A New Three-Step Mixed-Type Implicit Iterative Scheme with Errors for Common Fixed Points of Nonexpansive and Uniformly L-Lipschitzian Asymptotically Generalized ϕ-Hemicontractive Mappings

Austine Efut Ofem and Donatus Ikechi Igbokwe

Sahand Communications in Mathematical Analysis

Print ISSN: 2322-5807
Online ISSN: 2423-3900
Volume: 19
Number: 1
Pages: 15-40

Sahand Commun. Math. Anal.
DOI: 10.22130/scma.2021.140147.873

Sahand Communications in Mathematical Analysis (SCMA) Vol. 19 No. 1 (2022), 15-40
http://scma.maragheh.ac.ir
DOI: 10.22130/scma.2021.140147.873

A New Three-Step Mixed-Type Implicit Iterative Scheme with Errors for Common Fixed Points of Nonexpansive and Uniformly L–Lipschitzian Asymptotically Generalized Φ-Hemicontractive Mappings

Austine Efut Ofem[1*] and Donatus Ikechi Igbokwe[2]

ABSTRACT. In this paper, we introduce a three-step implicit iteration scheme with errors for finite families of nonexpansive and uniformly L-Lipschitzian asymptotically generalized Φ-hemicontractive mappings in real Banach spaces. Our new implicit iterative scheme properly includes several well known iterative schemes in the literature as its special cases. The results presented in this paper extend, generalize and improve well known results in the existing literature.

1. Introduction

Let E be an arbitrary real Banach space with dual E^*. We denote by J the *normalized duality* mapping from E into 2^{E^*} defined by

$$J(\xi) = \left\{ f^* \in E^* : \langle \xi, f^* \rangle = \|\xi\|^2 = \|f^*\|^2 \right\}, \quad \forall \xi \in E. \tag{1.1}$$

where $\langle ., . \rangle$ denotes the generalized duality pairing. The single-valued-normalized duality mapping is denoted by j and $F(T)$ denotes the set of fixed points of mapping $T : E \to E$, i.e., $F(T) = \{\xi \in E : T\xi = \xi\}$.

In the sequel, we give the following definitions which will be useful in this study.

Definition 1.1. Let K be a nonempty subset of a real Banach space E. A mapping $T : K \to K$ is said to be:

2020 *Mathematics Subject Classification.* 34B24, 34B27.

Key words and phrases. Fixed point, Nonexpasive mapping, Uniformly L–Lipschitzian asymptotically generalized Φ-Hemicontractive mapping, Strong convergence, Banach spaces, Normalized duality mapping.

Received: 19 November 2020, Accepted: 19 September 2021.

[*] Corresponding author.

(a) strongly pseudocontractive (Kim et al. [24]) if for all $\xi, \eta \in K$, there exists a constant $\lambda \in (0,1)$ and $j(\xi - \eta) \in J(\xi - \eta)$ satisfying

$$\langle T\xi - T\eta, j(\xi - \eta)\rangle \leq \lambda \|\xi - \eta\|^2; \tag{1.2}$$

(b) ϕ-strongly pseudocontractive (Kim et al. [24]) if for all $\xi, \eta \in K$, there exists a strictly increasing function $\phi : [0, \infty) \to [0, \infty)$ with $\phi(0) = 0$ and $j(\xi - \eta) \in J(\xi - \eta)$ satisfying

$$\langle T\xi - T\eta, j(\xi - \eta)\rangle \leq \|\xi - \eta\|^2 - \phi(\|\xi - \eta\|)\|\xi - \eta\|. \tag{1.3}$$

In [28], it is proved that the class of strongly pseudocontractive mappings is a proper subclass of ϕ-strongly pseudocontractive mappings, by setting $\phi(t) = tk$ for all $t \in [0, \infty)$, where $\phi : [0, \infty) \to [0, \infty)$ is a strictly increasing function with $\phi(0) = 0$. However, the converse fails.

(c) generalized Φ-pseudocontractive (Albert et al. [1], Chidume and Chidume [6]) if for all $\xi, \eta \in K$, there exists a strictly increasing function $\Phi : [0, \infty) \to [0, \infty)$ with $\Phi(0) = 0$ and $j(\xi - \eta) \in J(\xi - \eta)$ satisfying

$$\langle T\xi - T\eta, j(\xi - \eta)\rangle \leq \|\xi - \eta\|^2 - \Phi(\|\xi - \eta\|). \tag{1.4}$$

The class of generalized Φ-pseudocontractive mappings is also called uniformly pseudocontractive mappings (see [6]). Clearly, the class of generalized Φ-pseudocontractive mappings properly contains the class of ϕ-pseudocontractive mappings;

(d) generalized Φ-hemicontractive if $F(T) \neq \emptyset$ and there exists a strictly increasing function $\Phi : [0, \infty) \to [0, \infty)$ with $\Phi(0) = 0$, such that for all $\xi \in K$ and $p \in F(T)$, there exists $j(\xi - p) \in J(\xi - p)$ such that the following inequality holds:

$$\langle T\xi - p, j(\xi - p)\rangle \leq \|\xi - p\|^2 - \Phi(\|\xi - p\|). \tag{1.5}$$

Clearly, the class of generalized Φ-hemicontractive mappings includes the class of generalized Φ-pseudocontractive mappings where the fixed points set $F(T)$ is nonempty;

(e) asymptotically generalized Φ-pseudocontractive (Kim et al. [24]) with sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$, if for each $\xi, \eta \in K$, there exist a strictly increasing function $\Phi : [0, \infty) \to [0, \infty)$ satisfying $\Phi(0) = 0$ such that

$$\langle T^n\xi - T^n\eta, j(\xi - \eta)\rangle \leq k_n\|\xi - \eta\|^2 - \Phi(\|\xi - \eta\|). \tag{1.6}$$

The class of asymptotically generalized Φ-pseudocontractive mappings is a generalization of the class of generalized

Φ-peudocontractive mappings. The class of mapping was introduced by Kim et al. [24] in 2009;

(f) asymptotically generalized Φ–hemicontractive with sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ if there exist a strictly increasing function $\Phi : [0, \infty) \to [0, \infty)$ with $\Phi(0) = 0$, such that for each $\xi \in K,\ p \in F(T)$, there exists $j(\xi - p) \in J(\xi - p)$ such that the following inequality holds:

$$\langle T^n\xi - p, j(\xi - p)\rangle \leq k_n\|\xi - p\|^2 - \Phi(\|\xi - p\|). \tag{1.7}$$

Clearly, every asymptotically generalized Φ–pseudocontractive mapping with a nonempty fixed point set is an asymptotically generalized Φ–hemicontractive mapping. Hence, the class of asymptotically generalized Φ-hemicontractive mappings is the most general of the classes of mappings defined above since it includes the class of asymptotically generalized Φ-pseudocontractive mappings.

For recent results on the approximation of fixed points of mappings which are asymptotically generalized Φ-hemicontractive mappings (see for example, [4–6, 19, 20, 23, 33, 43, 46] and the references there in).

In 1974, Ishikawa [21] introduced an iteration process $\{\xi_n\}$ defined by:

$$\begin{cases} \xi_0 \in K, \\ \xi_{n+1} = (1 - \alpha_n)\xi_n + \alpha_n T\eta_n, \qquad \forall n \geq 1, \\ \eta_n = (1 - \delta_n)\xi_n + \delta_n T\xi_n, \end{cases} \tag{1.8}$$

where $\{\alpha_n\}$ and $\{\delta_n\}$ are sequences in $[0, 1]$. This iteration process reduces to Mann iteration [31] if $\delta_n = 0$ for all $n \geq 1$ as follows:

$$\begin{cases} \xi_0 \in K, \\ \xi_{n+1} = (1 - \alpha_n)\xi_n + \alpha_n T\xi_n, \end{cases} \qquad \forall n \geq 1, \tag{1.9}$$

where $\{\alpha_n\}$ is a sequence in $[0, 1]$.

In 1991, Schu [34] introduced the following Mann-type iterative process for an asymptotically nonexpansive in Hilbert spaces:

$$\begin{cases} \xi_0 \in K, \\ \xi_{n+1} = (1 - \alpha_n)\xi_n + \alpha_n T^n\xi_n, \end{cases} \qquad \forall n \geq 1, \tag{1.10}$$

where $\{\alpha_n\}$ is a sequences in $[0, 1]$.

Several authors have studied the approximation of (1.10) for the classes of asymptotically nonexpansive and asymptotically generalized Φ-hemicontractive mappings (see for example Chang [4], Cho et al. [9], Gu [19] , Kim [24], Ofoedu [26], Schu [35], Zeng [48, 49]). Also Chidume and Chidume [5, 6] considered the iteration process (1.10) with errors for uniformly continuous asymptotically generalized Φ-hemicontractive mappings in Banach spaces.

