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An Introduction to Spectral Theory of Bounded Linear Operators in Intuitionistic Fuzzy Pseudo Normed Linear Space

Bivas Dinda[1*], Santanu Kumar Ghosh[2] and Tapas Kumar Samanta[3]

Abstract. In this paper, focus is on the study of spectrum and the spectral properties of bounded linear operators in intuitionistic fuzzy pseudo normed linear spaces(IFPNLS). It is done by studying regular value, resolvent set, spectrum of a linear operator in IFPNLS. Also, some properties of spectrum and resolvent of strongly intuitionistic fuzzy bounded(IFB) linear operators in IFPNLS are being developed. It is observed that, for a linear operator P in an IFPNLS, the resolvent set $\rho(P)$ and spectrum $\sigma(P)$ are nonempty, $\rho(P)$ is open and $\sigma(P)$ is closed set.

1. Introduction

There is inherent uncertainty in almost all real world phenomena. It is very difficult for a human being to process imprecise, or incomplete, or uncertain information and make a correct decision. For dealing with uncertainty, in 1965, Zadeh initiated the idea of fuzzy set [18] as an extension of classical set. Fuzzy norm and fuzzy operator theory received lots of interest in past years [2–4, 8–11, 15, 17].

In 1986, K.T. Atanassov extended the notion of fuzzy set and proposed intuitionistic fuzzy set [1]. In 2006, Saadati and Park [14] introduced intuitionistic fuzzy norm. Chasing the concepts of S. Nădăban [13] and T. Bag and S.K. Samanta [2, 3], Dinda et. al. [5–7] extended the concept of intuitionistic fuzzy norm and developed intuitionistic fuzzy pseudo norm.

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For solving system of linear algebraic equation, or differential equation, or integral equation in intuitionistic fuzzy environment there is a problem related to inverse operations. The spectral theory in intuitionistic fuzzy pseudo normed linear space mainly deals with a systematic study of the inverse operators, their properties and their relations to the original operators.

In this paper, we endeavor to study the spectral theory in IFPNLS. In section 3, we studied resolvent, regular value, resolvent set and different spectrums of linear operators with illustrative examples in IFPNLS. Also, we studied the properties of resolvent set and spectrum. It is shown that resolvent set $\rho(P)$ and spectrum $\sigma(P)$ are non empty, $\rho(P)$ is open and $\sigma(P)$ is closed set for a linear opertor P in an IFPNLS. Also we obtained the condition for strongly intuitionistic fuzzy boundedness of $R_\lambda(P)$, where $\lambda \in \rho(P)$.

2. Preliminaries

Definition 2.1 ([16])**.** A pseudo norm on a linear space X over the field $\mathbb{K}(=\mathbb{R}$ or $\mathbb{C})$ is a real function $\|\cdot\| : X \to \mathbb{R}$ defined on X such that for any $x, y \in X$ and for all $c \in \mathbb{K}$ with $|c| \leq 1$,

- $(P.1)$ $\|x\| \geq 0$;
- $(P.2)$ $\|x\| = 0$ if and only if $x = \theta$;
- $(P.3)$ $\|cx\| \leq \|x\|$;
- $(P.4)$ $\|x + y\| \leq \|x\| + \|y\|$.

Definition 2.2 ([5])**.** An intuitionistic fuzzy pseudo norm(IFPN) on a linear space X over the field $\mathbb{K}(=\mathbb{R}$ or $\mathbb{C})$ is an intuitionistic fuzzy subset (μ, ν) of $(X \times \mathbb{R}, X \times \mathbb{R})$ satisfying the following conditions $\forall x, y \in X$,

