative of the generalized function $f \in D'$ is obtained from the relation $f = \{f'(x)\}+[f]_{x_0}\delta(x-x_0)$. Where $\{f'(x)\}$ is the classical derivative of the function $f(x)$ and $[f]_{x_0}$ is the mutation rate of the function $f(x)$ at the point $x = x_0$, which is represented as $[f]_{x_0} = f(x_0^+)$.

So, we get the necessary conditions and use the Theorem 4.2. For this case, we get the necessary conditions in two steps.

Step 1: Multiplying the fundamental solution by the sides of equation (5.5), and integrating of it in the interval $(0, 1)$:

In this step, at first, by putting the fundamental solution $Z_\varepsilon\,(t-\xi)$ in the homogeneous equation, we will have:

$$\begin{aligned} L_\varepsilon^* Z_\varepsilon(t-\xi) &= \varepsilon Z_\varepsilon''(t-\xi) - (a-\varepsilon)Z_\varepsilon'(t-\xi) + bZ_\varepsilon(t-\xi) \\ &= \delta(t-\xi) \end{aligned} \tag{5.5}$$

where $\delta(t-\xi)$ is a function of the Dirac Delta.

Now multiply $Z_\varepsilon(t-\xi)$ by the sides (5.5) and by integrating into the interval $(0, 1)$ we get:

$$\begin{aligned} \varepsilon \int_0^1 Y_\varepsilon''(t)Z_\varepsilon(t-\xi)dt + (a-\varepsilon)\int_0^1 Y_\varepsilon'(t)Z_\varepsilon(t-\xi)dt& \\ + b\int_0^1 Y_\varepsilon(t)Z_\varepsilon(t-\xi)dt& \\ = \int_0^1 f(t)Z_\varepsilon(t-\xi)dt& \end{aligned} \tag{5.6}$$

Here, using of equation (5.6) and the properties of the Dirac Delta function, and by putting the fundamental solution $Z_\varepsilon(t-\xi)$ and equation (4.3), we get:

$$\begin{aligned} -\varepsilon \int_0^1 Y_\varepsilon''(t)Z_\varepsilon(t-\xi)dt - (a-\varepsilon)\int_0^1 Y_\varepsilon'(t)Z_\varepsilon(t-\xi)dt& \\ - b\int_0^1 Y_\varepsilon(t)Z_\varepsilon(t-\xi)dt + \int_0^1 f(t)Z_\varepsilon(t-\xi)dt& \\ = \int_0^1 Y_\varepsilon(t)\delta(t-\xi)dt& \end{aligned}$$

$$= \begin{cases} Y_\varepsilon(\xi), & \xi \in (0,1) \\ \frac{1}{2}Y_\varepsilon(\xi), & \xi = 0, \quad \xi = 1 \\ 0, & \xi \notin [0,1] \end{cases}$$

Now, we have from integration by part,

$$\begin{aligned}
&-\varepsilon Y_\varepsilon'(t)\frac{\Theta(\xi-1)}{\varepsilon[\rho_2(\varepsilon)-\rho_1(\varepsilon)]}\left(e^{\rho_1(\varepsilon)(1-\xi)}-e^{\rho_2(\varepsilon)(1-\xi)}\right)\\
&\quad+\frac{\varepsilon Y_\varepsilon(0)\Theta(\xi)}{\varepsilon[\rho_2(\varepsilon)-\rho_1(\varepsilon)]}\left(e^{-\rho_1(\varepsilon)(-\xi)}-e^{\rho_2(\varepsilon)(-\xi)}\right)\\
&\quad+\frac{\varepsilon Y_\varepsilon(1)\Theta(\xi-1)}{\varepsilon[\rho_2(\varepsilon)-\rho_1(\varepsilon)]}\left(\rho_1(\varepsilon)e^{\rho_1(\varepsilon)(1-\xi)}-\rho_2(\varepsilon)e^{\rho_2(\varepsilon)(1-\xi)}\right)\\
&\quad-\frac{\varepsilon Y_\varepsilon(0)\Theta(\xi)}{\varepsilon[\rho_2(\varepsilon)-\rho_1(\varepsilon)]}\left(\rho_1(\varepsilon)e^{\rho_1(\varepsilon)(-\xi)}-\rho_2(\varepsilon)e^{\rho_2(\varepsilon)(-\xi)}\right)\\
&\quad-\frac{(a-\varepsilon)Y_\varepsilon(0)\Theta(\xi-1)}{\varepsilon[\rho_2(\varepsilon)-\rho_1(\varepsilon)]}\left(e^{\rho_1(\varepsilon)(1-\xi)}-e^{\rho_2(\varepsilon)(1-\xi)}\right)\\
&\quad+\frac{(a-\varepsilon)Y_\varepsilon(0)\Theta(\xi)}{\varepsilon[\rho_2(\varepsilon)-\rho_1(\varepsilon)]}\left(e^{\rho_1(\varepsilon)(-\xi)}-e^{\rho_2(\varepsilon)(-\xi)}\right)\\
&\quad+\int_0^1\frac{f(t)\Theta(\xi-t)}{\varepsilon[\rho_2(\varepsilon)-\rho_1(\varepsilon)]}\left(e^{\rho_1(\varepsilon)(t-\xi)}-e^{\rho_2(\varepsilon)(t-\xi)}\right)dt\\
&=\begin{cases} Y_\varepsilon(\xi), & \xi \in (0,1) \\ \frac{1}{2}Y_\varepsilon(\xi), & \xi = 0, \quad \xi = 1 \\ 0, & \xi \notin [0,1] \end{cases}
\end{aligned}$$