In 2001, Xu and Ori [44] introduced the following implicit iteration process for a finite family of nonexpansive mapping $\{T_i : i \in R\}$ (here $R = \{1, 2, 3, \ldots, N\}$), with a real sequence $\{\alpha_n\}$ in $(0, 1)$, and an initial point $\xi_0 \in K$:

$$\begin{aligned}
&\xi_1 = \alpha_1\xi_0 + (1 - \alpha_1)T_1\xi_1,\\
&\xi_2 = \alpha_2\xi_1 + (1 - \alpha_2)T_2\xi_2,\\
&\quad\vdots\\
&\xi_N = \alpha_N\xi_{N-1} + (1 - \alpha_N)T_N\xi_N,\\
&\xi_{N+1} = \alpha_{N+1}\xi_N + (1 - \alpha_{N+1})T_{N+1}\xi_{N+1},\\
&\quad\vdots
\end{aligned}$$

The scheme is expressed in a compact form as

$$\xi_n = \alpha_n\xi_{n-1} + (1 - \alpha_n)T_n\xi_n, \quad n \geq 1, \tag{1.11}$$

where $T_n = T_{n(\,mod\,N)}$ (here the $\mod N$ function takes values in R). Xu and Ori [44] proved the weak convergence of this process for a common fixed point of the finite family of nonexpansive mappings in a Hilbert space.

In 2007, Rafiq [32] introduced the following implicit iteration process:

$$\begin{cases} \xi_0 \in K, \\ \xi_n = \alpha_n\xi_{n-1} + (1 - \alpha_n)T\xi_n, \end{cases} \quad \forall n \geq 1, \tag{1.12}$$

where $\{\alpha_n\}$ is a sequence in $[0, 1]$. In [33], Rafiq and Imdad characterized conditions for the convergence of the implicit Mann iterative scheme (1.12) to the unique fixed point of generalized Φ-hemicontractive mappings defined on a nonempty convex subset of an arbitrary Banach space.

In 2003, Sun [38] modified the implicit iteration of Xu and Ori [44] and applied the modified averaging iteration process for the approximation of fixed points of asymptotically quasi-nonexpansive mappings. Precisely, Sun [38] introduced the following modified implicit iteration process:

$$\begin{cases} \xi_0 \in K, \\ \xi_n = \alpha_n\xi_{n-1} + (1 - \alpha_n)T_{i(n)}^{h(n)}\xi_n, \end{cases} \quad \forall n \geq 1, \tag{1.13}$$

where $\{\alpha_n\}$ is a sequence in $[0, 1]$, $n = (h - 1)N + i$, $i = n(i) \in I = \{1, 2, \ldots, N\}$, $h = h(n) \geq 1$ is some positive integers and $h(n) \to \infty$ as $n \to \infty$.

In 2008, Yang and Hu [47] proposed an implicit iteration process with errors as follows:

$$(1.14)\qquad \begin{cases} \xi_0 \in K, \\ \xi_n = (1-\alpha_n-\beta_n)\xi_{n-1} + \alpha_n T_{i(n)}^{h(n)}\xi_n + \beta_n u_n, \end{cases} \qquad \forall n \geq 1,$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $[0,1]$ and $\{u_n\}$ is a bounded sequence in K and $n = (h-1)N + i$, $i = i(n) \in \{1, 2, \ldots, N\}$, $h = h(n) \geq 1$ is some positive integers and $h(n) \to \infty$ as $n \to \infty$.

In 2007, Thahur [42] proposed the following composite implicit iteration process for a finite family of asymptotically nonexpansive mappings as follows:

$$(1.15)\qquad \begin{cases} \xi_0 \in K, \\ \xi_n = (1-\alpha_n)\xi_{n-1} + \alpha_n T_{i(n)}^{h(n)}\eta_n, \\ \eta_n = (1-\delta_n)\xi_{n-1} + \delta_n T_{i(n)}^{h(n)}\xi_n, \end{cases} \qquad \forall n \geq 1,$$

where $\{\alpha_n\}$ and $\{\delta_n\}$ are sequences in $[0,1]$ and $n = (h-1)N + i$, $i = i(n) \in \{1, 2, \ldots, N\}$, $h = h(n) \geq 1$ is some positive integers and $h(n) \to \infty$ as $n \to \infty$.

The above iteration processes deal with only one mapping. The case of two mappings in iterative processes was first studied by Das and Debata [13]. For some decades now, so many authors have studied both implicit and explicit iteration methods with two or more mappings (see for example, [22] and [41]). The problem of approximating common fixed points of finitely many mappings plays an important role in applied mathematics, especially in the theory of evolution equations and the minimization problems (see [10–12, 14–16, 40] and the references in them).

In 2013, Lv et al. [25] characterized conditions for the convergence of the following implicit Ishikawa iterative scheme with errors in the sense of Agarwal et al. [2] to a common fixed point of two generalized Φ-hemicontractive mappings in a nonempty convex subset of an arbitrary Banach space.

$$(1.16)\qquad \begin{cases} \xi_0 \in K, \\ \xi_n = (1-\alpha_n-\beta_n)\xi_{n-1} + \alpha_n T\eta_n + \beta_n u_n, \\ \eta_n = (1-\delta_n-\gamma_n)\xi_{n-1} + \delta_n S\xi_n + \gamma_n v_n, \end{cases} \qquad \forall n \geq 1,$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\delta_n\}$, $\{\gamma_n\}$ are four real sequences in $[0,1]$, $\{u_n\}$ and $\{v_n\}$ are bounded sequences in K.

Noor et al. [27] introduced the following three-step iteration process for solving non-linear operator equations in real Banach spaces:

$$\begin{cases} \xi_0 \in K, \\ \xi_{n+1} = (1-\alpha_n)\xi_n + \alpha_n T\eta_n, \\ \eta_n = (1-\delta_n)\xi_n + \delta_n T\tau_n, \\ \tau_n = (1-e_n)\xi_n + e_n T\xi_n, \end{cases} \quad \forall n \geq 1, \tag{1.17}$$

where $\{\alpha_n\}$, $\{\delta_n\}$ and $\{e_n\}$ are sequences in $[0,1]$.

Since then, Noor iteration scheme has been used to study the strong and weak convergence results of several kind mappings (see, e.g., [9, 39, 45]). It was proved by Bnouhachem et al. [3] that three-step method performs better in terms of rate of convergence than two-step and one-step methods for solving variational inequalities. Moreover, three-step schemes are natural generalizations of the splitting methods to solve partial differential equations, (see [36, 37, 39]).

On the other hand, Glowinski and Le-Tallec [17] used a three-step iterative process to solve elasto-viscoplasticity, liquid crystal and eigenvalue problems. They established that three-step iterative scheme performs better than one-step (Mann) and two-step (Ishikawa) iterative schemes. Haubruge et al. [18] studied the convergence analysis of the three-step iterative processes of Glowinski and Le-Tallec [17] and used the three-step iteration to obtain some new splitting type algorithms for solving variational inequalities, separable convex programming and minimization of a sum of convex functions. They also proved that three-steps iteration processes also lead to highly parallelized algorithms under certain conditions.

From the above facts, it clear that three-steps iterative processes play pivotal role in solving various problems in pure and applied sciences and perform better than Man (one-step) and Ishikawa (two-step) type schemes for solving some problems of nonlinear equations.

In the past two decades or so, many researchers in nonlinear analysis have constructed and studied sever implicit iteration processes for approximating the unique fixed point of some classes mappings. Although, it is understood that implicit methods are more complicated to use, but they are usually much more stable and so larger time steps can be implemented . In general, implicit methods are more complex to programme and require more computational effort in each iteration or solution step, they are used because many physical problems arising in practice are stiff, for which the use of explicit method requires small time steps to keep the errors in the result bounded. For example, in numerical stability, which has to do with behavior of the solution as the time–step is increased, if the solution remains well behaved for arbitrary large values of time step, the method is said to be unconditionally stable. This situation never occurs with explicit methods which are always conditionally stable. Therefore, for stiff problems, to achieve given accuracy, it takes much less computational time to use an implicit method with larger time steps. Implicit iterative schemes have been studied recently by several authors (see for example, [8, 25, 29, 38, 44] and the references there in)

Motivated and inspired by the above results, we introduce a modified three-step composite implicit iteration process with errors for finite family of nonexpansive and three uniformly L-Lipschitz asymptotically generalized Φ-hemicontractive mappings as follows:

$$\begin{cases} \xi_0 \in K, \\ \xi_n = (1-\alpha_n-\beta_n)\xi_{n-1} + \alpha_n T_{i(n)}^{h(n)}\eta_n + \beta u_n, \\ \eta_n = (1-\delta_n-\gamma_n)\xi_{n-1} + \delta_n S_{i(n)}^{h(n)}\tau_n + \gamma_n v_n, \\ \tau_n = (1-e_n-f_n)H_{i(n)}\xi_n + e_n G_{i(n)}^{h(n)}\xi_n + f_n w_n, \end{cases} \quad \forall n \geq 1, \tag{1.18}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\delta_n\}$, $\{\gamma_n\}$, $\{e_n\}$, $\{f_n\}$ are real sequences in $[0,1]$ satisfying $\alpha_n + \beta_n \leq 1$, $\delta_n + \gamma_n \leq 1$ and $e_n + f_n \leq 1$, $\{u_n\}$, $\{v_n\}$ and $\{w_n\}$ are bounded sequences in K and $n = (h-1)N + i$, $i = i(n) \in \{1, 2, \dots, N\}$, $h = h(n) \geq 1$ is some positive integers and $h(n) \to \infty$ as $n \to \infty$.