- **(IFP.1)** $\mu(x, t) + \nu(x, t) \leq 1$;
- **(IFP.2)** $\forall t \in \mathbb{R}$ with $t \leq 0, \mu(x, t) = 0$;
- **(IFP.3)** $\forall t \in \mathbb{R}^+, \mu(x, t) = 1$ if and only if $x = \theta$;
- **(IFP.4)** $\forall t \in \mathbb{R}^+, \mu(cx, t) \geq \mu(x, t)$ if $| c | \leq 1, \forall c \in \mathbb{K}$;
- **(IFP.5)** $\mu(x + y, s + t) \geq \min\{\mu(x, s), \mu(y, t)\}, \forall s, t \in \mathbb{R}^+$;
- **(IFP.6)** $\lim_{t\to\infty} \mu(x, t) = 1$;
- **(IFP.7)** if there exists $\alpha \in (0, 1)$ such that $\mu(x, t) > \alpha, \forall t \in \mathbb{R}^+$ then $x = \theta$;
- **(IFP.8)** $\forall x \in X, \mu(x, \cdot)$ is left continuous on $\mathbb{R}$;
- **(IFP.9)** $\forall t \in \mathbb{R}$ with $t \leq 0, \nu(x, t) = 1$;
- **(IFP.10)** $\forall t \in \mathbb{R}^+, \nu(x, t) = 0$ if and only if $x = \theta$;
- **(IFP.11)** $\forall t \in \mathbb{R}^+, \nu(cx, t) \leq \nu(x, t)$ if $| c | \leq 1, \forall c \in \mathbb{K}$;
- **(IFP.12)** $\nu(x + y, s + t) \leq \max\{\nu(x, s), \nu(y, t)\}, \forall s, t \in \mathbb{R}^+$;
- **(IFP.13)** $\lim_{t\to\infty} \nu(x, t) = 0$;

(IFP.14) if there exists $\alpha \in (0,1)$ such that $\nu(x,t) < \alpha, \forall t \in \mathbb{R}^+$ then $x = \theta$;

(IFP.15) $\forall x \in X, \nu(x,\cdot)$ is left continuous on $\mathbb{R}$.

Here (X,μ,ν) is called IFPNLS.

Example 2.3 ([5])**.** Let $(X, \|\cdot\|)$ be a pseudo normed linear space. Define $\mu,\nu : X \times \mathbb{R} \to [0,1]$ by

$$\mu(x,t) = \begin{cases} 1 & \text{if } t > x, t > 0 \\ \frac{t}{t+\|x\|} & \text{if } t \leq x, t > 0 \\ 0 & \text{if } t \leq 0. \end{cases}$$

(2.1)

$$\nu(x,t) = \begin{cases} 0 & \text{if } t > x, t > 0 \\ \frac{\|x\|}{t+\|x\|} & \text{if } t \leq x, t > 0 \\ 1 & \text{if } t \leq 0. \end{cases}$$

Then (μ,ν) is an IFPN on X and (X,μ,ν) is an IFPNLS.

Definition 2.4 ([5])**.** Let (X,μ,ν) be an IFPNLS. A sequence $\{a_n\}_n$ converges to $a \in X$ if and only if $\lim_{n\to\infty} \mu(a_n - a, t) = 1$ and $\lim_{n\to\infty} \nu(a_n - a, t) = 0$.

Definition 2.5 ([7])**.** Let $(X,\mu_1,\nu_1),(Y,\mu_2,\nu_2)$ be two IFPNLS. A linear operator $P : (X,\mu_1,\nu_1) \to (Y,\mu_2,\nu_2)$ is said to be strongly IFB if $\forall x \in X$ and $\forall t \in \mathbb{R}^+$,

$$\mu_2(P(x),t) \geq \mu_1(x,t), \qquad \nu_2(P(x),t) \leq \nu_1(x,t).$$

Definition 2.6 ([6])**.** Let (X,μ,ν) be an IFPNLS. An open ball $B(x,r,t)$ with center at x, radius $0 < r < 1$ and $t \in \mathbb{R}^+$ is defined by

$$B(x,r,t) = \{a \in X : \mu(x-a,t) > 1-r, \nu(x-a,t) < r\}.$$

Lemma 2.7 ([12])**.** *Let $(X, \|\cdot\|)$ be a complete pseudo normed linear space, and $P : X \to X$ be a linear operator, and $\lambda \in \rho(P)$. If P is closed or P is bounded, then $R_\lambda(P)$ defined on the whole space X and is bounded.*

3. Spectrum, Resolvent and Some Spectral Properties in Intuitionistic Fuzzy Pseudo Normed Linear Space

Definition 3.1. Let $(X(\neq \{0\}),\mu,\nu)$ be an IFPNLS over the field $\mathbb{C}$ and $P : (X,\mu,\nu) \to (X,\mu,\nu)$ be a linear operator. The operator $P_\lambda = P - \lambda I$ is associated with the operator P, where $\lambda \in \mathbb{C}$ and I is the identity operator of X.