Now, with replacement $\xi = 0$ and $\xi = 1$ in the above relation and using the property of the Heaviside Function, we have:

(5.7)

$$\begin{aligned}
Y_\varepsilon(0) &\equiv 0,\\
Y_\varepsilon(1) &= \frac{Y_\varepsilon'(0)\left(e^{-\rho_1(\varepsilon)}-e^{-\rho_2(\varepsilon)}\right)}{\rho_2(\varepsilon)-\rho_1(\varepsilon)}\\
&\quad+\frac{Y_\varepsilon(0)}{\rho_2(\varepsilon)-\rho_1(\varepsilon)}\\
&\quad\times\left[\frac{a-\varepsilon}{\varepsilon}\left(e^{-\rho_1(\varepsilon)}-e^{-\rho_2(\varepsilon)}\right)-\left(\rho_1(\varepsilon)e^{-\rho_1(\varepsilon)}-\rho_2(\varepsilon)e^{-\rho_2(\varepsilon)}\right)\right]\\
&\quad+\int_0^1\frac{e^{\rho_1(\varepsilon)(t-1)}-e^{\rho_2(\varepsilon)(t-1)}}{\varepsilon(\rho_2(\varepsilon)-\rho_1(\varepsilon))}f(t)dt
\end{aligned}$$

The above equation is one of the necessary conditions for the problem.

Step 2: By multiplying the derivative of the fundamental solution to the sides of equation (5.3) and by integrating into the interval $(0,1)$ we will have:

$$\begin{aligned}\varepsilon\int_0^1 Y''_\varepsilon(t)Z'_\varepsilon(t-\xi)dt+(a-\varepsilon)\int_0^1 Y'_\varepsilon(t)Z'_\varepsilon(t-\xi)dt&\\ +b\int_0^1 Y_\varepsilon(t)Z'_\varepsilon(t-\xi)dt&\\ =\int_0^1 f(t)Z'_\varepsilon(t-\xi)dt&\end{aligned}$$

Now using equation (5.3) and the properties of the Dirac delta function and by putting the derivative of the fundamental solution, we have:

$$\begin{aligned}&\frac{\varepsilon Y'_\varepsilon(1)\Theta(\xi-1)}{\varepsilon[\rho_2(\varepsilon)-\rho_1(\varepsilon)]}\left(\rho_1(\varepsilon)e^{\rho_1(\varepsilon)(1-\xi)}-\rho_2(\varepsilon)e^{\rho_2(\varepsilon)(1-\xi)}\right)\\ &\quad-\frac{\varepsilon Y'_\varepsilon(0)\Theta(\xi)}{\varepsilon[\rho_2(\varepsilon)-\rho_1(\varepsilon)]}\left(\rho_1(\varepsilon)e^{\rho_1(\varepsilon)(-\xi)}-\rho_2(\varepsilon)e^{\rho_2(\varepsilon)(-\xi)}\right)\\ &\quad+\frac{bY_\varepsilon(1)\Theta(\xi-1)}{\varepsilon[\rho_2(\varepsilon)-\rho_1(\varepsilon)]}\left(e^{\rho_1(\varepsilon)(1-\xi)}-e^{\rho_2(\varepsilon)(1-\xi)}\right)\\ &\quad-\frac{bY_\varepsilon(0)\Theta(\xi)}{\varepsilon[\rho_2(\varepsilon)-\rho_1(\varepsilon)]}\left(e^{\rho_1(\varepsilon)(-\xi)}-e^{\rho_2(\varepsilon)(-\xi)}\right)\\ &\quad-\int_0^1\frac{\Theta(\xi-t)\left(\rho_1(\varepsilon)e^{\rho_1(\varepsilon)(t-\xi)}-\rho_2(\varepsilon)e^{\rho_2(\varepsilon)(t-\xi)}\right)}{\varepsilon[\rho_2(\varepsilon)-\rho_1(\varepsilon)]}dt\\ &=\begin{cases}Y'_\varepsilon(\xi), & \xi\in(0,1)\\ \frac{1}{2}Y'_\varepsilon(\xi), & \xi=0,\quad \xi=1\\ 0, & \xi\notin[0,1]\end{cases}\end{aligned}$$

So, with replacement $\xi=0$ and $\xi=1$ in the above relation, we will have:

$$\begin{aligned}&Y'_\varepsilon(0)\equiv 0,\\ \frac{1}{2}Y'_\varepsilon(1)&=\frac{-Y'_\varepsilon(0)\left(\rho_1(\varepsilon)e^{-\rho_1(\varepsilon)}-\rho_2(\varepsilon)e^{-\rho_2(\varepsilon)}\right)}{\rho_2(\varepsilon)-\rho_1(\varepsilon)}\\ &\quad-\frac{bY_\varepsilon(0)(e^{-\rho_1(\varepsilon)}-e^{-\rho_2(\varepsilon)})}{\varepsilon(\rho_2(\varepsilon)-\rho_1(\varepsilon))}\\ &\quad-\int_0^1\frac{\rho_1(\varepsilon)e^{\rho_1(\varepsilon)(t-1)}-\rho_2(\varepsilon)e^{\rho_2(\varepsilon)(t-1)}}{\varepsilon(\rho_2(\varepsilon)-\rho_1(\varepsilon))}f(t)dt\end{aligned}\tag{5.8}$$

Hence, from the first and second steps, we can have the following theorem:

