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On 1-index of Unstable Spacelike Hypersurfaces in Pseudo-Euclidean Spheres

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Abstract. In mathematical physics, the stable hypersurfaces of constant mean curvature in pseudo-Euclidian spheres have been interested by many researchers on general relativity. As an extension, the notion of index of stability has been introduced for unstable ones. The stability index (as a rate of distance from being stable) is defined in terms of the Laplace operator Δ as the trace of Hessian tensor. In this paper, we study an extension of stability index (namely, 1-index) of hypersurfaces with constant scalar curvature in pseudo-Euclidian sphere $\mathbb{S}_1^{n+1}$. 1-index is defined based on the Cheng-Yau operator $\square$ as a natural extension of Δ.

1. Introduction

One of the simple invariants on (semi-)Euclidean spaces is the Ricci scalar (i.e. scalar curvature). It gives to each point a real number related to the locally intrinsic geometry of a neighborhood of that point. Sharply, this scalar says the amount by which the volume of a small geodesic ball in a semi-Euclidean space deviates from that of the standard ball in Euclidean space. In dimension 2, it characterizes the curvature of a surface, but in higher dimensions the curvature involves more independent quantities. In relativity theory, the Lagrangian density of Einstein-Hilbert action can be described by Ricci scalar. The Euler-Lagrange equations for this Lagrangian under metrical variations constitutes the vacuum Einstein field equations. The stationary metrics are known as Einstein metrics. The positive mass theorem of Schoen, Yau and Witten

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shows the power of Ricci scalar and it can give a clear understanding of manifolds with positive scalar curvature.

In the eighties, Cheng and Yau ([9, 10]) have studied the geometric properties of constant scalar curvature hypersurfaces of Minkowski spacetimes. They have generalized the Calabi-Bernstein problem on these hypersurfaces. Before them, Calabi ([7]) had showed that each maximal complete hypersurface in the m-dimensional Minkowski space (for $m < 6$) is totally geodesic. In 1977, A.J. Goddard has generalized Calabi's result as a conjecture, which says that each constant mean curvature complete spacelike hypersurface in $\mathbb{S}_1^m$ is totally umbilic ([12]). In 1984, S. Nishikawa ([16]) proved a proposition similar to Calabi's result on maximal complete hypersurfaces of Lorentz manifolds. In 1987, Ramanathan ([19]) and Akutagawa ([1]) (independently) have affirmed Goddard conjecture first for $m = 3$ with assumption $H^2 \leq 1$ and then for $m \geq 4$ with the general condition $H^2 \leq \frac{4(m-2)}{(m-1)^2}$. Of course, in 1988, Ishihara proved that this property dose not occur in pseudo-hyperbolic spaces ([13]). Simultaneously, the stability of hypersurfaces of constant mean curvature in Riemannian manifolds had attracted a great deal of mathematics and physics interest ([3–5]). Also, in Lorentz context Li ([14]) proved that every closed spacelike hypersurface in the pseudo-Euclidean sphere $\mathbb{S}_1^{n+1}$ having constant scalar curvature satisfying $(n-2)/n \leq R \leq 1$ has to be totally umbilic. Particularly, the closed ones of normal scalar curvature 1 in $\mathbb{S}_1^{n+1}$ have to be totally geodesic. Recently, Colombo et. al. ([11]) have showed that the complete maximal hypersurfaces of $\mathbb{S}_1^{n+1}$ are compact, totally geodesic and unstable.

The area functional first variation problem deals with minimal (and maximal) hypersurfaces as its critical points. But, in order to get an exact knowledge of minimal (and maximal) hypersurfaces we need to study the second variation problem which introduces the concept of stability. The stability is defined using the Laplace operator Δ which has a straightforward extension (namely 1-stability) defined based on the Cheng-Yau operator $\Box$ (as a natural extension of Δ). As another extensional stage, the stability index is defined on unstable hypersurfaces as a rate of distance from being stable. Intuitively, the stability index of a hypersurface is the dimension of a distribution containing all directions that the hypersurface fails to be of minimum area. In this context, the stability index of minimal hypersurfaces in Euclidean spheres (with special importance) has been studied in [2, 6].

In [2], Alias also has paid attention to weak stability index of constant mean curvature hypersurfaces. In [6], the authors have studied the stability index of compact non-totally geodesic minimal hypersurface in the Euclidean spheres. They have determined that the only minimal hypersurface of index $m+2$ in m-sphere is the Clifford tori and other oriented closed minimal hypersurface with negative 3rd mean curvature has the stability index greater than $2m+2$. In this paper, we focus on the stability 1-index of spacelike hypersurfaces with constant scalar curvature in pseudo-Euclidean sphere $\mathbb{S}_1^{n+1}$. We show that, the 1-index of complete totally geodesic hypersurfaces of normal scalar curvature 1 and positive mean curvature in $\mathbb{S}_1^{n+1}$ is 1. Also, in the non-totally case, we prove that the 1-index is greater than $n+2$.