If P_λ has an inverse then ${P_\lambda}^{-1}$ is called the resolvent operator of P or resolvent of P and denoted by $R_\lambda(P)$. Therefore, $R_\lambda(P) = {P_\lambda}^{-1} = (P - \lambda I)^{-1}$.

Spectral theory is concerned with the properties where P_λ and R_λ depend on λ.

Definition 3.2. Let $(X(\neq \{0\}), \mu, \nu)$ be an IFPNLS over the field $\mathbb{C}$, and $P : (X, \mu, \nu) \to (X, \mu, \nu)$ a linear operator. $\lambda \in \mathbb{C}$ is said to be a regular value of P if and only if

(i) $R_\lambda(P)$ exists,
(ii) Range(P_λ) is dense in (X, μ, ν),
(iii) $R_\lambda(P)$ is strongly IFB.

Definition 3.3. The set of all regular values of P is called the resolvent set and is denoted by $\rho(P)$.

Definition 3.4. The spectrum of P is denoted by $\sigma(P)$ and $\sigma(P) = \{\lambda \in \mathbb{C} : (P - \lambda I) \text{ is not invertible}\}$ i.e., $\sigma(P) = \mathbb{C} - \rho(P)$ and $\lambda \in \sigma(P)$ are called spectral value of P.

Definition 3.5. Let $(X(\neq \{0\}), \mu, \nu)$ be an IFPNLS over $\mathbb{C}$ and $P : X \to X$ be a strongly IFB linear operator. There are following three types of spectrums.

(a) The point spectrum (or, discrete spectrum) of P is the set of all $\lambda \in \sigma(P)$ such that $R_\lambda(P)$ does not exist and is denoted by $\sigma_p(P)$. $\lambda \in \sigma_p(P)$ are called eigenvalues of P.
(b) The continuous spectrum of P is the set of all $\lambda \in \sigma(P)$ such that $R_\lambda(P)$ exists and satisfies (ii) but not (iii) of the Definition 3.2 and is denoted by $\sigma_c(P)$. In other words, if $\lambda \in \sigma_c(P)$ then $R_\lambda(P)$ exists, unbounded and domain of $R_\lambda(P)$ is dense in X.
(c) The residual spectrum of P is the set of all $\lambda \in \sigma(P)$ such that $R_\lambda(P)$ exists (may be strongly IFB or unbounded) but does not satisfy (ii) of Definition 3.2 and is denoted by $\sigma_r(P)$.

Example 3.6. Let $(X(\neq \{0\}), \mu, \nu)$ be an IFPNLS over $\mathbb{C}$. Then $\sigma(I) = \{1\} = \sigma_p(I)$ and $\sigma(O) = \{0\} = \sigma_p(O)$, where I and O are the identity operator and the zero operator of X respectively.

Example 3.7. Let P be a linear operator in a finite dimensional space, then $\sigma_c(P) = \phi = \sigma_r(P)$ and $\sigma_p(P) = \sigma(P)$. In other words, for a linear operator in a finite dimensional space, every spectrum value is an eigenvalue.

Example 3.8. (Residual spectrum) Let l^2 be the linear space of all sequences $\xi = (\xi_1, \xi_2, \ldots)$ such that $\sum\limits_{n=1}^{\infty} \|\xi_n\|^2 < \infty$.

Let us define a pseudo norm on l^2 by $\|\xi\| = \left(\sum_{n=1}^{\infty} \|\xi_n\|^2\right)^{\frac{1}{2}}$.

Let us consider an operator $P : l^2 \to l^2$ by $P(\xi_1, \xi_2, \ldots) = (0, \xi_1, \xi_2, \ldots)$ and the IFPN $\mu, \nu : l^2 \times \mathbb{R}^+ \to [0, 1]$ defined by Equation 2.1.

Clearly P is a bounded linear operator in $\left(l^2, \|\cdot\|\right)$. Therefore P is one to one. Then $P^{-1} : \text{Range}(P) \to l^2$ exists.