Theorem 5.2. *If the solution $Y_\varepsilon(t)$ is established in the differential equation* (5.3) *in interval* $(0,1)$*, then* (5.7) *and* (5.8) *apply in the necessary conditions.*

6. Converting of Non-Local Boundary Conditions to Local Boundary Conditions

Using non-local boundary conditions (5.4) and essential conditions (5.7) and (5.8), we form the following linear system.

$$
\begin{cases}
\alpha_{11}Y_\varepsilon(0)+\alpha_{10}Y'_\varepsilon(0)+\beta_{11}Y_\varepsilon(1)+\beta_{10}Y'_\varepsilon(1)=\gamma_1\\
\alpha_{21}Y_\varepsilon(0)+\alpha_{20}Y'_\varepsilon(0)+\beta_{21}Y_\varepsilon(1)+\beta_{20}Y'_\varepsilon(1)=\gamma_2\\
\alpha_{31}Y_\varepsilon(0)+\alpha_{30}Y'_\varepsilon(0)-Y_\varepsilon(1)=\int_0^1\gamma_3 f(t)dt\\
\alpha_{41}Y_\varepsilon(0)+\alpha_{40}Y'_\varepsilon(0)-\frac{1}{2}Y'_\varepsilon(1)=\int_0^1\gamma_4 f(t)dt
\end{cases}
\tag{6.1}
$$

which is displayed as the following matrix:

$$
\begin{pmatrix}
\alpha_{11} & \alpha_{10} & \beta_{11} & \beta_{10}\\
\alpha_{21} & \alpha_{20} & \beta_{21} & \beta_{20}\\
\alpha_{31} & \alpha_{30} & -1 & 0\\
\alpha_{41} & \alpha_{40} & 0 & -\frac{1}{2}
\end{pmatrix}
\begin{pmatrix}
Y_\varepsilon(0)\\ Y'_\varepsilon(0)\\ Y_\varepsilon(1)\\ Y'_\varepsilon(1)
\end{pmatrix}
=
\begin{pmatrix}
\gamma_1\\ \gamma_2\\ \int_0^1\gamma_3 f(t)dt\\ \int_0^1\gamma_4 f(t)dt
\end{pmatrix}
$$

where $\alpha_{31},\alpha_{30},\alpha_{41},\alpha_{40},\gamma_3,\gamma_4$ are obtained as follows:

(6.2)

$$
\begin{aligned}
\alpha_{31} &= \frac{1}{\rho_2(\varepsilon)-\rho_1(\varepsilon)}\\
&\quad\times\left[\frac{a-\varepsilon}{\varepsilon}\left(e^{-\rho_1(\varepsilon)}-e^{-\rho_2(\varepsilon)}\right)-\left(\rho_2(\varepsilon)e^{-\rho_1(\varepsilon)}-\rho_2(\varepsilon)e^{-\rho_2(\varepsilon)}\right)\right],\\
\alpha_{30} &= \frac{e^{-\rho_1(\varepsilon)}-e^{-\rho_2(\varepsilon)}}{\rho_2(\varepsilon)-\rho_1(\varepsilon)},\\
\alpha_{41} &= \frac{-b\left(e^{-\rho_1(\varepsilon)}-e^{-\rho_2(\varepsilon)}\right)}{\varepsilon\left(\rho_2(\varepsilon)-\rho_1(\varepsilon)\right)},\\
\alpha_{40} &= \frac{\rho_2(\varepsilon)e^{-\rho_2(\varepsilon)}-\rho_1(\varepsilon)e^{-\rho_1(\varepsilon)}}{\rho_2(\varepsilon)-\rho_1(\varepsilon)},\\
\gamma_3 &= \frac{e^{\rho_2(\varepsilon)(t-1)}-e^{\rho_1(\varepsilon)(t-1)}}{\varepsilon(\rho_2(\varepsilon)-\rho_1(\varepsilon))},\\
\gamma_4 &= \frac{\rho_1(\varepsilon)e^{\rho_1(\varepsilon)(t-1)}-\rho_2(\varepsilon)e^{\rho_2(\varepsilon)(t-1)}}{\varepsilon(\rho_2(\varepsilon)-\rho_1(\varepsilon))}
\end{aligned}
$$

Now using the Cramer Rule, we get $Y_\varepsilon'(0), Y_\varepsilon(0), Y_\varepsilon(1)$ and $Y_\varepsilon'(1)$:

$$Y_\varepsilon'(0) = \frac{1}{\Delta}\begin{vmatrix} \alpha_{11} & \beta_{11} & \beta_{10} & \gamma_1 \\ \alpha_{21} & \beta_{21} & \beta_{20} & \gamma_2 \\ \alpha_{31} & -1 & 0 & \int_0^1 \gamma_3 f(t)dt \\ \alpha_{41} & 0 & -\frac{1}{2} & \int_0^1 \gamma_4 f(t)dt \end{vmatrix},$$

$$Y_\varepsilon(0) = \frac{1}{\Delta}\begin{vmatrix} \gamma_1 & \alpha_{10} & \beta_{11} & \beta_{10} \\ \gamma_2 & \alpha_{20} & \beta_{21} & \beta_{20} \\ \int_0^1 \gamma_3 f(t)dt & \alpha_{30} & -1 & 0 \\ \int_0^1 \gamma_4 f(t)dt & \alpha_{40} & 0 & -\frac{1}{2} \end{vmatrix},$$