Now, we will show that the iteration process (1.18) can be employed to approximate the unique common fixed point of finite families of nonexpansive and three asymptotically generalized Φ-hemicontractive mappings which are assumed to be Lipschitz continuous. To do this, we have to show that (1.18) is well defined. Let H_i be a finite family of nonexpansive mappings, T_i be a L_t^i-Lipschitz assymptotically generalized Φ-hemicontractive mappings with a sequence $\{\ell_n^i\} \subset [1,\infty)$ such that $\ell_n^i \to 1$ as $n \to \infty$, S_i be a L_s^i-Lipschitz assymptotically generalized Φ-hemicontractive mappings with a sequence $\{\varsigma_n^i\} \subset [1,\infty)$ such that $\varsigma_n^i \to 1$ as $n \to \infty$ and G_i be a L_g^i-Lipschitz assymptotically generalized Φ-hemicontractive mappings with a sequence $\{\nu_n^i\} \subset [1,\infty)$ such that $\nu_n^i \to 1$ as $n \to \infty$, for each $1 \leq i \leq N$.

Define a mapping $\Psi_n : K \to K$ by

$$\begin{aligned} \Psi_n(\xi) = (1-\alpha_n-\beta_n)\xi_{n-1} + \alpha_n T_{i(n)}^{h(n)} \Big\{ &(1-\delta_n-\gamma_n)\xi_{n-1} \\ &+ \delta_n S_{i(n)}^{h(n)} \Big[(1-e_n-f_n)H_{i(n)}\xi + e_n G_{i(n)}^{h(n)}\xi + f_n w_n \Big] + \gamma_n v_n \Big\} \\ &+ \beta_n u_n, \end{aligned}$$

for all $n \geq 1$. It follows that

$$\begin{aligned}
&\|\Psi_n(\xi)-\Psi_n(\eta)\|\\
&\quad=\alpha_n\Big\|T_{i(n)}^{h(n)}\Big\{(1-\delta_n-\gamma_n)\xi_{n-1}+\delta_nS_{i(n)}^{h(n)}\Big[(1-e_n-f_n)H_{i(n)}\xi\\
&\qquad+e_nG_{i(n)}^{h(n)}x+f_nw_n\Big]+\gamma_nv_n\Big\}-T_{i(n)}^{h(n)}\Big\{(1-\delta_n-\gamma_n)\xi_{n-1}\\
&\qquad+\delta_nS_{i(n)}^{h(n)}\Big[(1-e_n-f_n)H_{i(n)}\eta+e_nG_{i(n)}^{h(n)}\eta+f_nw_n\Big]+\gamma_nv_n\Big\}\Big\|\\
&\quad\le\alpha_nL\Big\|\delta_nS_{i(n)}^{h(n)}\Big[(1-e_n-f_n)H_{i(n)}\xi+e_nG_{i(n)}^{h(n)}\xi+f_nw_n\Big]+\gamma_nv_n\\
&\qquad-\delta_nS_{i(n)}^{h(n)}\Big[(1-e_n-f_n)\,H_{i(n)}\eta+e_nG_{i(n)}^{h(n)}\eta+f_nw_n\Big]\Big\|\\
&\quad\le\alpha_n\delta_nL^2\left\|(1-e_n-f_n)\left(H_{i(n)}\xi-H_{i(n)}\eta\right)+e_n\left(T\xi-T\eta\right)\right\|\\
&\quad\le\alpha_n\delta_nL^2\Big[(1-e_n-f_n)\left\|H_{i(n)}\xi-H_{i(n)}\eta\right\|+e_n\Big\|G_{i(n)}^{h(n)}\xi-G_{i(n)}^{h(n)}\eta\Big\|\Big]\\
&\quad\le\alpha_n\delta_nL^2\left[(1-e_n-f_n)\|\xi-\eta\|+e_nL\|\xi-\eta\|\right]\\
&\quad\le\alpha_n\delta_nL^2\left[1-e_n+e_nL\right]\|\xi-\eta\|\\
&\quad=\alpha_n\delta_nL^2\left[1+e_n(L-1)\right]\|\xi-\eta\|,\quad\forall\xi,\eta\in K,
\end{aligned}$$

where $L=\max\left\{L_t^i,\dots,L_t^N,L_s^i,\dots,L_s^N,L_g^i,\dots,L_g^N\right\}$.

If $\alpha_n\delta_nL^2[1+e_n(L-1)]<1$ for all $n\ge1$, then Ψ_n is a contraction. By Banach contraction mapping principle, we can see that there exists a unique fixed point $\xi_n\in K$ such that

$$\begin{aligned}
&\xi_n=\Psi_n(\xi)\\
&\quad=(1-\alpha_n-\beta_n)\xi_{n-1}+\alpha_nT_{i(n)}^{h(n)}\Big\{(1-\delta_n-\gamma_n)\xi_{n-1}\\
&\qquad+\delta_nS_{i(n)}^{h(n)}\Big[(1-e_n-f_n)H_{i(n)}\xi+e_nG_{i(n)}^{h(n)}\xi+f_nw_n\Big]+\gamma_nv_n\Big\}+\beta_nu_n,
\end{aligned}$$

for all $n\ge1$. This shows that the implicit iteration sequence (1.18) is well defined.

Let I denote the identity mapping, then our new iteration process (1.18) reduces to:

- (1.9) when $\beta_n=\delta_n=\gamma_n=e_n=f_n=0$, $T^n=T$, $N=1$.
- (1.10) when $\beta_n=\delta_n=\gamma_n=e_n=f_n=0$, $N=1$.
- (1.11) when $\beta_n=\gamma_n=e_n=f_n=0$, $\delta_n=1$, $S_i=I$, $H_i=I$, $T^n=T$.
- (1.12) when $\beta_n=\gamma_n=e_n=f_n=0$, $\delta_n=1$, $S_i=I$, $H_i=I$, $T^n=T$, $N=1$.
- (1.13) when $\beta_n=\gamma_n=e_n=f_n=0$, $\delta_n=1$, $S_i=I$, $H_i=I$.
- (1.14) when $\gamma_n=e_n=f_n=0$, $\delta_n=1$, $S_i=I$, $H_i=I$
- (1.15) when $\beta_n=\gamma_n=e_n=f_n=0$, $H_i=I$.

- (1.16) when $e_n = f_n = 0$, $H_i = I$, $T^n = T$, $N = 1$.

From the above demonstration, we can easily see part of the novelty of the new implicit iteration process (1.18) as it properly includes the iteration processes (1.9)–(1.16) and several other implicit and explicit iteration processes in the existing literation and hence, our results will improve, generalize, extend and unify several other results in the existing literation in this direction.

It is our purpose in this paper to use a simple and quite different method, to study the strong convergence of our new implicit iterative sequence $\{\xi_n\}$ defined by (1.18) to a common fixed points for finite family of nonexpansive and 3N uniformly L-Lipschitzian asymptotically generalized Φ-hemicontractive mappings in Banach spaces. Our results extend and improve some recent results in [4, 7, 9, 19, 24, 26, 33, 38, 43, 44, 47–49] and several others in the existing literature.

2. Preliminaries

In the sequel, we will need the following Lemmas.