If Range(P) is not dense in l^2 with respect to $\left(l^2, \mu, \nu\right)$ then $0 \in \sigma_r(P)$. Let $\xi_n = \left(0, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\right) \in \text{Range}(P)$ and $\xi = (1, 0, 0, \ldots) \in l^2$. Now,

$$\begin{aligned}
\|\xi_n - \xi\| &= \left\| \left(0, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\right) - (1, 0, 0, 0, \ldots) \right\| \\
&= \left\| \left(-1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\right) \right\| \\
&= \left(1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots\right)^{\frac{1}{2}} \\
&= \sqrt{\frac{\pi^2}{6}} \\
&= \frac{\pi}{\sqrt{6}}.
\end{aligned}$$

Let $t = 1$. Then $t < \|\xi_n - \xi\|$. Hence

$$\begin{aligned}
\mu(\xi_n - \xi, t) &= \frac{t}{t + \|\xi_n - \xi\|} \\
&= \frac{1}{1 + \frac{\pi}{\sqrt{6}}} \\
&= \frac{\sqrt{6}}{\pi + \sqrt{6}},
\end{aligned}$$

$$\begin{aligned}
\nu(\xi_n - \xi, t) &= \frac{\|\xi_n - \xi\|}{t + \|\xi_n - \xi\|} \\
&= \frac{\frac{\pi}{\sqrt{6}}}{1 + \frac{\pi}{\sqrt{6}}} \\
&= \frac{\pi}{\pi + \sqrt{6}}
\end{aligned}$$

Therefore ξ_n does not converges to ξ. Hence $\xi \notin \text{Range}(P)$, thus Range($P$) is not dense in l^2. Therefore $0 \in \sigma_r(P)$.

Theorem 3.9. *Let $(X(\neq \{0\}), \|\cdot\|)$ be a complete pseudo normed linear space over $\mathbb{C}$. Let (μ, ν) be the IFPN on X defined by Equation 2.1, and*

$P : (X, \mu, \nu) \to (X, \mu, \nu)$ be strongly IFB linear operator. Then $\rho(P)$ is open and $\sigma(P)$ is closed set in $\mathbb{C}$.

Proof. Since P be a strongly IFB linear operator on X, then $\forall x \in X$ and $\forall t \in \mathbb{R}^+$,

$$\mu(P(x), t) \geq \mu(x, t), \qquad \nu(P(x), t) \leq \nu(x, t)$$

Hence

$$\frac{t}{t + \|P(x)\|} \geq \frac{t}{t + \|x\|} \quad \Rightarrow \quad \|P(x)\| \leq \|x\|\,;$$

Also,

$$\frac{\|P(x)\|}{t + \|P(x)\|} \leq \frac{\|x\|}{t + \|x\|} \quad \Rightarrow \quad t\,\|P(x)\| + \|P(x)\|\,\|x\| \leq t\,\|x\| + \|P(x)\|\,\|x\|$$
$$\Rightarrow \quad \|P(x)\| \leq \|x\|\,.$$

Thus P is bounded linear operator with respect to $(X, \|\cdot\|)$.

Then by Theorem 7.3-2 of [12], $\rho(P)$ is an open set; i.e., for each $\lambda \in \rho(P)$ there exists $\epsilon > 0$ such that $B(\lambda, \epsilon) \subset \rho(P)$, where

$$B(\lambda, \epsilon) = \{c \in \mathbb{C} : |\lambda - c| < \epsilon\}.$$

Now $\forall t \in \mathbb{R}^+$ and $0 < r < 1$ we prove that $B(\lambda, t) = B(\lambda, r, t)$.

Let $\epsilon = \frac{tr}{1-r} > 0, \forall t \in \mathbb{R}^+$ and $0 < r < 1$. Let $c \in B(\lambda, \epsilon)$, then $|\lambda - c| < \epsilon$. Now,

$$\begin{aligned}
\mu(\lambda - c, t) &= \frac{t}{t + |\lambda - c|} \\
&> \frac{t}{t + \epsilon} \\
&= \frac{t}{t + \frac{tr}{1-r}} \\
&= \frac{t(1-r)}{t} \\
&= 1 - r,
\end{aligned}$$

$$\begin{aligned}
\nu(\lambda - c, t) &= \frac{|\lambda - c|}{t + |\lambda - c|} \\
&< \frac{\epsilon}{t + \epsilon} \\
&= \frac{\frac{tr}{1-r}}{t + \frac{tr}{1-r}} \\
&= \frac{tr}{t}
\end{aligned}$$

$$= r.$$

Therefore, $c \in B(\lambda, r, t)$. Hence $B(\lambda, \epsilon) \subseteq B(\lambda, r, t)$.