$$Y_\varepsilon(1) = \frac{1}{\Delta}\begin{vmatrix} \alpha_{11} & \alpha_{10} & \beta_{10} & \gamma_1 \\ \alpha_{21} & \alpha_{20} & \beta_{20} & \gamma_2 \\ \alpha_{31} & \alpha_{30} & 0 & \int_0^1 \gamma_3 f(t)dt \\ \alpha_{41} & \alpha_{40} & -\frac{1}{2} & \int_0^1 \gamma_4 f(t)dt \end{vmatrix},$$

$$Y_\varepsilon'(1) = \frac{1}{\Delta}\begin{vmatrix} \alpha_{11} & \alpha_{10} & \beta_{11} & \gamma_1 \\ \alpha_{21} & \alpha_{20} & \beta_{21} & \gamma_2 \\ \alpha_{31} & \alpha_{30} & -1 & \int_0^1 \gamma_3 f(t)dt \\ \alpha_{41} & \alpha_{40} & 0 & \int_0^1 \gamma_4 f(t)dt \end{vmatrix}$$

where

$$\Delta = \begin{vmatrix} \alpha_{11} & \alpha_{10} & \beta_{11} & \beta_{10} \\ \alpha_{21} & \alpha_{20} & \beta_{21} & \beta_{20} \\ \alpha_{31} & \alpha_{30} & -1 & 0 \\ \alpha_{41} & \alpha_{40} & 0 & -\frac{1}{2} \end{vmatrix}$$

In other words, we have the third and fourth equations from the linear system (6.1):

$$Y_\varepsilon(1) = \alpha_{31} Y_\varepsilon(0) + \alpha_{30} Y_\varepsilon'(0) - \int_0^1 \gamma_3 f(t)dt \tag{6.3}$$

$$Y_\varepsilon'(1) = 2\alpha_{41} Y_\varepsilon(0) + 2\alpha_{40} Y_\varepsilon'(0) - 2\int_0^1 \gamma_4 f(t)dt \tag{6.4}$$

Here by placing above relations in the first and second equations of linear system (6.1), we simplify the system as follows:

$$\begin{cases} (\alpha_{11}+\beta_{11}\alpha_{31}+2\beta_{10}\alpha_{41})Y_\varepsilon(0)+(\alpha_{10}+\beta_{11}\alpha_{30}+2\beta_{10}\alpha_{40})Y'_\varepsilon(0) \\ \qquad =\gamma_1+\beta_{11}\int_0^1\gamma_3 f(t)dt+2\beta_{10}\int_0^1\gamma_4 f(t)dt, \\ (\alpha_{21}+\beta_{21}\alpha_{31}+2\beta_{20}\alpha_{41})Y_\varepsilon(0)+(\alpha_{20}+\beta_{21}\alpha_{30}+2\beta_{20}\alpha_{40})Y'_\varepsilon(0) \\ \qquad =\gamma_2+\beta_{21}\int_0^1\gamma_3 f(t)dt+2\beta_{20}\int_0^1\gamma_4 f(t)dt \end{cases}$$

Then, $Y_\varepsilon(0)$ and $Y'_\varepsilon(0)$ respectively are obtained as follows:

(6.5)

$$Y_\varepsilon(0)=\frac{1}{w}\begin{vmatrix}\gamma_1+\beta_{11}\int_0^1\gamma_3 f(t)dt+2\beta_{10}\int_0^1\gamma_4 f(t)dt & \alpha_{10}+\beta_{11}\alpha_{30}+2\beta_{10}\alpha_{40} \\ \gamma_2+\beta_{21}\int_0^1\gamma_3 f(t)dt+2\beta_{10}\int_0^1\gamma_4 f(t)dt & \alpha_{20}+\beta_{21}\alpha_{30}+2\beta_{20}\alpha_{40}\end{vmatrix};$$

(6.6)

$$Y'_\varepsilon(0)=\frac{1}{w}\begin{vmatrix}\alpha_{11}+\beta_{11}\alpha_{31}+2\beta_{10}\alpha_{41} & \gamma_1+\beta_{11}\int_0^1\gamma_3 f(t)dt+2\beta_{10}\int_0^1\gamma_4 f(t)dt \\ \alpha_{21}+\beta_{21}\alpha_{31}+2\beta_{20}\alpha_{41} & \gamma_2+\beta_{21}\int_0^1\gamma_3 f(t)dt+2\beta_{20}\int_0^1\gamma_4 f(t)dt\end{vmatrix};$$

in which

$$w=\begin{vmatrix}\alpha_{11}+\beta_{11}\alpha_{31}+2\beta_{10}\alpha_{41} & \alpha_{10}+\beta_{11}\alpha_{30}+2\beta_{10}\alpha_{40} \\ \alpha_{21}+\beta_{21}\alpha_{31}+2\beta_{20}\alpha_{41} & \alpha_{20}+\beta_{21}\alpha_{30}+2\beta_{20}\alpha_{40}\end{vmatrix}$$

Now by placing $Y_\varepsilon(0)$ and $Y'_\varepsilon(0)$ in equations (6.3) and (6.4) we will have:

(6.7)

$$\begin{aligned} Y_\varepsilon(1)=\frac{\alpha_{31}}{w}&\begin{vmatrix}\gamma_1+\beta_{11}\int_0^1\gamma_3 f(t)dt+2\beta_{10}\int_0^1\gamma_4 f(t)dt & \alpha_{10}+\beta_{11}\alpha_{30}+2\beta_{10}\alpha_{40} \\ \gamma_2+\beta_{21}\int_0^1\gamma_3 f(t)dt+2\beta_{20}\int_0^1\gamma_4 f(t)dt & \alpha_{20}+\beta_{21}\alpha_{30}+2\beta_{20}\alpha_{40}\end{vmatrix} \\ &+\frac{\alpha_{31}}{w}\begin{vmatrix}\alpha_{11}+\beta_{11}\alpha_{31}+2\beta_{10}\alpha_{41} & \gamma_1+\beta_{11}\int_0^1\gamma_3 f(t)dt+2\beta_{10}\int_0^1\gamma_4 f(t)dt \\ \alpha_{21}+\beta_{21}\alpha_{31}+2\beta_{20}\alpha_{41} & \gamma_2+\beta_{21}\int_0^1\gamma_3 f(t)dt+2\beta_{20}\int_0^1\gamma_4 f(t)dt\end{vmatrix} \\ &-\int_0^1\gamma_3 f(t)dt, \end{aligned}$$

(6.8)

$$Y'_\varepsilon(1) = \frac{2\alpha_{41}}{w}\begin{vmatrix} \gamma_1 + \beta_{11}\int_0^1 \gamma_3 f(t)dt + 2\beta_{10}\int_0^1 \gamma_4 f(t)dt & \alpha_{10} + \beta_{11}\alpha_{30} + 2\beta_{10}\alpha_{40} \\ \gamma_2 + \beta_{21}\int_0^1 \gamma_3 f(t)dt + 2\beta_{20}\int_0^1 \gamma_4 f(t)dt & \alpha_{20} + \beta_{21}\alpha_{30} + 2\beta_{20}\alpha_{40} \end{vmatrix}$$
$$+ \frac{2\alpha_{41}}{w}\begin{vmatrix} \alpha_{11} + \beta_{11}\alpha_{31} + 2\beta_{10}\alpha_{41} & \gamma_1 + \beta_{11}\int_0^1 \gamma_3 f(t)dt + 2\beta_{10}\int_0^1 \gamma_4 f(t)dt \\ \alpha_{21} + \beta_{21}\alpha_{31} + 2\beta_{20}\alpha_{41} & \gamma_2 + \beta_{21}\int_0^1 \gamma_3 f(t)dt + 2\beta_{20}\int_0^1 \gamma_4 f(t)dt \end{vmatrix}$$
$$- 2\int_0^1 \gamma_4 f(t)dt$$

So, we consider the following theorem, which states the relation between local and non-local boundary conditions:

Theorem 6.1. *If the solution $Y_\varepsilon(t)$ is established in the differential equation* (5.3) *with the boundary condition* (5.4) *in interval* $(0,1)$*, with non-local boundary conditions,* (5.4) *together with essential conditions* (5.7) *and* (5.8) *form a linear system* (6.1)*, in which $w \neq 0$ is established, then solution $Y_\varepsilon(t)$ applies to local boundary conditions* (6.5), (6.6), (6.7), *and* (6.8).

7. Sufficient Conditions for the Self-Adjoint of the Perturbation Problem

In this section, we first use the Lagrange Identity and the two-lined form $B(Y_\varepsilon, Z_\varepsilon)$ to look for conditions in which the condition is self-adjoint, on the other hand we know that a problem is self-adjoint when it is also self-adjoint differential equation. Also, its self-adjoint must be established by the boundary condition. Now we consider:

$$\begin{aligned}(L_\varepsilon Y_\varepsilon, Z_\varepsilon) &= \varepsilon \int_0^1 Y''_\varepsilon(t)\, Z_\varepsilon(t)\, dt + (a-\varepsilon)\int_0^1 Y'_\varepsilon(t)\, Z_\varepsilon(t)\, dt \\ &\quad + b\int_0^1 Y_\varepsilon(t)\, Z_\varepsilon(t)\, dt \\ &= B(Y_\varepsilon, Z_\varepsilon) + (Y_\varepsilon, L^*_\varepsilon Z_\varepsilon)\end{aligned}$$

Hence, the two-lined form is as follows:

$$\begin{aligned}B(Y_\varepsilon, Z_\varepsilon) &= \varepsilon Y'_\varepsilon(1)Z_\varepsilon(1) - \varepsilon Y'_\varepsilon(0)Z_\varepsilon(0) - \varepsilon Y_\varepsilon(1)Z'_\varepsilon(1) + \varepsilon Y_\varepsilon(0)Z'_\varepsilon(0) \\ &\quad + (a-\varepsilon)Y_\varepsilon(1)Z_\varepsilon(1) - (a-\varepsilon)Y_\varepsilon(0)Z_\varepsilon(0)\end{aligned}$$

Now assuming $B(Y_\varepsilon, Z_\varepsilon) = 0$ we will have:

$$\begin{aligned}&[\varepsilon Y'_\varepsilon(1) + (a-\varepsilon)Y_\varepsilon(1)]Z_\varepsilon(1) + [-\varepsilon Y_\varepsilon(1)]Z'_\varepsilon(1) \\ &\quad + [-\varepsilon Y'_\varepsilon(0) - (a-\varepsilon)Y_\varepsilon(0)]Z_\varepsilon(0) + [\varepsilon Y_\varepsilon(0)]Z'_\varepsilon(0)\end{aligned}$$

$= 0$

where $Y_\varepsilon(0), Y'_\varepsilon(0), Y_\varepsilon(1), Y'_\varepsilon(1)$ are obtained from relations (6.5), (6.6), (6.7), and (6.8) respectively.