Lemma 2.1 ([4]). *Let $J : E \to 2^{E^*}$ be the normalized duality mapping. Then for any $\xi, \eta \in E$, one has*

$$\|\xi + \eta\|^2 \leq \|\xi\|^2 + 2\langle \eta, j(\xi + \eta)\rangle, \quad \forall j(\xi + \eta) \in J(\xi + \eta).$$

Lemma 2.2 ([30]). *Let $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ be nonnegative real sequences satisfying the following conditions:*

$$a_{n+1} \leq (1 + b_n)a_n + c_n, \quad \forall n \geq n_0.$$

If $\sum_{n=0}^{\infty} c_n < \infty$, $\sum_{n=0}^{\infty} b_n < \infty$. Then,

(i) *the $\lim_{n\to\infty} a_n$ exits.*

(ii) *In addition if there exists a subsequence $\{a_{n_i}\} \subset \{a_n\}$such that $a_{n_i} \to 0$, then $a_n \to \infty$ (as $n \to \infty$).*

Lemma 2.3 ([46]). *Let $\Phi : [0, \infty] \to [0, \infty)$ be a strictly increasing function with $\Phi(0) = 0$ and let $\{\rho_n\}, \{\lambda_n\}, \{\varpi_n\}, \{\omega_n\}$ be nonnegative real sequences such that $\sum_{n=1}^{\infty} \lambda_n = \infty$, $\lim_{n\to\infty} \varpi_n = 0$, $\sum_{n=1}^{\infty} \omega_n < \infty$. Suppose that*

$$\rho_{n+1}^2 \leq \rho_n^2 - \lambda_n \Phi(\rho_{n+1}) + \lambda_n \varpi_n + \omega_n, \quad n \geq 1.$$

Then

$$\lim_{n\to\infty} \rho_n = 0.$$

3. Main Results

Theorem 3.1. *Let K be a nonempty closed convex subset of a real Banach space E. Let $N \geq 1$ be a positive integer and $R = \{1, 2, 3, \ldots, N\}$. Let H_i be a nonexpansive mapping, $T_i : K \to K$ a uniformly L_t^i–Lipschitz and asymptotically generalized Φ-hemicontractive mappings with sequence $\{\ell_n^i\} \subset [1, \infty)$, where $\ell_n^i \to 1$ as $n \to \infty$, $S_i : K \to K$ a uniformly L_s^i–Lipschitz and asymptotically generalized Φ-hemicontractive mappings with sequence $\{\varsigma_n^i\} \subset [1, \infty)$, where $\varsigma_n^i \to 1$ as $n \to \infty$ and $G_i : K \to K$ a uniformly L_g^i–Lipschitz and asymptotically generalized Φ-hemicontractive mappings with sequence $\{\nu_n^i\} \subset [1, \infty)$, where $\nu_n^i \to 1$ as $n \to \infty$, for each $i \in R$. Let $k_n = \max\{\ell_n, \varsigma_n, \nu_n\}$, where $\ell_n = \max\{\ell_n^i : i \in R\}$, $\varsigma_n = \max\{\varsigma_n^i : i \in R\}$ and $\nu_n = \max\{\nu_n^i : i \in R\}$. Suppose that $\boldsymbol{F} = \left(\bigcap_{i=1}^N F(H_i)\right) \bigcap \left(\bigcap_{i=1}^N F(T_i)\right) \bigcap \left(\bigcap_{i=1}^N F(S_i)\right) \bigcap \left(\bigcap_{i=1}^N F(G_i)\right) \neq \emptyset$. Let $\{u_n\}$, $\{v_n\}$, $\{w_n\}$ be bounded sequences in K and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\delta_n\}$, $\{\gamma_n\}$, $\{e_n\}$ and $\{f_n\}$ be sequences in $[0, 1]$ such that $\alpha_n + \beta_n \leq 1$, $\delta_n + \gamma_n \leq 1$ and $e_n + f_n \leq 1$, for each $n \geq 1$. Let $\{\xi_n\}$ be a sequence generated in (1.18). Let $\Phi(\wp) = \max\{\Phi_i(\wp) : i \in R\}$, for each $\wp \geq 0$. Assume that the following conditions are satisfied:*

(i) $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(ii) $\sum_{n=1}^{\infty} \alpha_n^2 < \infty$, $\sum_{n=1}^{\infty} \alpha_n \beta_n < \infty$, $\sum_{n=1}^{\infty} \alpha_n \delta_n < \infty$, $\sum_{n=1}^{\infty} \alpha_n \gamma_n < \infty$, $\sum_{n=1}^{\infty} \alpha_n \delta_n f_n < \infty$;

(iii) $\sum_{n=1}^{\infty} \beta_n < \infty$;

(iv) $\alpha_n \delta_n L^2 [1 + e_n(L - 1)] < 1$, *where*
$$L = \max\left\{L_t^i, \ldots, L_t^N, L_s^i, \ldots, L_s^N, L_g^i, \ldots, L_g^N\right\}.$$

Then the sequence $\{\xi_n\}$ converges strongly to a point in $\boldsymbol{F}$.

Proof. We divide the proof into 2 steps as follows:
Step 1: First, we show that $\lim_{n\to\infty} \|\xi_n - p\|$ exists for $p \in \mathbf{F}$. From (1.18), we have

(3.1)
$$\begin{aligned}
&\|\tau_n - p\| \\
&\quad = \left\|(1 - e_n - f_n) H_{i(n)}\xi_n + e_n G_{i(n)}^{h(n)}\xi_n + f_n w_n - p\right\| \\
&\quad = \left\|(1 - e_n - f_n)\left(H_{i(n)}\xi_n - p\right) + e_n \left(G_{i(n)}^{h(n)}\xi_n - p\right) + f_n(w_n - p)\right\| \\
&\quad \leq (1 - e_n - f_n)\left\|H_{i(n)}\xi_n - H_{i(n)}p\right\| + e_n \left\|G_{i(n)}^{h(n)}\xi_n - p\right\| + f_n \|w_n - p\|
\end{aligned}$$

$$
\begin{aligned}
&\leq \|\xi_n - p\| + e_n L \|\xi_n - p\| + f_n \|w_n - p\| \\
&\leq (1 + e_n L) \|\xi_n - p\| + f_n \|w_n - p\| \\
&\leq (1 + L) \|\xi_n - p\| + f_n \|w_n - p\| .
\end{aligned}
$$

Using (1.18) and (3.1), we obtain,

(3.2)

$$
\begin{aligned}
\|\eta_n - p\| &= \left\|(1 - \delta_n - \gamma_n)\, \xi_{n-1} + \delta_n S_{i(n)}^{h(n)} \tau_n + \gamma_n v_n - p\right\| \\
&= \left\|(1 - \delta_n - \gamma_n)\, (\xi_{n-1} - p) + \delta_n \left(S_{i(n)}^{h(n)} \tau_n - p\right) + \gamma_n (v_n - p)\right\| \\
&\leq (1 - \delta_n - \gamma_n) \|\xi_{n-1} - p\| + \delta_n \left\|S_{i(n)}^{h(n)} \tau_n - p\right\| + \gamma_n \|v_n - p\| \\
&\leq \|\xi_{n-1} - p\| + \delta_n L \|\tau_n - p\| + \gamma_n \|v_n - p\| \\
&= \|\xi_{n-1} - p\| + \delta_n L \left\{(1 + L) \|\xi_n - p\| + f_n \|w_n - p\|\right\} \\
&\quad + \gamma_n \|v_n - p\| \\
&= \|\xi_{n-1} - p\| + \delta_n L (1 + L) \|\xi_n - p\| + \delta_n f_n L \|w_n - p\| \\
&\quad + \gamma_n \|v_n - p\| .
\end{aligned}
$$

Again, from (1.18) we obtain that

(3.3)

$$
\begin{aligned}
\|\eta_n - \xi_n\| &= \|\eta_n - \xi_{n-1} + \xi_{n-1} - \xi_n\| \\
&\leq \|\eta_n - x_{n-1}\| + \|\xi_{n-1} - \xi_n\| \\
&= \left\|(1 - \delta_n - \gamma_n) \xi_{n-1} + \delta_n S_{i(n)}^{h(n)} \tau_n + \gamma_n v_n - \xi_{n-1}\right\| \\
&\quad + \left\|\xi_{n-1} - \left[(1 - \alpha_n - \beta_n)\, \xi_{n-1} + \alpha_n T_{i(n)}^{h(n)} \eta_n + \beta_n u_n\right]\right\| \\
&= \left\|\delta_n \left(S_{i(n)}^{h(n)} \tau_n - \xi_{n-1}\right) + \gamma_n (v_n - \xi_{n-1})\right\| \\
&\quad + \left\|\alpha_n \left(\xi_{n-1} - T_{i(n)}^{h(n)} \eta_n\right) + \beta_n (\xi_{n-1} - u_n)\right\| \\
&= \left\|\delta_n \left(S_{i(n)}^{h(n)} \tau_n - p + p - \xi_{n-1}\right) + \gamma_n (v_n - p + p - \xi_{n-1})\right\| \\
&\quad + \left\|\alpha_n \left(\xi_{n-1} - p + p - T_{i(n)}^{h(n)} \eta_n\right) + \beta_n (\xi_{n-1} - p + p - u_n)\right\| \\
&\leq \delta_n \left\|S_{i(n)}^{h(n)} \tau_n - p\right\| + \delta_n \|p - \xi_{n-1}\| + \gamma_n \|v_n - p\| \\
&\quad + \gamma_n \|p - \xi_{n-1}\| + \alpha_n \|\xi_{n-1} - p\| + \alpha_n \left\|p - T_{i(n)}^{h(n)} \eta_n\right\| \\
&\quad + \beta_n \|\xi_{n-1} - p\| + \beta_n \|p - u_n\| \\
&\leq \delta_n L \|\tau_n - p\| + \delta_n \|p - \xi_{n-1}\| + \gamma_n \|v_n - p\| + \gamma_n \|p - \xi_{n-1}\| \\
&\quad + \alpha_n \|\xi_{n-1} - p\| + \alpha_n L \|p - \eta_n\| + \beta_n \|\xi_{n-1} - p\| + \beta_n \|p - u_n\|
\end{aligned}
$$