2. Preliminaries

We recall the notations and prerequisite concepts from [8, 17]. The Euclidean m-space $\mathbb{R}^m$ endowed with the scalar product

$$\mathbf{d}(x,y) := -\sum_{j=1}^{\alpha} x_j y_j + \sum_{j=\alpha+1}^{m} x_j y_j$$

will be denoted by $\mathbb{R}^m_\alpha$ (where $\alpha \leq m$ is a non-negative integer number). We use only two special cases $\alpha = 0, 1$. $\mathbb{R}^m = \mathbb{R}^m_0$ is the ordinary Euclidean space (with dot product) and $\mathbb{L}^m = \mathbb{R}^m_1$ is called the pseudo-Euclidean (Lorentz) space. For $\rho > 0$,

$$\mathbb{S}^{n+1}_\alpha(\rho) = \left\{ y \in \mathbb{R}^{n+2}_\alpha | \mathbf{d}(y,y) = \rho^2 \right\}$$

denotes the Euclidean sphere (when $\alpha = 0$) and pseudo-Euclidean sphere (when $\alpha = 1$) of radius ρ and curvature $1/\rho^2$, and

$$\mathbb{H}^{n+1}_\alpha(-\rho) = \left\{ y \in \mathbb{R}^{n+2}_{\alpha+1} | \mathbf{d}(y,y) = -\rho^2 \right\}$$

denotes the hyperbolic space (when $\alpha = 0$) and pseudo-hyperbolic space (when $\alpha = 1$) of radius ρ and curvature $-1/\rho^2$. In general, $\tilde{M}^{n+1}_\alpha(c)$ stands for $\mathbb{R}^{n+1}_\alpha$, $\mathbb{S}^{n+1}_\alpha = \mathbb{S}^{n+1}_\alpha(1)$ and $\mathbb{H}^{n+1}_\alpha = \mathbb{H}^{n+1}_\alpha(-1)$ when c is 0, 1 and -1, respectively.

We consider $\mathbf{x} : M^n \to \mathbb{S}^{n+1}_1$ as an isometrically immersed orientable spacelike hypersurface. Let ∇^0, $\bar{\nabla}$ and ∇ denote the Levi-Civita connection respectively on $\mathbb{L}^{n+2}$, $\mathbb{S}^{n+1}_1$ and M^n. First, we remember two formulae due to Weingarten and Gauss respectively as

$$\begin{aligned} AX &= -\bar{\nabla}_X \mathbf{n} = -\nabla^0_X \mathbf{n}, \\ \nabla^0_X Y &= \bar{\nabla}_X Y - \mathbf{d}(X,Y)\mathbf{x} \\ &= \nabla_X Y - \mathbf{d}(AX,Y)\mathbf{n} - \mathbf{d}(X,Y)\mathbf{x}, \end{aligned}$$

for every $X, Y \in \mathcal{K}(M^n)$. As the usual notations, $\mathcal{K}(M^n)$ stands for the set of all smooth vector fields on M^n and A denotes the second fundamental form on M^n related to a chosen unit normal timelike vector field $\mathbf{n}$. Since $\mathbb{S}^{n+1}_1$ is time-oriented and M^n is orientable in $\mathbb{S}^{n+1}_1$, we can assume that $\mathbf{n}$ is a unit normal timelike vector on M^n with time orientation of $\mathbb{S}_1^{n+1}$.

By definition, the curvature tensor $\boldsymbol{c}$ is defined as

$$\boldsymbol{c}(X,Y)Z = \nabla_{[X,Y]}Z - [\nabla_X, \nabla_Y]Z.$$

([,] stands for the Lie bracket) for every $X, Y, Z \in \mathcal{K}(M^n)$. By the Gauss equation it has a simple formula as

$$\boldsymbol{c}(X,Y)Z = \mathbf{d}(X,Z)Y - \mathbf{d}(Y,Z)X - \mathbf{d}(AX,Z)AY + \mathbf{d}(AY,Z)AX.$$

Also, the Codazzi equation of M^n is given by

$$(\nabla_X A)Y = (\nabla_Y A)X.$$

We can assume that, with respect to $\mathcal{E} = \{e_i\}_{i=1}^n$ as the tangent frame of unite principal directions on M^n, the operator A is diagonal. The hypersurface $\mathbf{x} : M^n \to \mathbb{S}_1^{n+1}$ is said to be totally umbilic if A is a multiple of the identity operator I on $\mathcal{K}(M^n)$ and similarly, it is called totally geodesic if A vanishes. Denoting the eigenfunctions of A (as the principal curvatures of M^n) by $\kappa_1, \ldots, \kappa_n$, we define the kth elementary function

$$s_k := \sum_{1 \le j_1 < \cdots < j_k \le n} \kappa_{j_1} \cdots \kappa_{j_k}$$

(for $k = 1, 2, \ldots, n$). The kth mean curvature of M^n is defined by $H_k := \frac{(-1)^k}{\binom{n}{k}} s_k$. By definition, $\mathbf{x} : M^n \to \mathbb{S}_1^{n+1}$ is said to be k-maximal if $H_{k+1} \equiv 0$ on M^n. A 0-maximal hypersurface is called maximal. Specially, we define $H_0 := 1$. We will denote the ordinary mean curvature vector filed on M^n by $\mathbf{H} := H_1 \mathbf{n}$. The second mean curvature H_2 and the normalized scalar curvature R satisfy the equality $H_2 := n(n-1)(1-R)$.

We will use a transformation $T : \mathcal{K}(M^n) \to \mathcal{K}(M^n)$ defined by $T := A - \mathrm{tr}(A)I = A - s_1 I$. For $i = 1, 2, \ldots, n$, by equalities $Ae_i = \kappa_i e_i$ we have $Te_i = \tau_i e_i$ where $\tau_i = -\sum_{j \ne i} \kappa_j$. Also, T satisfies the following identities:

$$\begin{aligned} \mathrm{tr}(T) = (1-n)s_1, \mathrm{tr}(A \circ T) \\ = -s_2, \mathrm{tr}(A^2 \circ T) \\ = 3s_3 - s_1 s_2. \end{aligned}$$

In the rest, we use two height functions $\phi_{\mathbf{a}} := \mathbf{d}(\mathbf{x}, \mathbf{a})$ and $\psi_{\mathbf{a}} := \mathbf{d}(\mathbf{n}, \mathbf{a})$ on M^n, associated to any fixed vector $\mathbf{a} \in \mathbb{L}^{n+2}$. The dimension of the following linear subspaces of $\mathcal{K}(M^n)$ is one of technical challenges.