We consider

$$A = \begin{vmatrix} \gamma_1 + \beta_{11}\int_0^1 \gamma_3 f(t)dt + 2\beta_{10}\int_0^1 \gamma_4 f(t)dt & \alpha_{10} + \beta_{11}\alpha_{30} + 2\beta_{10}\alpha_{40} \\ \gamma_2 + \beta_{21}\int_0^1 \gamma_3 f(t)dt + 2\beta_{20}\int_0^1 \gamma_4 f(t)dt & \alpha_{20} + \beta_{21}\alpha_{30} + 2\beta_{20}\alpha_{40} \end{vmatrix},$$

$$B = \begin{vmatrix} \alpha_{11} + \beta_{11}\alpha_{31} + 2\beta_{10}\alpha_{41} & \gamma_1 + \beta_{11}\int_0^1 \gamma_3 f(t)dt + 2\beta_{10}\int_0^1 \gamma_4 f(t)dt \\ \alpha_{21} + \beta_{21}\alpha_{31} + 2\beta_{20}\alpha_{41} & \gamma_2 + \beta_{21}\int_0^1 \gamma_3 f(t)dt + 2\beta_{20}\int_0^1 \gamma_4 f(t)dt \end{vmatrix},$$

$$C = \int_0^1 \gamma_3 f(t)dt, \qquad D = \int_0^1 \gamma_4 f(t)dt$$

And by putting the above relations in $B(Y_\varepsilon, Z_\varepsilon) = 0$, we have:

$$\begin{aligned} &\left[\frac{2\varepsilon\alpha_{41}}{w}A + \frac{2\varepsilon\alpha_{41}}{w}B - 2\varepsilon D + (a-\varepsilon)\frac{\alpha_{31}}{w}A + (a-\varepsilon)\frac{\alpha_{30}}{w}B - (a-\varepsilon)C\right] Z_\varepsilon(1) \\ &\quad + \left[-\frac{\varepsilon\alpha_{31}}{w}A - \frac{\varepsilon\alpha_{30}}{w}B + \varepsilon C\right] Z'_\varepsilon(1) + \left[-\frac{\varepsilon}{w}B - \frac{(a-\varepsilon)}{w}A\right] Z_\varepsilon(0) \\ &\quad + \left[\frac{\varepsilon}{w}A\right] Z'_\varepsilon(0) \\ &= 0 \end{aligned}$$

Now, assuming $C = D = 0$, and by factoring the above relation from sentences A, B, we have by writing the rewritten form as the following system:

$$\begin{cases} \left(\frac{2\varepsilon\alpha_{41}}{w} + (a-\varepsilon)\frac{\alpha_{31}}{w}\right) Z_\varepsilon(1) + \left(-\frac{\varepsilon\alpha_{31}}{w}\right) Z'_\varepsilon(1) \\ \qquad - \left(\frac{(a-\varepsilon)}{w}\right) Z_\varepsilon(0) + \left(\frac{\varepsilon}{w}\right) Z'_\varepsilon(0) = 0, \\ \left(\frac{2\varepsilon\alpha_{41}}{w} + (a-\varepsilon)\frac{\alpha_{30}}{w}\right) Z_\varepsilon(1) + \left(-\frac{\varepsilon\alpha_{30}}{w}\right) Z'_\varepsilon(1) \\ \qquad - \left(\frac{\varepsilon}{w}\right) Z_\varepsilon(0) = 0 \end{cases} \tag{7.1}$$

Here, by adapting the non-local boundary conditions (5.4) to the other boundary conditions (7.1), we present the necessary and sufficient conditions for the self-adjoint of problem (5.3) in the form of a theorem:

Theorem 7.1. *Problem* (5.3) *is self-adjoint with non-local boundary conditions* (5.4) *if and only if we have the following conditions:*

(i) $a = \varepsilon, \quad C = D = 0$

(ii) $\alpha_{10} = \frac{\varepsilon}{w}, \quad \beta_{10} = -\frac{\varepsilon}{w}\alpha_{31}, \quad \alpha_{11} = -\frac{a-\varepsilon}{w},$
$\beta_{11} = \frac{2\varepsilon\alpha_{41}}{w} + (a-\varepsilon)\frac{\alpha_{31}}{w}, \quad \gamma_1 = 0$

(iii) $\alpha_{20} = 0, \quad \beta_{20} = -\frac{\varepsilon}{w}\alpha_{30}, \quad \alpha_{21} = -\frac{\varepsilon}{w},$
$\beta_{21} = \frac{2\varepsilon\alpha_{41}}{w} + (a-\varepsilon)\frac{\alpha_{30}}{w}, \quad \gamma_2 = 0$

8. Obtaining the Condition of Non-Formation a Boundary Layer Near the Boundary Points

In this section, we investigate the formation or non-formation of the boundary layer near the boundary points for the singular perturbation problem that had non-local boundary conditions and the boundary conditions were localized using the 4-step method [6, 10–12]. In this case, we can determine the formation or non-formation of the boundary layer near the boundary points with the help of the Frequent uniform limit method.