$$= \alpha_n L\|\eta_n - p\| + \delta_n L\|\tau_n - p\| + (\alpha_n + \beta_n + \delta_n + \gamma_n)\|\xi_{n-1} - p\| + \beta_n\|u_n - p\| + \gamma_n\|v_n - p\|$$

Substituting (3.1) and (3.2) into (3.3) we obtain

(3.4)

$$\begin{aligned}
&\|\eta_n - \xi_n\| \\
&\le \alpha_n L\left\{\|\xi_{n-1} - p\| + \delta_n L(1+L)\|\xi_n - p\| + \delta_n f_n L\|w_n - p\| + \gamma_n\|v_n - p\|\right\} \\
&\quad + \delta_n L\{(1+L)\|\xi_n - p\| + f_n\|w_n - p\|\} + (\alpha_n + \beta_n + \delta_n + \gamma_n)\|\xi_{n-1} - p\| \\
&\quad + \beta_n\|u_n - p\| + \gamma_n\|v_n - p\| \\
&= \alpha_n L\|\xi_{n-1} - p\| + \alpha_n\delta_n L^2(1+L)\|\xi_n - p\| + \alpha_n\delta_n f_n L^2\|w_n - p\| \\
&\quad + \alpha_n\gamma_n L\|v_n - p\| + \delta_n L(1+L)\|\xi_n - p\| + \delta_n f_n L\|w_n - p\| \\
&\quad + (\alpha_n + \beta_n + \delta_n + \gamma_n)\|\xi_{n-1} - p\| + \beta_n\|u_n - p\| + \gamma_n\|v_n - p\| \\
&= (\alpha_n L + \alpha_n + \beta_n + \delta_n + \gamma_n)\|\xi_{n-1} - p\| + \left(\alpha_n\delta_n L^2 + \delta_n L\right)(1+L)\|\xi_n - p\| \\
&\quad + (\alpha_n\delta_n f_n L^2 + \delta_n f_n L)\|w_n - p\| + (\alpha_n\gamma_n L + \gamma_n)\|v_n - p\| + \beta_n\|u_n - p\| \\
&= [\alpha_n(L+1) + \beta_n + \delta_n + \gamma_n]\|\xi_{n-1} - p\| + \delta_n L(\alpha_n L + 1)(1+L)\|\xi_n - p\| \\
&\quad + \delta_n f_n L(\alpha_n L + 1)\|w_n - p\| + \gamma_n(\alpha L + 1)\|v_n - p\| + \beta_n\|u_n - p\| \\
&\le [\alpha_n(L+1) + \beta_n + \delta_n + \gamma_n]\|\xi_{n-1} - p\| + \delta_n L(L+1)^2\|\xi_n - p\| \\
&\quad + \delta_n f_n L(L+1)\|w_n - p\| + \gamma_n(L+1)\|v_n - p\| + \beta_n\|u_n - p\|
\end{aligned}$$

From (1.18) and Lemma 2.1, we obtain that

(3.5)

$$\begin{aligned}
&\|\xi_n - p\|^2 \\
&= \left\|(1-\alpha_n-\beta_n)\xi_{n-1} + \alpha_n T_{i(n)}^{h(n)}\eta_n + \beta_n u_n - p\right\|^2 \\
&= \left\|(1-\alpha_n-\beta_n)(\xi_{n-1} - p) + \alpha_n\left(T_{i(n)}^{h(n)}\eta_n - p\right) + \beta_n(u_n - p)\right\|^2 \\
&\le (1-\alpha_n-\beta_n)^2\|\xi_{n-1} - p\|^2 \\
&\quad + 2\left\langle \alpha_n\left(T_{i(n)}^{h(n)}\eta_n - p\right) + \beta_n(u_n - p), j(\xi_n - p)\right\rangle \\
&= (1-\alpha_n-\beta_n)^2\|\xi_{n-1} - p\|^2 + 2\alpha_n\left\langle T_{i(n)}^{h(n)}\eta_n - p, j(\xi_n - p)\right\rangle \\
&\quad + 2\beta_n\langle u_n - p, j(\xi_n - p)\rangle \\
&= (1-\alpha_n-\beta_n)^2\|\xi_{n-1} - p\|^2 \\
&\quad + 2\alpha_n\left\langle T_{i(n)}^{h(n)}\eta_n - T_{i(n)}^{h(n)}\xi_n + T_{i(n)}^{h(n)}\xi_n - p, j(\xi_n - p)\right\rangle \\
&\quad + 2\beta_n\langle u_n - p, j(\xi_n - p)\rangle
\end{aligned}$$

$$
\begin{aligned}
&= (1-\alpha_n-\beta_n)^2\|\xi_{n-1}-p\|^2 + 2\alpha_n\left\langle T_{i(n)}^{h(n)}\eta_n - T_{i(n)}^{h(n)}\xi_n, j(\xi_n-p)\right\rangle \\
&\quad + 2\alpha_n\left\langle T_{i(n)}^{h(n)}\xi_n - p, j(x_n-p)\right\rangle + 2\beta_n\langle u_n-p, j(\xi_n-p)\rangle \\
&\leq (1-\alpha_n-\beta_n)^2\|\xi_{n-1}-p\|^2 + 2\alpha_n\left\|T_{i(n)}^{h(n)}\eta_n - T_{i(n)}^{h(n)}\xi_n\right\|\|\xi_n-p\| \\
&\quad + 2\beta_n\|u_n-p\|\|\xi_n-p\| + 2\alpha_n\left\langle T_{i(n)}^{h(n)}\xi_n - p, j(\xi_n-p)\right\rangle \\
&\leq (1-\alpha_n)^2\|\xi_{n-1}-p\|^2 + 2\alpha_n L\|\eta_n-\xi_n\|\|\xi_n-p\| \\
&\quad + 2\beta_n\|u_n-p\|\|\xi_n-p\| + 2\alpha_n\left\langle T_{i(n)}^{h(n)}\xi_n - p, j(\xi_n-p)\right\rangle.
\end{aligned}
$$

Substituting (3.4) into (3.5) we obtain

$$
\begin{aligned}
&\|\xi_n-p\|^2 \\
&\leq (1-\alpha_n)^2\|\xi_{n-1}-p\|^2 + 2\alpha_n L\{[\alpha_n(L+1)+\beta_n+\delta_n+\gamma_n]\|\xi_{n-1}-p\| \\
&\quad + \delta_n L(L+1)^2\|\xi_n-p\| + \delta_n f_n L(L+1)\|w_n-p\| + \gamma_n(L+1)\|v_n-p\| \\
&\quad + \beta_n\|u_n-p\|\}\|\xi_n-p\| + 2\beta_n\|u_n-p\|\|\xi_n-p\| \\
&\quad + 2\alpha_n\left\langle T_{i(n)}^{h(n)}\xi_n - p, j(x_n-p)\right\rangle \\
&= (1-\alpha_n)^2\|\xi_{n-1}-p\|^2 \\
&\quad + 2\alpha_n L\left[\alpha_n(L+1)+\beta_n+\delta_n+\gamma_n\right]\|\xi_{n-1}-p\|\|\xi_n-p\| \\
&\quad + 2\alpha_n\delta_n L^2(L+1)^2\|\xi_n-p\|^2 + 2\alpha_n\delta_n f_n L^2(L+1)\|w_n-p\|\|\xi_n-p\| \\
&\quad + 2\alpha_n\gamma_n L(L+1)\|v_n-p\|\|\xi_n-p\| + 2\alpha_n\beta_n L\|u_n-p\|\|\xi_n-p\| \\
&\quad + 2\beta_n\|u_n-p\|\|\xi_n-p\| + 2\alpha_n\left\langle T_{i(n)}^{h(n)}\xi_n - p, j(\xi_n-p)\right\rangle \\
&\leq (1-\alpha_n)^2\|\xi_{n-1}-p\|^2 \\
&\quad + 2\alpha_n L[\alpha_n(L+1)+\beta_n+\delta_n+\gamma_n]\|\xi_{n-1}-p\|\|\xi_n-p\| \\
&\quad + 2\alpha_n\delta_n L^2(L+1)^2\|\xi_n-p\|^2 + 2\alpha_n\delta_n f_n L^2(L+1)\|w_n-p\|\|\xi_n-p\| \\
&\quad + 2\alpha_n\gamma_n L(L+1)\|v_n-p\|\|\xi_n-p\| + 2\beta_n(L+1)\|u_n-p\|\|\xi_n-p\| \\
&\quad + 2\alpha_n\left\langle T_{i(n)}^{h(n)}\xi_n - p, j(\xi_n-p)\right\rangle.
\end{aligned}
$$