Using subspaces $\Phi := \left\{\phi_{\mathbf{a}} | \mathbf{a} \in \mathbb{L}^{n+2}\right\}$ and $\Psi := \left\{\psi_{\mathbf{a}} | \mathbf{a} \in \mathbb{L}^{n+2}\right\}$ of $\mathcal{C}^\infty(M^n)$ we introduce four additional subspaces as $\hat{\Phi} := \mathrm{span}(\Phi \cup \{1\})$, $\hat{\Psi} := \mathrm{span}(\Psi \cup \{1\})$, $\Theta := \mathrm{span}(\Phi \cup \Psi)$ and $\hat{\Theta} := \mathrm{span}(\Theta \cup \{1\})$.

Note that the linear subspaces Φ and Ψ are generated by $\mathcal{B} := \{\phi_{e_i}\}_{i=1}^{n+2}$ and $\hat{\mathcal{B}} := \{\psi_{e_i}\}_{i=1}^{n+2}$, respectively, where $\{e_i\}_{i=1}^{n+2}$ is the canonical basis of $\mathbb{L}^{n+2}$.

Now, we recall the Cheng-Yau operator which has been introduced in [10]. The Cheng-Yau operator $\Box : \mathcal{C}^\infty(M^n) \to \mathcal{C}^\infty(M^n)$ is a differential operator of order 2 defined as $\Box f := \mathrm{tr}(T \circ \nabla^2 f)$ for every $f \in \mathcal{C}^2(M^n)$, where $\nabla^2 f$ is the Hessian of f given by

$$\begin{aligned}\mathbf{d}\left(\nabla^2 f(V), W\right) &= \mathrm{Hess}(f)(V, W)\\ &= V(W(f)) - (\nabla_V W)(f)\end{aligned}$$

for every $V, W \in \mathcal{K}(M^n)$. $\Box f$ has a computational formula as

$$\Box f = \sum_{i=1}^{n} (nH - \kappa_i) \nabla^2 f(e_i, e_i),$$

which gives a simple equality on umbilical points (where $\kappa_1 = \kappa_2 = \cdots = \kappa_n$) as

$$\begin{aligned}\Box f &= (n-1)H \sum_{i=1}^{n} \nabla^2 f(e_i, e_i)\\ &= (n-1)H\Delta f.\end{aligned} \tag{2.1}$$

Remember that, Δ is the Laplace operator defined by $\Delta f = \mathrm{tr}(\nabla^2 f)$.

Definition 2.1. For a connected orientable spacelike hypersurface $\mathbf{x} : M^n \to \mathbb{S}_1^{n+1}$, a differentiable map $\mathbf{X} : M^n \times (-\delta, \delta) \to \mathbb{S}_1^{n+1}$ is said to be a *variation* of $\mathbf{x}$ if both of the following conditions hold:

- (i) The map $\mathbf{X}_t : M^n \to \mathbb{S}_1^{n+1}$ by rule $\mathbf{X}_t(p) := \mathbf{X}(p, t)$ for every $t \in (-\delta, \delta)$ is an isometric immersion,
- (ii) $\mathbf{X}_0 = \mathbf{x}$ and $\mathbf{X}_t|_{\mathrm{bd}(M^n)} = \mathbf{x}|_{\mathrm{bd}(M^n)}$ for every $t \in (-\delta, \delta)$.

The variational field associated with a variation $\mathbf{X}$ is the vector field $\frac{\partial \mathbf{X}}{\partial t}\big|_{t=0}$. Putting $f := -\mathbf{d}\left(\frac{\partial \mathbf{X}}{\partial t}, \mathbf{n}_t\right)$, we have

$$\frac{\partial \mathbf{X}}{\partial t} = f\mathbf{n}_t + \left(\frac{\partial \mathbf{X}}{\partial t}\right)^\top.$$

The superscript $\top$ stands for tangent component and $\mathbf{n}_t$ denotes the unit normal vector field along $\mathbf{X}_t$. For every compactly supported smooth real function f on M^n (i.e. $f \in \mathcal{C}_0^\infty(M^n)$ satisfying equality $\int_{M^n} f dM^n = 0$, one can find a volume-preserving normal variation of M^n with variational field $f\mathbf{n}$ (see [21]). In this case, if the function f has compact support (i.e. $f \in \mathcal{C}_0^\infty(M)$), X is called a compactly supported normal variation.

Associated to each variation $X : M^n \times (-\delta, \delta) \to \mathbb{S}_1^{n+1}$, using the differentiable Jacobi operator $J := -\mathrm{tr}(T \circ A^2)I + \mathrm{tr}(T)I + \square$, we define a real function $B : \mathcal{C}_c^\infty(M^n) \to \mathbb{R}$ by the rule $B(f) := \int_{M^n} fJfdM^n$. A spacelike hypersurface $\mathbf{x} : M^n \to \mathbb{S}_1^{n+1}$ that maximizes B is called 1-stable.