Now with respect to the limit states of $\rho_j(\varepsilon)$, we have:

$$\lim_{\varepsilon\to 0, j=1,2} \rho_j(\varepsilon) = \begin{cases} \lim\limits_{\varepsilon\to 0, j=1} \rho_1(\varepsilon) = \begin{cases} -\infty, & a<0 \\ \frac{b}{a}, & a>0 \end{cases} = \begin{cases} \frac{b}{a}>0, & b>0, \\ \frac{b}{a}<0, & b<0 \end{cases} \\ \lim\limits_{\varepsilon\to 0, j=2} \rho_j(\varepsilon) = \begin{cases} \frac{b}{a}, & a<0 = \begin{cases} \frac{b}{a}>0, & b<0 \\ \frac{b}{a}<0, & b>0 \end{cases} \\ +\infty, & a>0 \end{cases} \end{cases}$$

And that we have considered the fundamental solution of the first case, in which $\rho_1(\varepsilon)$ and $\rho_2(\varepsilon)$ are both positive.

Now from the Frequent uniform limit method, we have the following theorem:

Theorem 8.1. *According to the limit states of the relations $Y'_\varepsilon(0)$, $Y_\varepsilon(0)$, $Y_\varepsilon(1)$ and $Y'_\varepsilon(1)$ when $\varepsilon \to 0$, if the following condition is met,*

$$\frac{b}{a}\begin{vmatrix} \gamma_1 + \beta_{11}\int_0^1 -\frac{1}{a}e^{-\frac{b}{a}(1-t)}f(t)dt + 2\beta_{10}\int_0^1 \frac{b}{a^2}e^{-\frac{b}{a}(1-t)}f(t)dt & \alpha_{10} \\ \gamma_2 + \beta_{21}\int_0^1 -\frac{1}{a}e^{-\frac{b}{a}(1-t)}f(t)dt + 2\beta_{20}\int_0^1 \frac{b}{a^2}e^{-\frac{b}{a}(1-t)}f(t)dt & \alpha_{20} \end{vmatrix}$$

$$= \begin{vmatrix} \gamma_1 + \beta_{11}\int_0^1 -\frac{1}{a}e^{-\frac{b}{a}(1-t)}f(t)dt + 2\beta_{10}\int_0^1 \frac{b}{a^2}e^{-\frac{b}{a}(1-t)}f(t)dt & \alpha_{11} + \beta_{11}e^{-\frac{b}{a}} + 2\beta_{10} - \frac{b}{a}e^{-\frac{b}{a}} \\ \gamma_2 + \beta_{21}\int_0^1 -\frac{1}{a}e^{-\frac{b}{a}(1-t)}f(t)dt + 2\beta_{20}\int_0^1 \frac{b}{a^2}e^{-\frac{b}{a}(1-t)}f(t)dt & \alpha_{21} + \beta_{21}e^{-\frac{b}{a}} + 2\beta_{20} - \frac{b}{a}e^{-\frac{b}{a}} \end{vmatrix}$$

then no boundary layer is formed at any of the boundary points.

Proof. According to the limit state $\rho_j(\varepsilon)$, which are in positive form, ie

$$\lim_{\varepsilon\to 0}\rho_1(\varepsilon)=\frac{b}{a}>0,\qquad \lim_{\varepsilon\to 0}\rho_2(\varepsilon)=+\infty,$$

when $\varepsilon\to 0$, the limit states of relation (6.2), are obtained as follows:

$$\begin{array}{lll}\lim\limits_{\varepsilon\to 0}\alpha_{31}=e^{-\frac{b}{a}}, & \lim\limits_{\varepsilon\to 0}\alpha_{30}=0, & \lim\limits_{\varepsilon\to 0}\gamma_3=-\frac{1}{a}e^{-\frac{b}{a}(1-t)},\\ \lim\limits_{\varepsilon\to 0}\alpha_{41}=-\frac{b}{a}e^{-\frac{b}{a}}, & \lim\limits_{\varepsilon\to 0}\alpha_{40}=0, & \lim\limits_{\varepsilon\to 0}\gamma_4=-\frac{b}{a^2}e^{-\frac{b}{a}(1-t)}\end{array}$$

Now the limit states of the relationships $Y_\varepsilon(0),Y'_\varepsilon(0),Y_\varepsilon(1),Y'_\varepsilon(1)$ when $\varepsilon\to 0$, with the help of the above limit relations, we have:

$$Y_0(0)=\frac{1}{w}\begin{vmatrix}\gamma_1+\beta_{11}\int_0^1-\frac{1}{a}e^{-\frac{b}{a}(1-t)}f(t)dt+2\beta_{10}\int_0^1\frac{b}{a^2}e^{-\frac{b}{a}(1-t)}f(t)dt & \alpha_{10}\\ \gamma_2+\beta_{21}\int_0^1-\frac{1}{a}e^{-\frac{b}{a}(1-t)}f(t)dt+2\beta_{20}\int_0^1\frac{b}{a^2}e^{-\frac{b}{a}(1-t)}f(t)dt & \alpha_{20}\end{vmatrix};$$

$$Y'_0(0)=\frac{1}{w}\begin{vmatrix}\alpha_{11}+\beta_{11}e^{-\frac{b}{a}} & \gamma_1+\beta_{11}\int_0^1-\frac{1}{a}e^{-\frac{b}{a}(1-t)}f(t)dt+2\beta_{10}\int_0^1\frac{b}{a^2}e^{-\frac{b}{a}(1-t)}f(t)dt\\ \alpha_{21}+\beta_{21}(-\frac{b}{a}e^{-\frac{b}{a}}) & \gamma_2+\beta_{21}\int_0^1-\frac{1}{a}e^{-\frac{b}{a}(1-t)}f(t)dt+2\beta_{20}\int_0^1\frac{b}{a^2}e^{-\frac{b}{a}(1-t)}f(t)dt\end{vmatrix};$$