Setting $M = \max\{\sup\{u_n - p\|\}, \sup\{v_n - p\|\}, \sup\{w_n - p\|\}, n \geq 1\}$ and noting that

$$\left.\begin{aligned}
&\|\xi_{n-1} - p\|\|\xi_n - p\| \leq \tfrac{1}{2}\left(\|\xi_{n-1} - p\|^2 + \|\xi_n - p\|^2\right),\\
&\|u_n - p\|\|\xi_n - p\| \leq \tfrac{1}{2}\left(\|u_n - p\|^2 + \|\xi_n - p\|^2\right),\\
&\|v_n - p\|\|\xi_n - p\| \leq \tfrac{1}{2}\left(\|v_n - p\|^2 + \|\xi_n - p\|^2\right),\\
&\|w_n - p\|\|\xi_n - p\| \leq \tfrac{1}{2}\left(\|w_n - p\|^2 + \|\xi_n - p\|^2\right).
\end{aligned}\right\}, \quad \forall n \geq 1,$$

and also, from asymptotically generalized Φ-hemicontractivity we have

(3.6)

$$\begin{aligned}
&\|\xi_n - p\|^2\\
&\leq (1-\alpha_n)^2\|\xi_{n-1} - p\|^2\\
&\quad + 2\alpha_n L\left[\alpha_n(L+1) + \beta_n + \delta_n + \gamma_n\right] \times \frac{1}{2}\left(\|\xi_{n-1} - p\|^2 + \|\xi_n - p\|^2\right)\\
&\quad + 2\alpha_n\delta_n L^2(L+1)^2\|\xi_n - p\|^2 + 2\alpha_n\delta_n f_n L^2(L+1)\\
&\quad \times \frac{1}{2}\left(\|w_n - p\|^2 + \|\xi_n - p\|^2\right)\\
&\quad + 2\alpha_n\gamma_n L(L+1) \times \frac{1}{2}\left(\|v_n - p\|^2 + \|\xi_n - p\|^2\right)\\
&\quad + 2\beta_n(L+1) \times \frac{1}{2}\left(\|u_n - p\|^2 + \|\xi_n - p\|^2\right)\\
&\quad + 2\alpha_n\left\langle T_{i(n)}^{h(n)}\xi_n - \xi_n, j(\xi_n - p)\right\rangle\\
&\leq \left\{(1-\alpha_n)^2 + \alpha_n L\left[\alpha_n(L+1) + \beta_n + \delta_n + \gamma_n\right]\right\}\|\xi_{n-1} - p\|^2\\
&\quad + \left\{\alpha_n L[\alpha_n(L+1) + \beta_n + \delta_n + \gamma_n] + 2\alpha_n\delta_n L^2(L+1)^2\right.\\
&\quad \left. + \alpha_n\delta_n f_n L^2(L+1) + \alpha_n\gamma_n L(L+1) + \beta_n(L+1)\right\}\|\xi_n - p\|^2\\
&\quad + \alpha_n\delta_n f_n L^2(L+1)M_1 + \alpha_n\gamma_n L(L+1)M_1 + \beta_n(L+1)M_1\\
&\quad + 2\alpha_n\left\langle T_{i(n)}^{h(n)}\xi_n - p, j(\xi_n - p)\right\rangle\\
&\leq \{(1-\alpha_n)^2 + 2\alpha_n L[\alpha_n(L+1) + \beta_n + \delta_n + \gamma_n]\}\|\xi_{n-1} - p\|^2\\
&\quad + \{\alpha_n L[\alpha_n(L+1) + \beta_n + \delta_n + \gamma_n] + 2\alpha_n\delta_n L^2(L+1)^2\\
&\quad + \alpha_n\delta_n f_n L^2(L+1) + \alpha_n\gamma_n L(L+1) + \beta_n(L+1)\}\|\xi_n - p\|^2\\
&\quad + \alpha_n\delta_n f_n L^2(L+1)M_1 + \alpha_n\gamma_n L(L+1)M_1 + \beta_n(L+1)M_1\\
&\quad + 2\alpha_n(k_n\|\xi_n - p\|^2 - \Phi(\|\xi_n - p\|))\\
&= \{(1-\alpha_n)^2 + 2\alpha_n L[\alpha_n(L+1) + \beta_n + \delta_n + \gamma_n]\}\|\xi_{n-1} - p\|^2
\end{aligned}$$

$$
\begin{aligned}
&+ \{\alpha_n L[\alpha_n(L+1) + \beta_n + \delta_n + \gamma_n] + 2\alpha_n\delta_n L^2(L+1)^2 \\
&+ \alpha_n\delta_n f_n L^2(L+1) + \alpha_n\gamma_n L(L+1) + \beta_n(L+1) + 2\alpha_n k_n\}\|\xi_n - p\|^2 \\
&+ \alpha_n\delta_n f_n L^2(L+1)M_1 + \alpha_n\gamma_n L(L+1)M_1 + \beta_n(L+1)M_1 \\
&- 2\alpha_n\Phi(\|\xi_n - p\|)) \\
= {}& \vartheta_n\|\xi_n - p\|^2 - 2\alpha_n\Phi(\|\xi_n - p\|) + \sigma_n\|\xi_n - p\|^2 + \alpha_n\mu_n + \beta_n(L+1)M_1,
\end{aligned}
$$

where

$$
\begin{aligned}
\vartheta_n &= (1-\alpha_n)^2 + 2\alpha_n L[\alpha_n(L+1) + \beta_n + \delta_n + \gamma_n] \\
\sigma_n &= \alpha_n L[\alpha_n(L+1) + \beta_n + \delta_n + \gamma_n] + 2\alpha_n\delta_n L^2(L+1)^2 \\
&\quad + \alpha_n\delta_n f_n L^2(L+1) + \alpha_n\gamma_n L(L+1) + \beta_n(L+1) + 2\alpha_n k_n \\
\mu_n &= \delta_n f_n L^2(L+1)M_1 + \gamma_n L(L+1)M_1.
\end{aligned}
$$

It follows from transposing and simplifying (3.6) that we obtain

(3.7)

$$
\begin{aligned}
\|\xi_n - p\|^2 \leq{}& \left[\frac{\vartheta_n}{1-\sigma_n}\right]\|\xi_{n-1} - p\|^2 - \left[\frac{2\alpha_n}{1-\sigma_n}\right]\Phi(\|\xi_n - p\|) \\
&+ \left[\frac{\alpha_n\mu_n + \beta_n M_1}{1-\sigma_n}\right] \\
={}& \left[1 + \frac{\vartheta_n + \sigma_n - 1}{1-\sigma_n}\right]\|\xi_{n-1} - p\|^2 - \left[\frac{2\alpha_n}{1-\sigma}\right]\Phi(\|\xi_n - p\|) \\
&+ \left[\frac{\alpha_n\mu_n + \beta_n M_1}{1-\eta_n}\right].
\end{aligned}
$$