Definition 2.2. Let $x : M^n \to \mathbb{S}_1^{n+1}$ be a connected spacelike hypersurface in the de Sitter space.

(i) M^n is said to be *1-stable*, if $B(f) \leq 0$ for every function $f \in \mathcal{C}_0^\infty(M^n)$,

(ii) Otherwise, the *1-index* of stability of M^n, denoted by $\mathrm{Ind}_1(M^n)$ is the dimension of $\{f \in \mathcal{C}_0^\infty(M^n) | B(f) > 0\}$.

3. Auxiliary Lemmas

On a connected oriented spacelike hypersurface $\mathbf{x} : M^n \to \mathbb{S}_1^{n+1}$, the Laplace operator satisfies a self-adjoint condition as

$$\int_{M^n} f(\Delta g)dM^n = \int_{M^n} g(\Delta f)dM^n$$

for every $f, g \in \mathcal{C}_0^\infty(M^n)$. In [10], the same property has been showed for $\square$ as

$$\int_{M^n} f(\square g)dM^n = \int_{M^n} g(\square f)dM^n.$$

Also, one can see that for every function $f \in \mathcal{C}^\infty(M^n)$ with compact support, we have

$$\begin{aligned} &\int_{M^n} \square f dM^n = 0, \\ &\int_{M^n} \left(f\square f + \mathbf{d}\left(T \ \mathrm{grad}(f), \mathrm{grad}(f)\right)\right) dM^n = 0. \end{aligned} \tag{3.1}$$

The first lemma states the role of $\square$ in a variation equality.

Lemma 3.1. *Let $\boldsymbol{x} : M^n \to \mathbb{S}_1^{n+1}$ be a connected spacelike hypersurface, $\boldsymbol{X}$ be a variation of $\boldsymbol{x}$ and $f := -\boldsymbol{d}\left(\frac{\partial \boldsymbol{X}}{\partial t}, \boldsymbol{n}_t\right)$. Then, we have:*

$$\frac{\partial s_2}{\partial t} = \square f + \mathrm{tr}(T)f - \mathrm{tr}\left(A^2 \circ T\right) f + \boldsymbol{d}\left(\left(\frac{\partial X}{\partial t}\right)^{\perp}, \nabla s_2\right).$$

Associated to each variation of $\mathbf{x} : M^n \to \mathbb{S}_1^{n+1}$, the *area functional* $A : (-\delta, \delta) \to \mathbb{R}$ is defined by $A(t) := n \int_{M^n} H dM_t$. If $s_2 = 0$, there is a function $f \in \mathcal{C}_0^\infty(M^n)$ such that the second variation of $A(t)$ is as in the following lemma.

Lemma 3.2. *Let* $\boldsymbol{x} : M^n \to \mathbb{S}_1^{n+1}$ *be a connected spacelike hypersurface with normalized scalar curvature 1, and* $\boldsymbol{X}$ *be compactly supported normal variation of* $\boldsymbol{x}$ *and* $f(t) := -\boldsymbol{d}\left(\frac{\partial \boldsymbol{X}}{\partial t}, \boldsymbol{n}_t\right)$. *Then, we have*

$$\mathrm{A}''(t) = 2\int_{M^n} \left[f\Box f + ((1-n)s_1 - (3s_3 - s_1 s_2))f^2 \right] dM_t$$

where, dM_t *denotes the volume element of* M^n *with the metric induced by* $\boldsymbol{X}_t$.

Lemma 3.3. *Let* $\boldsymbol{x} : M^n \to \mathbb{S}_1^{n+1}$ *be a connected oriented spacelike hypersurface and* $n \geq 2$. *Then, for each* $\boldsymbol{a} \in \mathbb{L}^{n+2}$. *Then, we have:*

$$\Box\lambda_a = 2s_2\gamma_a + (n-1)s_1\lambda_a, \tag{3.2}$$

$$\Box\gamma_a = \boldsymbol{d}\left(\mathrm{grad}H_2, \boldsymbol{a}^\top\right) + (3s_3 - s_1 s_2)\gamma_a - 2s_2\lambda_a. \tag{3.3}$$

Proof. Every $\mathbf{a} \in \mathbb{L}^{n+2}$ has a decomposition as $\mathbf{a} = \mathbf{a}^\top - \gamma_{\mathbf{a}}\mathbf{n} + \lambda_{\mathbf{a}}\mathbf{x}$. For each $V \in \mathcal{K}(M^n)$ we have

$$\begin{aligned} V\lambda_{\mathbf{a}} &= \mathbf{d}(V, \mathbf{a}) \\ &= \mathbf{d}\left(V, \mathbf{a}^\top\right), \end{aligned}$$

and

$$\begin{aligned} V\gamma_{\mathbf{a}} &= -\mathbf{d}(AV, \mathbf{a}) \\ &= -\mathbf{d}\left(V, A\mathbf{a}^\top\right). \end{aligned}$$

Hence, we have $\mathrm{grad}\lambda_{\mathbf{a}} = \mathbf{a}^\top$ and $\mathrm{grad}\gamma_{\mathbf{a}} = -A\mathbf{a}^\top$. So, we get

$$\begin{aligned} \text{(i) } \nabla_V(\mathrm{grad}\lambda_{\mathbf{a}}) &= \nabla_V \mathbf{a}^\top \\ &= -\gamma_{\mathbf{a}}AV - \lambda_{\mathbf{a}}V, \\ \text{(ii) } \nabla_V(\mathrm{grad}\gamma_{\mathbf{a}}) &= -\nabla_V\left(A\mathbf{a}^\top\right) \\ &= -\nabla_V(A)\mathbf{a}^\top - \gamma_{\mathbf{a}}A^2V + \lambda_{\mathbf{a}}AV. \end{aligned} \tag{3.4}$$

which, by the Codazzi identity, gives

$$\begin{aligned} \nabla A\left(\mathbf{a}^\top, V\right) &= \nabla A\left(V, \mathbf{a}^\top\right) \\ &= \left(\nabla_{\mathbf{a}^\top} A\right) V. \end{aligned}$$