Let $b \in B(\lambda, r, t)$ and $r = \frac{\epsilon}{1+\epsilon}$ and $t = 1 \in \mathbb{R}^+$. Now,

$$\begin{aligned}\mu(\lambda - b, t) &> 1 - r \\ &= 1 - \frac{\epsilon}{1+\epsilon} \\ &= \frac{1}{1+\epsilon}\end{aligned}$$

then

$$\frac{1}{1 + |\lambda - b|} > \frac{1}{1+\epsilon} \quad \Rightarrow \quad |\lambda - b| < \epsilon,$$

and

$$\nu(\lambda - b, t) < r = \frac{\epsilon}{1+\epsilon}$$

then

$$\frac{|\lambda - b|}{1 + |\lambda - b|} < \frac{\epsilon}{1+\epsilon} \quad \Rightarrow \quad |\lambda - b| < \epsilon.$$

Therefore, $b \in B(\lambda, \epsilon)$. Hence $B(\lambda, r, t) \subseteq B(\lambda, \epsilon)$. Thus $B(\lambda, \epsilon) = B(\lambda, r, t)$ for $0 < r < 1$ and $t \in \mathbb{R}^+$.

Therefore, $B(\lambda, r, t) \subset \rho(P)$ and hence $\rho(P)$ is an open set in $\mathbb{C}$. Also, $\sigma(P) = \mathbb{C} - \rho(P)$ is a closed set in $\mathbb{C}$. □

In the next example we found the point spectrum or eigenvalues of an operator.

Example 3.10. Let $P : \left(l^2, \mu, \nu\right) \to \left(l^2, \mu, \nu\right)$ be defined by

$$P\left(\xi_1, \xi_2, \xi_3, \ldots\right) = \left(\xi_2, \xi_3, \xi_4, \ldots\right).$$

Suppose $P\left(\xi_1, \xi_2, \xi_3, \ldots\right) = \lambda\left(\xi_1, \xi_2, \xi_3, \ldots\right)$. Then

$$\left(\xi_2, \xi_3, \xi_4, \ldots\right) = \lambda\left(\xi_1, \xi_2, \xi_3, \ldots\right)$$

therefore

$$\lambda\xi_1 = \xi_2, \quad \lambda\xi_2 = \xi_3, \quad \lambda\xi_3 = \xi_4, \ldots \quad \Rightarrow \quad \xi_n = \lambda^{n-1}\xi_1,$$

for each $n \geq 2$.

Clearly, $(\xi_n)_{n=1}^{\infty} \in l^2$ if and only if $\sum\limits_{n=1}^{\infty} \|\xi_n\|^2 = \sum\limits_{n=1}^{\infty} |\lambda|^{2n}\|\xi_1\|$ converges. And $\sum\limits_{n=1}^{\infty} |\lambda|^{2n}\|\xi_1\|$ converges if $|\lambda| < 1$. Now,

$$B(\lambda, \epsilon) = \{c \in \mathbb{C} : |\lambda - c| < \epsilon\} \quad \Rightarrow \quad B(0,1) = \{\lambda \in \mathbb{C} : |\lambda| < 1\}.$$

Also, from the proof of the Theorem 3.9, $B(0,1) = B(0, r, 1)$ for $0 < r < 1$. Therefore,

$$\sigma_p(P) = \{\lambda \in \mathbb{C} : \mu(\lambda, 1) > 1 - r, \nu(\lambda, 1) < r\}$$

$$= B(0, r, 1).$$

Theorem 3.11. *Let $(X, \|\cdot\|)$ be a pseudo normed linear space, and $(X(\neq \{0\}), \mu, \nu)$ be an IFPNLS defined by Equation 2.1. Then the closure of a subspace A in $(X, \|\cdot\|)$ and the closure of A in (X, μ, ν) are equal.*

Proof. Let $\overline{A}$ be the closure of A in $(X, \|\cdot\|)$. Then for every $a \in \overline{A}$ there exits a sequence $\{a_n\}_n$ in A such that $\lim_{n\to\infty} \|a_n - a\| = 0$. Therefore $\forall t \in \mathbb{R}^+$,

$$\begin{aligned}\lim_{n\to\infty} \mu(a_n - a, t) &= \lim_{n\to\infty} \frac{t}{t + \|a_n - a\|} \\ &= 1,\end{aligned}$$

and

$$\begin{aligned}\lim_{n\to\infty} \nu(a_n - a) &= \frac{\|a_n - a\|}{t + \|a_n - a\|} \\ &= 0.\end{aligned}$$