Notice that

$$
\begin{aligned}
\vartheta_n + \sigma_n - 1 &= \alpha_n^2 + 3\alpha_n L[\alpha_n(L+1) + \beta_n + \delta_n + \gamma_n] + 2\alpha_n\delta_n L^2(L+1)^2 \\
&\quad + \alpha_n\delta_n f_n L^2(L+1) + \alpha_n\gamma_n L(L+1) + 2\alpha_n(k_n - 1) + \beta_n(L+1) \\
&= \alpha_n[\alpha_n + 3L[\alpha_n(L+1) + \beta_n + \delta_n + \gamma_n] + 2\delta_n L^2(L+1)^2 \\
&\quad + \delta_n f_n L^2(L+1) + \gamma_n L(L+1) + (k_n - 1)] + \beta_n(L+1) \\
&= \alpha_n\zeta_n + \beta_n(L+1),
\end{aligned}
$$

where

$$
\begin{aligned}
\zeta_n ={}& \alpha_n + 3L[\alpha_n(L+1) + \beta_n + \delta_n + \gamma_n] + \delta_n L^2(L+1)^2 \\
&+ \delta_n f_n L^2(L+1) + \gamma_n L(L+1) + (k_n - 1).
\end{aligned}
$$

From (3.7) we obtain

$$
\|\xi_n - p\|^2 \leq \left[1 + \frac{\alpha_n\zeta_n + \beta_n(L+1)}{1-\sigma_n}\right]\|\xi_{n-1} - p\|^2 \tag{3.8}
$$

$$- \left[\frac{2\alpha_n}{1-\sigma_n} \right] \Phi(\|\xi_n - p\|) + \left[\frac{\alpha_n \mu_n + \beta_n (L+1) M_1}{1-\sigma_n} \right].$$

Since $\alpha_n \to 0$ and $k_n \to 1$ as $n \to \infty$, it follows from condition conditions (ii)-(iii) that

$$\begin{aligned}\sigma_n &= \alpha_n L(\alpha_n(L+1) + \beta_n + \delta_n + \gamma_n) + 2\alpha_n \delta_n L^2 (L+1)^2 + \alpha_n \delta_n f_n L^2 (L+1) \\ &\quad + \alpha_n \gamma_n L(L+1) + \beta_n (L+1) + 2\alpha_n k_n \to 0 \quad \text{as } n \to \infty,\end{aligned}$$

therefore, there exist a positive n_0 such that $1 - \sigma_n \geq \frac{1}{2}$, for any $n \geq n_0$.

Hence from (3.8) we have that

$$\begin{aligned}\|\xi_n - p\|^2 &\leq [1 + 2(\alpha_n \zeta_n + \beta_n)] \|\xi_{n-1} - p\|^2 - 2\alpha_n \Phi(\|\xi_n - p\|) \\ &\quad + 2[\alpha_n \mu_n + \beta_n (L+1) M_1], \quad \forall n \geq n_0.\end{aligned} \tag{3.9}$$

Since $\Phi(q) \geq 0$ for all $q \geq 0$, then for all $n \geq n_0$, we obtain (3.9) that

$$\begin{aligned}\|\xi_n - p\|^2 &\leq [1 + 2(\alpha_n \zeta_n + \beta_n (L+1))] \|\xi_{n-1} - p\|^2 \\ &\quad + 2[\alpha_n \mu_n + \beta_n (L+1) M_1] \\ &= [1 + b_n] \|\xi_{n-1} - p\|^2 + c_n,\end{aligned} \tag{3.10}$$

where $b_n = 2(\alpha_n \zeta_n + \beta_n (L+1))$ and $c_n = 2(\alpha_n \mu_n + \beta_n (L+1) M_1)$. From conditions (ii) and (iii) it is easy to see that $\sum_{n=1}^{\infty} b_n < \infty$ and $\sum_{n=1}^{\infty} c_n < \infty$. Thus using (3.10) and Lemma 2.2 we have that $\lim_{n\to\infty} \|\xi_n - p\|$ exists. Therefore, the sequence $\{\|\xi_n - p\|\}$ is bounded.

Step 2: We now prove that $\{\xi_n\}$ converges strongly to a point in $\mathbf{F}$. Without loss of generality, we can assume that $\|\xi_n - p\|^2 \leq M_2$, where M_2 is a positive constant.

It follows from (3.9) we obtain that

$$\begin{aligned}\|\xi_n - p\|^2 &\leq \|\xi_{n-1} - p\|^2 - 2\alpha_n \Phi(\|\xi_n - p\|) + 2\alpha_n (\vartheta_n M_2 + \mu_n) \\ &\quad + 2\beta_n (L+1)(M_1 + M_2).\end{aligned} \tag{3.11}$$

Notice that (3.11) can be written as:

$$\rho_{n+1}^2 \leq \rho_n^2 - \lambda_n \Phi(\rho_{n+1}) + \lambda_n \varpi_n + \omega_n, \quad n \geq n_0,$$

where

$$\begin{aligned}\rho_{n+1} &= \|\xi_n - p\|, \\ \rho_n &= \|\xi_{n-1} - p\|, \\ \lambda_n &= 2\alpha_n, \\ \varpi_n &= \vartheta_n M_2 + \mu_n, \\ \omega_n &= 2\beta_n (L+1)(M_1 + M_2), \quad \forall\, n \geq 1.\end{aligned}$$

Since $\lim_{n\to\infty} \alpha_n = 0$, then with help of conditions (i),(ii) and (iii), we have

$$\sum_{n=1}^{\infty} \lambda_n = \infty, \quad \lim_{n\to\infty} \varpi_n = 0, \quad \sum_{n=1}^{\infty} \omega_n < \infty.$$

Therefore, all the conditions of Lemma 2.3 are satisfied. Hence,

$$\lim_{n\to\infty} \|\xi_n - p\| = 0.$$

This completes the proof of Theorem 3.1. □

The following results are obtain immediately from Theorem 3.1.

Corollary 3.2. *Let K be a nonempty closed convex subset of a real Banach space E. Let $N \geq 1$ be a positive integer and $R = \{1, 2, 3, \dots, N\}$. Let $T_i : K \to K$ be a uniformly L_t^i–Lipschitz and asymptotically generalized Φ-hemicontractive mappings with sequence $\{\ell_n^i\} \subset [1, \infty)$, where $\ell_n^i \to 1$ as $n \to \infty$ and $S_i : K \to K$ a uniformly L_s^i–Lipschitz and asymptotically generalized Φ-hemicontractive mappings with sequence $\{\varsigma_n^i\} \subset [1, \infty)$, where $\varsigma_n^i \to 1$ as $n \to \infty$, for each $i \in R$. Let $k_n = \max\{\ell_n, \varsigma_n\}$, where $\ell_n = \max\{\ell_n^i : i \in R\}$ and $\varsigma_n = \max\{\varsigma_n^i : i \in R\}$. Suppose that $\boldsymbol{F} = \left(\bigcap_{i=1}^N F(T_i)\right) \bigcap \left(\bigcap_{i=1}^N F(S_i)\right) \neq \emptyset$. Let $\{u_n\}$ and $\{v_n\}$ be bounded sequences in K and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\delta_n\}$ and $\{\gamma_n\}$, be sequences in $[0, 1]$ such that $\alpha_n + \beta_n \leq 1$ and $\delta_n + \gamma_n \leq 1$, for each $n \geq 1$. Let $\Phi(\wp) = \max\{\Phi_i(\wp) : i \in R\}$, for each $\wp \geq 0$. Assume that the following conditions are satisfied:*

(i) $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(ii) $\sum_{n=1}^{\infty} \alpha_n^2 < \infty$, $\sum_{n=1}^{\infty} \alpha_n\beta_n < \infty$, $\sum_{n=1}^{\infty} \alpha_n\delta_n < \infty$, $\sum_{n=1}^{\infty} \alpha_n\gamma_n < \infty$;

(iii) $\sum_{n=1}^{\infty} \beta_n < \infty$;

(iv) $\alpha_n\delta_n L^2 < 1$, *where* $L = \max\{L_t^i, \dots, L_t^N, L_s^i, \dots, L_s^N\}$.

Let $\{\xi_n\}$ be a sequence generated by

$$\begin{cases} \xi_0 \in K, \\ \xi_n = (1 - \alpha_n - \beta_n)\xi_{n-1} + \alpha_n T_{i(n)}^{h(n)} \eta_n + \beta u_n, \quad \forall n \geq 1. \\ \eta_n = (1 - \delta_n - \gamma_n)\xi_{n-1} + \delta_n S_{i(n)}^{h(n)} \xi_n + \gamma_n v_n, \end{cases}$$

Then the sequence $\{\xi_n\}$ converges strongly to a point in $\boldsymbol{F}$.