By equations (3.1), from equations (3.4) we get

$$\begin{aligned} \Box\lambda_{\mathbf{a}} &= \mathrm{tr}(A \circ T)\gamma_{\mathbf{a}} - \mathrm{tr}(T)\lambda_{\mathbf{a}} \\ &= n(n-1)\left[H_2\gamma_a - H_1\lambda_a\right] \\ &= 2s_2\gamma_a + (n-1)s_1\lambda_a, \end{aligned}$$

and

$$\begin{aligned}\square\gamma_{\mathbf{a}} &= -\mathrm{tr}\,(T\circ\nabla_{\mathbf{a}^\top}A) - \mathrm{tr}(A^2\circ T)\gamma_{\mathbf{a}} + \mathrm{tr}(A\circ T)\lambda_{\mathbf{a}}\\ &= \frac{1}{2}n(n-1)\left[\mathbf{d}\left(\mathrm{grad}(H_2),\mathbf{a}^\top\right) + (nH_1H_2-(n-2)H_3)\gamma_a - 2H_2\lambda_a\right]\\ &= \mathbf{d}\left(\mathrm{grad}H_2,\mathbf{a}^\top\right) + (3s_3 - s_1s_2)\gamma_a - 2s_2\lambda_a.\end{aligned}$$

□

Lemma 3.4 ([8]). *On each closed spacelike hypersurface $\boldsymbol{x}: M^n \to \mathbb{S}_1^{n+1}$ with principal curvatures $\kappa_1,\dots,\kappa_n$, we have:*
$H^2 \geq H_2$, and the equality happens if and only if $\kappa_1 = \dots = \kappa_n$.

Lemma 3.5 ([8]). *On every spacelike hypersurface in $\mathbb{S}_1^{n+1}$ with $H_2 = H_3 \equiv 0$ we have $H_k = 0$ on M^n for $k = 4,\dots,n$.*

Lemma 3.6. (i) *Each complete compact spacelike hypersurface in $\mathbb{S}_1^{n+1}$ (where $n\geq 2$) is diffeomorphic to S^n,*
(ii) *Every compact totally umbilic spacelike hypersurface in $\mathbb{S}_1^{n+1}$ (for $n \geq 2$) is a round n-sphere.*

4. 1-stable Hypersurfaces

Proposition 4.1 ([11]). *Let $\boldsymbol{x}: M^n \to \mathbb{S}_1^{n+1}$ be a spacelike hypersurface that is complete, compact, oriented and 1-maximal (where $n \geq 2$) and its mean curvature is positive and its shape operator has* $\mathrm{rank}(A)\geq 2$. *Then it is 1-unstable.*

Now, we state some results on 1-stability of closed hypersurfaces.

Theorem 4.2. *Let $n \geq 2$ and M^n be a totally umbilic closed oriented spacelike hypersurface of $\mathbb{S}_1^{n+1}$ with constant mean curvature H_1 satisfying $0 < H_1 < \sqrt{\frac{2}{n-1}}$. Then M^n is 1-stable.*

Proof. By the assumption, the ordinary mean curvature H_1 of M^n is constant and satisfies $0 < H_1 < \sqrt{\frac{2}{n-1}}$. By the assumption M^n is totally umbilic, so we have $\kappa_1 = \kappa_2 = \dots = \kappa_n = -H_1 < 0$ on M^n. Hence, κ_i and all $H_i = (-1)^i\kappa_1^i$ (for $i = 1,2,\dots,n$) are constant on M^n. By the definition of 1-stability we have to prove

$$\begin{aligned}B(f) &= \int_{M^n} fJ(f)dM^n\\ &= \int_{M^n}\left[\square f + \mathrm{tr}(T)f - \mathrm{tr}\left(A^2\circ T\right)f\right]fdM^n\\ &\leq 0,\end{aligned}$$

for every $f \in \mathcal{C}^\infty(M^n)$.

Since M^n is totally umbilical, the first eigenvalue of the Laplace operator Δ on M^n is $\lambda(M^n) = n(1 + H_1^2)$. By equation (2.1) we have

$$\begin{aligned}\Box f &= (n-1)H_1 \Delta f \\ &= (n-1)H_1(1+H_1^2)f,\end{aligned}$$

for every $f \in \mathcal{C}^\infty(M)$. Also, we have

$$\mathrm{tr}(T) = (n-1)H_1,$$

and

$$\begin{aligned}\mathrm{tr}(A^2 \circ T) &= \frac{n(n-1)}{2}\left[nH_1H_2 - (n-2)H_3\right] \\ &= \frac{n(n-1)}{2}\left[nH_1H_1^2 - (n-2)H_1^3\right] \\ &= n(n-1)H_1^3.\end{aligned}$$

So, we get

$$B(f) = \left[(n-1)H_1(2 + (1-n)H_1^2)\right] \int_M f^2 dM \leq 0,$$

which means that M^n is 1-stable. $\Box$

Before the next proposition we need to recall a computational lemma.