Hence $\overline{A}$ be the closure of A in (X, μ, ν). □

Theorem 3.12. *Let $(X(\neq \{0\}), \|\cdot\|)$ be a complete pseudo normed linear space over $\mathbb{C}$, and (μ, ν) be the IFPN on X defined by Equation 2.1. Let $P : (X, \mu, \nu) \to (X, \mu, \nu)$ be a linear operator on X, and $\lambda \in \rho(P)$. If P is strongly IFB linear operator then $R_\lambda(P)$ is strongly IFB linear operator on (X, μ, ν).*

Proof. Since P be a strongly IFB linear operator on X, then $\forall x \in X$ and $\forall t \in \mathbb{R}^+$,

$$\mu(P(x), t) \geq \mu(x, t), \qquad \nu(P(x), t) \leq \nu(x, t)$$

Therefore, by the proof of the Theorem 3.9, $\|P(x)\| \leq \|x\|$. Hence P is bounded with respect to $(X, \|\cdot\|)$. Also, $\lambda \in \rho(P)$. Therefore, by Lemma 2.7, $R_\lambda(P)$ is defined on the whole space X and $R_\lambda(P)$ is bounded with respect to $(X, \|\cdot\|)$. Then $\forall t \in \mathbb{R}^+$ and $\forall x \in X$,

$$\begin{aligned}\|R_\lambda(P)(x)\| \leq \|x\| \quad &\Rightarrow \quad t + \|R_\lambda(P)(x)\| \leq t + \|x\| \\ &\Rightarrow \quad \frac{1}{t + \|R_\lambda(P)(x)\|} \geq \frac{1}{t + \|x\|} \\ &\Rightarrow \quad \frac{t}{t + \|R_\lambda(P)(x)\|} \geq \frac{t}{t + \|x\|} \\ &\Rightarrow \quad \mu(R_\lambda(P)(x), t) \geq \mu(x, t),\end{aligned}$$

$$\begin{aligned}&\|R_\lambda(P)(x)\| \leq \|x\| \\ &\quad \Rightarrow \quad t.\, \|R_\lambda(P)(x)\| + \|R_\lambda(P)(x)\| \,.\, \|x\| \leq t.\, \|x\| + \|R_\lambda(P)(x)\| \,.\, \|x\|\end{aligned}$$

$$\Rightarrow \quad \frac{\|R_\lambda(P)(x)\|}{t+\|R_\lambda(P)(x)\|} \leq \frac{\|x\|}{t+\|x\|}$$
$$\Rightarrow \quad \nu(R_\lambda(P)(x),t) \leq \nu(x,t).$$

Hence $R_\lambda(P)$ is strongly IFB on X. □

Theorem 3.13. *Let $(X(\neq\{0\}),\|\cdot\|)$ be a complete pseudo normed linear space over $\mathbb{C}$, and (μ,ν) be the IFPN on X defined by Equation 2.1. Let $P:(X,\mu,\nu)\to(X,\mu,\nu)$ be a linear operator on X, and $\lambda\in\rho(P)$. If P is strongly IFB on X, and $\lim_{n\to\infty}\mu(a_n,t)=1$ and $\lim_{n\to\infty}\nu(a_n,t)=0$ for each $t\in\mathbb{R}^+$, then*

$$\lim_{n\to\infty}\mu(R_\lambda(P)(a_n),t)=1, \qquad \lim_{n\to\infty}\nu(R_\lambda(P)(a_n),t)=0.$$

Proof. Since $\lim_{n\to\infty}\mu(a_n,t)=1$ and $\lim_{n\to\infty}\nu(a_n,t)=0$, then for $\epsilon\in(0,1)$ and for each $t>0$ there exists $n_0\in\mathbb{N}$ such that $\mu(a_n,t)>1-\epsilon$ and $\nu(a_n,t)<\epsilon,\forall n\geq n_0$.

Since P is strongly IFB and $\lambda\in\rho(P)$, then by Theorem 3.12, $R_\lambda(P)$ is strongly IFB in (X,μ,ν).