$$\begin{aligned}Y_0(1)=&\frac{e^{-\frac{b}{a}}}{w}\\ &\times\begin{vmatrix}\gamma_1+\beta_{11}\int_0^1-\frac{1}{a}e^{-\frac{b}{a}(1-t)}f(t)dt+2\beta_{10}\int_0^1\frac{b}{a^2}e^{-\frac{b}{a}(1-t)}f(t)dt & 0\\ \gamma_2+\beta_{21}\int_0^1-\frac{1}{a}e^{-\frac{b}{a}(1-t)}f(t)dt+2\beta_{20}\int_0^1\frac{b}{a^2}e^{-\frac{b}{a}(1-t)}f(t)dt & \alpha_{20}\end{vmatrix}\\ &+\int_0^1\frac{1}{a}e^{-\frac{b}{a}(1-t)}f(t)dt,\end{aligned}$$

$$\begin{aligned}Y'_0(1)=&\frac{-2\frac{b}{a}e^{-\frac{b}{a}}}{w}\\ &\times\begin{vmatrix}\gamma_1+\beta_{11}\int_0^1-\frac{1}{a}e^{-\frac{b}{a}(1-t)}f(t)dt+2\beta_{10}\int_0^1\frac{b}{a^2}e^{-\frac{b}{a}(1-t)}f(t)dt & \alpha_{10}\\ \gamma_2+\beta_{21}\int_0^1-\frac{1}{a}e^{-\frac{b}{a}(1-t)}f(t)dt+2\beta_{20}\int_0^1\frac{b}{a^2}e^{-\frac{b}{a}(1-t)}f(t)dt & \alpha_{20}\end{vmatrix}\\ &+\frac{-2\frac{b}{a}e^{-\frac{b}{a}}}{w}\\ &\times\begin{vmatrix}\alpha_{11}+\beta_{11}e^{-\frac{b}{a}}+2\beta_{10}-\frac{b}{a}e^{-\frac{b}{a}} & \gamma_1+\beta_{11}\int_0^1-\frac{1}{a}e^{-\frac{b}{a}(1-t)}f(t)dt+2\beta_{10}\int_0^1\frac{b}{a^2}e^{-\frac{b}{a}(1-t)}f(t)dt\\ \alpha_{21}+\beta_{21}e^{-\frac{b}{a}}+2\beta_{20}-\frac{b}{a}e^{-\frac{b}{a}} & \gamma_2+\beta_{21}\int_0^1-\frac{1}{a}e^{-\frac{b}{a}(1-t)}f(t)dt+2\beta_{10}\int_0^1\frac{b}{a^2}e^{-\frac{b}{a}(1-t)}f(t)dt\end{vmatrix}\\ &-2\int_0^1\frac{b}{a^2}e^{-\frac{b}{a}(1-t)}f(t)dt\end{aligned}$$

In which

$$w=\begin{vmatrix} \alpha_{11}+\beta_{11}e^{-\frac{b}{a}}-2\beta_{10}\frac{b}{a}e^{-\frac{b}{a}} & \alpha_{10} \\ \\ \alpha_{21}+\beta_{21}e^{-\frac{b}{a}}-2\beta_{20}\frac{b}{a}e^{-\frac{b}{a}} & \alpha_{20} \end{vmatrix}$$

Here, with the help of the necessary condition limit when $\varepsilon \to 0$, and by putting it in the following conditions for the boundary points,

$$\lim_{\varepsilon\to 0}\lim_{t\to t_0} Y_{\varepsilon}\left(t\right)=\lim_{t\to t_0}\lim_{\varepsilon\to 0} Y_{\varepsilon}\left(t\right)$$

$$\lim_{\varepsilon\to 0}\lim_{t\to t_0} Y'_{\varepsilon}\left(t\right)=\lim_{t\to t_0}\lim_{\varepsilon\to 0} Y'_{\varepsilon}\left(t\right)$$

So that the above relations are established, we have the following condition:

$$\frac{b}{a}\begin{vmatrix} \gamma_1+\beta_{11}\int_0^1-\frac{1}{a}e^{-\frac{b}{a}(1-t)}f(t)dt+2\beta_{10}\int_0^1\frac{b}{a^2}e^{-\frac{b}{a}(1-t)}f(t)dt & \alpha_{10} \\ \\ \gamma_2+\beta_{21}\int_0^1-\frac{1}{a}e^{-\frac{b}{a}(1-t)}f(t)dt+2\beta_{20}\int_0^1\frac{b}{a^2}e^{-\frac{b}{a}(1-t)}f(t)dt & \alpha_{20} \end{vmatrix}$$

$$=\begin{vmatrix} \gamma_1+\beta_{11}\int_0^1-\frac{1}{a}e^{-\frac{b}{a}(1-t)}f(t)dt+2\beta_{10}\int_0^1\frac{b}{a^2}e^{-\frac{b}{a}(1-t)}f(t)dt & \alpha_{11}+\beta_{11}e^{-\frac{b}{a}}+2\beta_{10}-\frac{b}{a}e^{-\frac{b}{a}} \\ \gamma_2+\beta_{21}\int_0^1-\frac{1}{a}e^{-\frac{b}{a}(1-t)}f(t)dt+2\beta_{20}\int_0^1\frac{b}{a^2}e^{-\frac{b}{a}(1-t)}f(t)dt & \alpha_{21}+\beta_{21}e^{-\frac{b}{a}}+2\beta_{20}-\frac{b}{a}e^{-\frac{b}{a}} \end{vmatrix}$$

This determines that no boundary layer is formed near the boundary points. □

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