Lemma 4.3. *Let $n > 2$ and $c_1, c_2, \ldots, c_n$ be constant real numbers, $\sum_{i=1}^{n} c_i = 0$ and $\sum_{i=1}^{n} c_i^2 = d^2$ where $d \geq 0$. Then, we have*

$$\left|\sum_{i=1}^{n} c_i^3\right| \leq \frac{n-2}{\sqrt{n(n-1)}} d^3, \tag{4.1}$$

and, the equality holds if and only if at least $(n-1)$ of c_is are equal.

Proposition 4.4. *Let $n > 2$ and $\boldsymbol{x} : M^n \to \mathbb{S}_1^{n+1}$ be an isometric immersion of a complete connected 1-maximal spacelike hypersurface into the de Sitter space $\mathbb{S}_1^{n+1}$ with a chosen timelike unit normal vector filed $\boldsymbol{n}$ on $\boldsymbol{x}(M^n)$. If there exists a constant vector $\boldsymbol{a} \in \mathbb{L}^{n+2}$ such that $b = \boldsymbol{d}(\boldsymbol{n}, \boldsymbol{a})$ is constant and $\boldsymbol{d}(\boldsymbol{a}, \boldsymbol{a}) > -b^2$, then $\boldsymbol{x}(M^n)$ is a totally geodesic round sphere in $\mathbb{S}_1^{n+1}$.*

Proof. Taking $b := \gamma_{\mathbf{a}} = \mathbf{d}(\mathbf{n}, \mathbf{a})$ as a constant real number, we have

$$\begin{aligned}\mathbf{d}(b\mathbf{n} + \mathbf{a}, b\mathbf{n} + \mathbf{a}) &= b^2\mathbf{d}(\mathbf{n}, \mathbf{n}) + 2b\mathbf{d}(\mathbf{n}, \mathbf{a}) + \mathbf{d}(\mathbf{a}, \mathbf{a}) \\ &= -b^2 + 2b^2 + \mathbf{d}(\mathbf{a}, \mathbf{a}) \\ &= b^2 + \mathbf{d}(\mathbf{a}, \mathbf{a}) \\ &> 0.\end{aligned}$$

So, when $\mathbf{a}$ is not a spacelike vector, b is non-zero. By equality (3.3) and assumption $H_2 \equiv 0$, we get $\Box\gamma_a = 0$ and then

$$\begin{aligned}(4.2)\quad \frac{1}{2}n(n-1)\Big[&(grad(H_2),\mathbf{a}^\top) + (nH_1H_2 - (n-2)H_3)\gamma_a - 2H_2\lambda_a\Big] \\ &= \frac{-1}{2}n(n-1)(n-2)H_3,\end{aligned}$$

and then, $H_3 = 0$. From $H_2 = H_3 = 0$, by Proposition 3.5, we get

$$H_i = 0 \tag{4.3}$$

for $i = 2, 3, \ldots, n$.

Now, we use an auxiliary operator defined by $\bar{\Box} := A + H_1 I$. By using result (4.3) we have

$$\begin{aligned}\operatorname{tr}\left(A^3\right) &= s_1^3 - 3s_1s_2 + 3s_3 \\ &= s_1^3 \\ &= -n^3H_1^3,\end{aligned}$$

and

$$\begin{aligned}|\bar{\Box}|^2 &= \sum_{i=1}^{n}(\kappa_i + H_1)^2 \\ &= (nH_1)^2 - nH_1^2 \\ &= n(n-1)H_1^2,\end{aligned}$$

and then

$$\begin{aligned}\operatorname{tr}\left(\bar{\Box}^3\right) &= 3s_3 - s_1s_2 - (n-3)H_1|\bar{\Box}|^2 - n(n-1)H_1^3 \\ &= -(n-3)H_1|\bar{\Box}|^2 - n(n-1)H_1^3 \\ &= -(n-2)H_1|\bar{\Box}|^2,\end{aligned}$$

and

$$\left|\operatorname{tr}(\bar{\Box}^3)\right| = \frac{n-2}{\sqrt{n(n-1)}}\left|\bar{\Box}^3\right|.$$

So, by Lemma 4.1, at least $n-1$ of the eigenvalues of $\bar{\Box}$ are equal. Then, at least $n-1$ of the principal curvatures of M^n are equal. Without loss of generality, we assume that $\kappa_1 = \kappa_2 = \cdots = \kappa_{n-1}$. From $H_n = 0$ we get $\kappa_1^{n-1}\kappa_n = 0$. If $\kappa_1 \neq 0$, then $\kappa_n = 0$, on the other hand, from $H_2 = 0$ we obtain $\binom{n-1}{2}\kappa_1^2 + (n-1)\kappa_1\kappa_n = 0$, then $(n-1)\kappa_1[\frac{n}{2}\kappa_1 + \kappa_n] = 0$ which gives $\kappa_n = -\frac{n}{2}\kappa_1 = 0$. Therefore, we get a contradiction which implies that $\kappa_1 = 0$. Hence, the sectional curvature of $\mathbf{x}(M^n)$ is 1 identically, and $\mathbf{x}(M^n)$ is totally geodesic round sphere in $\mathbb{S}_1^{n+1}$. Similarly, when $b = 0$, we have $\mathbf{d}(\mathbf{a},\mathbf{a}) > 0$ (i.e. $\mathbf{a}$ is spacelike) and $H_i = 0$ for $i = 2, 3, \ldots, n$. By formula (3.4)(i) we have

$$\begin{aligned}\mathrm{Hess}(\lambda_{\mathbf{a}})(V,W) &= \mathbf{d}\left(\nabla_V(\mathrm{grad}\lambda_{\mathbf{a}}), W\right)\\ &= -\lambda_{\mathbf{a}}\mathbf{d}(V,W)\end{aligned}$$

for every $V \in \chi(M)$, which implies that $\mathbf{x}(M^n)$ is totally umbilical ([15]). So, admitting a non-constant function $f = \lambda_{\mathbf{a}}$ with $\nabla_V df = -f$, the hypersurface M^n is isometric to a totally geodesic round sphere of $\mathbb{S}_1^{n+1}$. □