Hence

$$\mu(R_\lambda(P)(a_n),t)\geq\mu(a_n,t)>1-\epsilon$$

and

$$\nu(R_\lambda(P)(a_n),t)\leq\nu(a_n,t)<\epsilon, \quad \forall n\geq n_0.$$

Thus

$$\lim_{n\to\infty}\mu(R_\lambda(P)(a_n),t)=1, \qquad \lim_{n\to\infty}\nu(R_\lambda(P)(a_n),t)=0.$$

□

Theorem 3.14. *Let $(X(\neq\{0\}),\|\cdot\|)$ be a complete pseudo normed linear space over $\mathbb{C}$, and (μ,ν) be the IFPN on X defined by Equation 2.1. Let $P:(X,\mu,\nu)\to(X,\mu,\nu)$ be a linear operator on X, and $\lambda\in\rho(P)$. If P is strongly IFB on X then $\rho(P)$ and $\sigma(P)$ are nonempty sets.*

Proof. Since P be a strongly IFB linear operator on X, then $\forall x\in X$ and $\forall t\in\mathbb{R}^+$,

$$\mu(P(x),t)\geq\mu(x,t), \qquad \nu(P(x),t)\leq\nu(x,t)$$

Therefore, by the proof of the Theorem 3.9, $\|P(x)\|\leq\|x\|$. Hence P is bounded with respect to $(X,\|\cdot\|)$. Also, $\lambda\in\rho(P)$. Then by Lemma 2.7, $R_\lambda(P)(=P_\lambda^{-1})$ is defined on the whole space X and bounded with respect to $(X,\|\cdot\|)$. Hence Range(P_λ) is dense in $(X,\|\cdot\|)$.

Now, $\forall x\in X$ and $\forall t\in\mathbb{R}^+$,

$$\|R_\lambda(P)(x)\|\leq\|x\| \quad \Rightarrow \quad t+\|R_\lambda(P)(x)\|\leq t+\|x\|$$

$$\begin{aligned}
&\Rightarrow \quad \frac{1}{t + \|R_\lambda(P)(x)\|} \geq \frac{1}{t + \|x\|} \\
&\Rightarrow \quad \frac{t}{t + \|R_\lambda(P)(x)\|} \geq \frac{t}{t + \|x\|} \\
&\Rightarrow \quad \mu(R_\lambda(P)(x), t) \geq \mu(x, t),
\end{aligned}$$

$$\begin{aligned}
&\|R_\lambda(P)(x)\| \leq \|x\| \\
&\quad \Rightarrow \quad t.\, \|R_\lambda(P)(x)\| + \|R_\lambda(P)(x)\| \,.\, \|x\| \leq t.\, \|x\| + \|R_\lambda(P)(x)\| \,.\, \|x\| \\
&\quad \Rightarrow \quad \frac{\|R_\lambda(P)(x)\|}{t + \|R_\lambda(P)(x)\|} \leq \frac{\|x\|}{t + \|x\|} \\
&\quad \Rightarrow \quad \nu(R_\lambda(P)(x), t) \leq \nu(x, t).
\end{aligned}$$

Thus $R_\lambda(P)$ is strongly IFB on X.

Now, since Range(P_λ) is dense in $(X, \|\cdot\|$, then Range(P_λ) is dense in (X, μ, ν), [by Theorem 3.11].

Therefore, for any $\lambda \in \mathbb{C}$, $R_\lambda(P)$ satisfies all the conditions of Definition 3.2 and hence $\rho(P)$ is a nonempty set. Therefore, $\sigma(P)$ is also a nonempty set. □

Theorem 3.15. *Let $(X(\neq \{0\}), \|\cdot\|)$ be a complete pseudo normed linear space over $\mathbb{C}$, and (μ, ν) be the IFPN on X defined by Equation 2.1. Let $P : (X, \mu, \nu) \to (X, \mu, \nu)$ be a linear operator on X, and $\lambda, \psi \in \rho(P)$. Then*

- (i) *the resolvent R_λ of P satisfies the resolvent equation or Hilbert relation: $R_\psi - R_\lambda = (\psi - \lambda) R_\psi R_\lambda$.*
- (ii) *If $F : (X, \mu, \nu) \to (X, \mu, \nu)$ be any strongly IFB linear operator on X and $FP = PF$, then $R_\lambda F = F R_\lambda$ i.e., R_λ commutes with F.*
- (iii) *$R_\lambda R_\psi = R_\psi R_\lambda$.*

Proof. Since P be a strongly IFB linear operator on X, then $\forall x \in X$ and $\forall t \in \mathbb{R}^+$,

$$\mu(P(x), t) \geq \mu(x, t), \qquad \nu(P(x), t) \leq \nu(x, t)$$

Therefore, by the proof of the Theorem 3.9, $\|P(x)\| \leq \|x\|$. Hence P is bounded with respect to $(X, \|\cdot\|)$. Also, $\lambda \in \rho(P)$. Therefore, from Lemma 2.7, $R_\lambda(P)$ defined in the whole space X i.e., Range(P_λ) is dense in X with respect to $\|\cdot\|$.

Hence by Theorem 3.11, Range(P_λ) is dense in X with respect to (X, μ, ν). Therefore, ${P_\lambda}^{-1} = R_\lambda$ i.e., $P_\lambda R_\lambda = I$ and also $R_\psi P_\psi = I$. Hence

$$\begin{aligned}
R_\psi - R_\lambda &= R_\psi P_\lambda R_\lambda - R_\psi P_\psi R_\lambda \\
&= R_\psi (P_\lambda - P_\psi) R_\lambda
\end{aligned}$$

$$= R_\psi \left(P - \lambda I - (P - \psi I)\right) R_\lambda$$
$$= (\psi - \lambda) R_\psi R_\lambda.$$

(ii)

$$\begin{aligned} FP = PF \quad &\Rightarrow \quad FP - \lambda F = PF - \lambda F \\ &\Rightarrow \quad F(P - \lambda I) = (P - \lambda I)F \\ &\Rightarrow \quad FP_\lambda = P_\lambda F. \end{aligned}$$

Also by (i) we have $R_\lambda P_\lambda = P_\lambda R_\lambda = I$. Hence

$$\begin{aligned} R_\lambda F &= R_\lambda F P_\lambda R_\lambda \\ &= R_\lambda P_\lambda F R_\lambda \\ &= F R_\lambda. \end{aligned}$$

(iii) Since P is a strongly IFB linear operator on X, by (ii), R_λ commutes with P. Therefore by (ii) and Theorem 3.12, R_ψ commutes with R_λ. □

4. Conclusion

In this paper we studied about resolvent operator, regular value, resolvent set and spectrum of linear operators with examples in IFPNLS. We also proposed the spectral properties of strongly IFB linear operators in the IFPN defined in Example 2.3.

Let us consider the following intuitionistic fuzzy pseudo norms:

Example 4.1 ([5])**.** Let $(X, \|\cdot\|)$ be a pseudo normed linear space. Define $\mu, \nu : X \times \mathbb{R} \to [0, 1]$ by

$$\mu(x,t) = \begin{cases} 1 & \text{if } t > \|x\| \\ 0 & \text{if } t \leq \|x\| \end{cases}$$

$$\nu(x,t) = \begin{cases} 0 & \text{if } t > \|x\| \\ 1 & \text{if } t \leq \|x\|. \end{cases}$$

Then (μ, ν) is an intuitionistic fuzzy pseudo norm.

Example 4.2. Let $(X, \|\cdot\|)$ be a pseudo normed linear space. Define $\mu, \nu : X \times \mathbb{R} \to [0, 1]$ by

$$\mu(x,t) = \begin{cases} \dfrac{t^2 - \|x\|^2}{t^2 + \|x\|^2} & \text{if } t > \|x\| \\ 0 & \text{if } t \leq \|x\| \end{cases}$$

$$\nu(x,t) = \begin{cases} \dfrac{\|x\|^2}{t^2 + \|x\|^2} & \text{if } t > \|x\| \\ 1, & \text{if } t \leq \|x\|. \end{cases}$$

Then (μ, ν) is an intuitionistic fuzzy pseudo norm.

If we take the intuitionistic fuzzy pseudo norms defined in the Example 4.1 or in the Example 4.2, instead of the IFPN defined in Example 2.3, then all the results of this paper can also be proved.

Actually, all the theorems of this paper can be proved for all intuitionistic fuzzy pseudo norms (μ, ν) in X for which the linear operator P is bounded with respect to $(X, \|\cdot\|)$, where $\|\cdot\| : X \to \mathbb{R}$ is a pseudo norm and $P : (X, \mu, \nu) \to (X, \mu, \nu)$ is strongly intuitionistic fuzzy bounded linear operator.

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