5. Stability 1-index

Theorem 5.1. *On each connected orientable non-totally geodesic spacelike hypersurface* $\boldsymbol{x}\colon M^n \to \mathbb{S}_1^{n+1}$ *we have:*

(i) $\dim(\bar{\Lambda}) = 1 + \dim(\Lambda) = n + 3$,
(ii) $\dim(\bar{\Gamma}) = 1 + \dim(\Gamma) = n + 3$.

Proof. (i) We consider Λ as a linear space generated by $\mathcal{B} := \{\lambda_{e_i}\}_{i=1}^{n+2}$, where $\{e_i\}_{i=1}^{n+2}$ is the canonical basis of $\mathbb{L}^{n+2}$. First, we show the linearly independence of $\mathcal{B}$. If $\mathcal{B}$ is linearly dependent, then we have a nonzero finite sequence of real numbers $\{r_i\}_{i=1}^{n+2}$ such that $\sum_{i=1}^{n+2} r_i\lambda_{e_i} \equiv 0$. Putting $v := \sum_{i=1}^{n+2} r_i e_i$ we have $\lambda_v = \mathbf{d}(\mathbf{x}, v) \equiv 0$. One can assume that v is a nonzero vector with $\mathbf{d}(v,v) \in \{-1, 0, 1\}$. On the other hand, the image of M by the isometric immersion $\mathbf{x}$ lies in the spacelike hyperplane with a timelike normal vector. Remember that, the totally geodesic spacelike hypersurfaces in $\mathbb{S}_1^{n+1}$ are obtained as intersection of a spacelike hyperplane of $\mathbb{L}^{n+2}$ with $\mathbb{S}_1^{n+1}$, and the causal character of the hyperplane determines the type of the hypersurface. Since $\mathbf{d}(\mathbf{x},\mathbf{x}) = 1$, we obtain $\mathbf{d}(v,v) = -1$ and $\mathbf{x}(M^n)$ is $\mathbb{S}^n \subset \mathbb{S}_1^{n+1}$, totally geodesic hypersurface in de Sitter space (see [18]), which contradicts with the assumption. Hence, $\mathcal{B}$ is a linearly independent subset of $\mathcal{C}(M)$. Therefore, $\dim(\Lambda) = n + 2$.

By definition, $\bar{\Lambda} := \mathrm{span}(\Lambda \cup \{1\})$ is the linear space generated by $\mathcal{B} \cup \{1\} := \{\lambda_{e_i}\}_{i=1}^{n+2} \cup \{1\}$. It is enough to show that $\mathcal{B}\cup\{1\}$ is linearly independent. Assume that it is linearly dependent. So, by independence of $\mathcal{B}$, there exists a finite sequence of real numbers as $\{r_i\}_{i=1}^{n+2}$ such that $\sum_{i=1}^{n+2} r_i\lambda_{e_i} \equiv 1$. Putting $v := \sum_{i=1}^{n+2} r_i e_i$ we have $\lambda_v \equiv 1$. Since $\mathbf{x}(M^n)$ is spacelike, v can not be spacelike and it has to be timelike and then, we can assume that $\mathbf{d}(v, v) = -1$. Hence, $\mathbf{x}(M^n)$ is a totally geodesic hypersurface in $\mathbb{S}_1^{n+1}$, which is in the contradiction with the assumption. Therefore, $\mathcal{B} \cup \{1\}$ is a linearly independent and $\dim(\bar{\Lambda}) = n + 3$.

(ii) Γ is a linear space generated by $\hat{\mathcal{B}} := \{\gamma_{e_i}\}_{i=1}^{n+2}$. In order to prove the linearly independence of $\hat{\mathcal{B}}$, we assume that there is a nonzero finite sequence $\{r_i\}_{i=1}^{n+2}$ of real numbers such that $\sum_{i=1}^{n+2} r_i \gamma_{e_i} \equiv 0$. Putting $u := \sum_{i=1}^{n+2} r_i e_i$ we have $\gamma_u = \mathbf{d}(\mathbf{n}, u) \equiv 0$, which means that, $\mathbf{n}(M)$, the Gauss image of M, is contained in the hyperplane P with normal vector u. By the sign of P, its normal vector is positive definite. Without loss of generality, it is enough to consider the case $\mathbf{d}(u, u) = 1$. So, $\mathbf{n}(M)$ lies in $P \cap \mathbb{H}_0^{n+1}$. According to completeness of $\mathbf{n}(M)$ we obtain that, $\mathbf{n}(M)$ lies in a connected component of the hyperbolic space (i.e. $\mathbb{H}^n$). Similar to the well-known theorem of Nomizu-Smyth, one can see that in this case $\mathbf{n}(M)$ is a fixed vector and $\mathbf{x}(M)$ is a totally geodesic spacelike hypersurface of the de Sitter space $\mathbb{S}_1^{n+1}$. This contradiction with the assumptions implies that $\hat{\mathcal{B}}$ is linearly independent and then $\dim(\Gamma) = n + 2$.

In similar way, we show that $\hat{\mathcal{B}} \cup \{1\}$ is linearly independent. If not, because of linearly independence of $\hat{\mathcal{B}}$, there are real numbers s_i $(i = 1, 2, \dots, n + 2)$ such that $\sum_{i=1}^{n+2} s_i \gamma_{e_i} \equiv 1$. Putting $u := \sum_{i=1}^{n+2} s_i e_i$ we have $\gamma_u \equiv 1$, which means that, $\mathbf{n}(M)$, the Gauss image of M, is contained in the intersection of a connected component of the hyperbolic space $\mathbb{H}_0^{n+1}$ with the hyperplane $Q = \{p \in \mathbb{L}^{n+2} | \mathbf{d}(p, u) = 1\}$. So, according to completeness and connectedness of $\mathbf{n}(M)$, we obtain that $\mathbf{n}(M)$ lies in a connected component of the hyperbolic space. Similar to the well-known theorem of Nomizu-Smyth, one can see that in this case, $\mathbf{x}(M)$ is a totally geodesic spacelike hypersurface of the de Sitter space $\mathbb{S}_1^{n+1}$. This contradiction with the assumptions implies that $\bar{\mathcal{B}} \cup \{1\}$ is linearly independent and then $\dim(\bar{\Gamma}) = n + 3$.

□

Proposition 5.2. *Let $x : M^n \to \mathbb{S}_1^{n+1}$ be a connected non-totally geodesic spacelike hypersurface with $s_1 < 0$ that its shape operator has rank*

≥ 2. *Then for any two linearly independent vectors $u, v \in \mathbb{R}^{n+2}_1$, the set $\{\lambda_u, \gamma_v, 1\}$ is linearly independent.*

Proof. First we show that $\{\lambda_u, \gamma_v\}$ is linearly independent. Suppose that $\lambda_u = \varepsilon\gamma_v$ for some non-zero vectors $u, v \in \mathbb{R}^{n+2}_1$ and $\varepsilon \in \mathbb{R}$. If $\lambda_u = 0$, then M^n has to be totally geodesic in $\mathbb{S}^{n+1}_1$, which is a contradiction. Assume that $\lambda_u \neq 0$. Then from $\lambda_u = \varepsilon\gamma_v$ we get $\varepsilon \neq 0$ and $L_1(\lambda_u) = \varepsilon L_1(\gamma_v)$. So, we have $H\lambda_u = 0$ which is a contradiction again. Therefore, $\{\lambda_u, \gamma_v\}$ is linearly independent.

In the second stage, we show the linearly independence of $\{\lambda_u, \gamma_v, 1\}$. It is enough to consider the case where $\lambda_u^2 + \gamma_v^2 > 0$. Suppose that $\lambda_u = \varepsilon_1\gamma_v + \varepsilon_2$ for some real numbers $\varepsilon_1, \varepsilon_2 \in \mathbb{R}$. By the first part of the proof, we know that $\varepsilon_2 \neq 0$. So, we have $H\lambda_u = 0$, which is a contradiction. Therefore, $\{\lambda_u, \gamma_v, 1\}$ is linearly independent. □

Theorem 5.3. *Let $x : M^n \to \mathbb{S}^{n+1}_1$ be a 1-maximal complete closed simply connected totally geodesic spacelike hypersurface space $\mathbb{S}^{n+1}_1$ with $s_1 < 0$ and $rank(A) \geq 2$. Then $Ind_1(M^n) = 1$.*

Proof. The result is derived from a similar version of Theorem 5.1.1 in [20]. □

Theorem 5.4. *Let $x : M^n \to \mathbb{S}^{n+1}_1$ be an 1-maximal complete connected non-totally geodesic spacelike hypersurface with $s_1 < 0$ and* $\operatorname{rank}(A) \geq 2$. *Then* $\operatorname{Ind}_1(M^n) \geq n + 3$.

Proof. Since $s_2 = s_3 = 0$, we have

$$\begin{aligned} B(f) &= \int_M (Jf)f \\ &= \int_M \left(f\Box f + (1-n)s_1 f^2\right) dM, \end{aligned}$$

for any $f \in \mathcal{C}^\infty_0(M^n)$. Putting $f = t + \lambda_u + \gamma_v$ we get

$$J(f) = \Box(\lambda_u) + \Box(\gamma_v) - t(n-1)s_1 - (n-1)s_1\lambda_u - (n-1)s_1\gamma_v.$$

Then, we get $J(f) = -(n-1)s_1(t + \gamma_v)$ and therefore we obtain

$$fJ(f) = -(n-1)s_1(t + \gamma_v)^2 - (n-1)s_1\lambda_u(t + \gamma_v).$$

Hence, we have

$$B(f) = -n(n-1)\int_M s_1(\gamma_v + t)^2 dM - n(n-1)\int_M s_1\lambda_u(t + \gamma_v) dM.$$

Taking into account that $s_1 < 0$, by putting $u = 0$, we have

$$\begin{aligned} B(f) &= -(n-1)\int_M s_1(\gamma_v + t)^2 dM \\ &> 0. \end{aligned} \tag{5.1}$$

Therefore, $B(f) > 0$ for all $f \in span\left\{1, \gamma_{e_1}, \ldots, \gamma_{e_{n+2}}\right\}$. So, by Theorem 5.3, we obtain the result. □

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