Proof. Set $e_n = f_n = 0$, $H_i = I$ in Theorem 3.1. □

Corollary 3.3. *Let K be a nonempty closed convex subset of a real Banach space E. Let $N \geq 1$ be a positive integer and $R = \{1, 2, 3, \dots, N\}$.*

Let $T_i : K \to K$ be a uniformly L_t^i–Lipschitz and asymptotically generalized Φ-hemicontractive mappings with sequence $\{\ell_n^i\} \subset [1,\infty)$, where $\ell_n^i \to 1$ as $n \to \infty$ and $S_i : K \to K$ a uniformly L_s^i–Lipschitz and asymptotically generalized Φ-hemicontractive mappings with sequence $\{\varsigma_n^i\} \subset [1,\infty)$, where $\varsigma_n^i \to 1$ as $n \to \infty$, for each $i \in R$. Let $k_n = \max\{\ell_n, \varsigma_n\}$, where $\ell_n = \max\{\ell_n^i : i \in R\}$ and $\varsigma_n = \max\{\varsigma_n^i : i \in R\}$. Suppose that $\boldsymbol{F} = \left(\bigcap_{i=1}^N F(T_i)\right) \bigcap \left(\bigcap_{i=1}^N F(S_i)\right) \neq \emptyset$. Let $\{\alpha_n\}$, $\{\delta_n\}$ and be sequences in $[0,1]$, for each $n \geq 1$. Let $\Phi(\wp) = \max\{\Phi_i(\wp) : i \in R\}$, for each $\wp \geq 0$. Assume that the following conditions are satisfied:

(i) $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(ii) $\sum_{n=1}^{\infty} \alpha_n^2 < \infty$, $\sum_{n=1}^{\infty} \alpha_n \delta_n < \infty$;

(iii) $\alpha_n \delta_n L^2 < 1$, where $L = \max\{L_t^i, \dots, L_t^N, L_s^i, \dots, L_s^N\}$.

Let $\{\xi_n\}$ be a sequence generated by

$$\begin{cases} \xi_0 \in K, \\ \xi_n = (1-\alpha_n)\xi_{n-1} + \alpha_n T_{i(n)}^{h(n)} \eta_n, \quad \forall n \geq 1. \\ \eta_n = (1-\delta_n)\xi_{n-1} + \delta_n S_{i(n)}^{h(n)} \xi_n, \end{cases}$$

Then the sequence $\{\xi_n\}$ converges strongly to a point in $\boldsymbol{F}$.

Proof. Set $\beta_n = \gamma_n = 0$ in Corollary 3.2. □

Corollary 3.4. *Let K be a nonempty closed convex subset of a real Banach space E. Let $N \geq 1$ be a positive integer and $R = \{1, 2, 3, \dots, N\}$. Let $T_i : K \to K$ be a uniformly L_t^i–Lipschitz and asymptotically generalized Φ-hemicontractive mappings with sequence $\{\ell_n^i\} \subset [1,\infty)$, where $\ell_n^i \to 1$ as $n \to \infty$, for each $i \in I$. Let $k_n = \max\{\ell_n^i : i \in R\}$. Suppose that $\boldsymbol{F} = \bigcap_{i=1}^N F(T_i) \neq \emptyset$. Let $\{u_n\}$ and $\{v_n\}$ be bounded sequences in K and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\delta_n\}$ and $\{\gamma_n\}$, be sequences in $[0,1]$ such that $\alpha_n + \beta_n \leq 1$ and $\delta_n + \gamma_n \leq 1$, for each $n \geq 1$. Let $\Phi(\wp) = \max\{\Phi_i(\wp) : i \in R\}$, for each $\wp \geq 0$. Assume that the following conditions are satisfied:*

(i) $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(ii) $\sum_{n=1}^{\infty} \alpha_n^2 < \infty$, $\sum_{n=1}^{\infty} \alpha_n \beta_n < \infty$, $\sum_{n=1}^{\infty} \alpha_n \delta_n < \infty$, $\sum_{n=1}^{\infty} \alpha_n \gamma_n < \infty$;

(iii) $\sum_{n=1}^{\infty} \beta_n < \infty$;

(iv) $\alpha_n \delta_n L^2 < 1$, where $L = \max\{L_t^i, \dots, L_t^N\}$.

Let $\{\xi_n\}$ *be a sequence generated by*

$$\begin{cases} \xi_0 \in K, \\ \xi_n = (1-\alpha_n-\beta_n)\xi_{n-1} + \alpha_n T_{i(n)}^{h(n)} \eta_n + \beta u_n, \quad \forall n \geq 1. \\ \eta_n = (1-\delta_n-\gamma_n)\xi_{n-1} + \delta_n T_{i(n)}^{h(n)} \xi_n + \gamma_n v_n, \end{cases}$$

Then the sequence $\{\xi_n\}$ *converges strongly to a point in* $\boldsymbol{F}$.

Proof. Put $T_i = S_i$ in Corollary 3.2. □

Corollary 3.5. *Let* K *be a nonempty closed convex subset of a real Banach space* E. *Let* $N \geq 1$ *be a positive integer and* $R = \{1, 2, 3, \ldots, N\}$. *Let* $T_i : K \to K$ *be a uniformly* L_t^i*–Lipschitz and asymptotically generalized* Φ*-hemicontractive mappings with sequence* $\{\ell_n^i\} \subset [1, \infty)$, *where* $\ell_n^i \to 1$ *as* $n \to \infty$, *for each* $i \in I$. *Let* $k_n = \max\{\ell_n^i : i \in R\}$. *Suppose that* $\boldsymbol{F} = \bigcap_{i=1}^{N} F(T_i) \neq \emptyset$. *Let* $\{\alpha_n\}$ *and* $\{\delta_n\}$ *be sequences in* $[0, 1]$, *for each* $n \geq 1$. *Let* $\Phi(\wp) = \max\{\Phi_i(\wp) : i \in R\}$, *for each* $\wp \geq 0$. *Assume that the following conditions are satisfied:*

(i) $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(ii) $\sum_{n=1}^{\infty} \alpha_n^2 < \infty$, $\sum_{n=1}^{\infty} \alpha_n \delta_n < \infty$;
(iii) $\alpha_n \delta_n L^2 < 1$, *where* $L = \max\{L_t^i, \ldots, L_t^N\}$.

Let $\{\xi_n\}$ *be a sequence generated by*

$$\begin{cases} \xi_0 \in K, \\ \xi_n = (1-\alpha_n)\xi_{n-1} + \alpha_n T_{i(n)}^{h(n)} \eta_n, \qquad \forall n \geq 1. \\ \eta_n = (1-\delta_n)\xi_{n-1} + \delta_n T_{i(n)}^{h(n)} \xi_n, \end{cases}$$

Then the sequence $\{\xi_n\}$ *converges strongly to a point in* $\boldsymbol{F}$.

Proof. Set $\beta_n = \gamma_n = 0$ in Corollary 3.4. □

Corollary 3.6. *Let* K *be a nonempty closed convex subset of a real Banach space* E. *Let* $N \geq 1$ *be a positive integer and* $R = \{1, 2, 3, \ldots, N\}$. *Let* $T_i : K \to K$ *be a uniformly* L_t^i*–Lipschitz and asymptotically generalized* Φ*-hemicontractive mappings with sequence* $\{\ell_n^i\} \subset [1, \infty)$, *where* $\ell_n^i \to 1$ *as* $n \to \infty$, *for each* $i \in R$. *Let* $k_n = \max\{\ell_n^i : i \in R\}$. *Suppose that* $\boldsymbol{F} = \bigcap_{i=1}^{N} F(T_i) \neq \emptyset$. *Let* $\{u_n\}$ *be a bounded sequence in* K *and* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *be sequences in* $[0, 1]$ *such that* $\alpha_n + \beta_n \leq 1$, *for each* $n \geq 1$. *Let* $\Phi(\wp) = \max\{\Phi_i(\wp) : i \in R\}$, *for each* $\wp \geq 0$. *Assume that the following conditions are satisfied:*

(i) $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(ii) $\sum_{n=1}^{\infty} \alpha_n^2 < \infty$, $\sum_{n=1}^{\infty} \alpha_n\beta_n < \infty$;

(iii) $\sum_{n=1}^{\infty} \beta_n < \infty$.

Let $\{\xi_n\}$ *be a sequence generated by*

$$\begin{cases} \xi_0 \in K, \\ \xi_n = (1-\alpha_n-\beta_n)\xi_{n-1} + \alpha_n T_{i(n)}^{h(n)}\xi_{n-1} + \beta u_n, \end{cases} \quad \forall n \geq 1.$$

Then the sequence $\{\xi_n\}$ *converges strongly to a point in* $\boldsymbol{F}$.

Proof. Take $\delta_n = \gamma_n = 0$ in Corollary 3.4. □

Corollary 3.7. *Let* K *be a nonempty closed convex subset of a real Banach space* E. *Let* $N \geq 1$ *be a positive integer and* $R = \{1, 2, 3, \ldots, N\}$. *Let* $T_i : K \to K$ *a uniformly* L_t^i*–Lipschitz and asymptotically generalized* Φ*-hemicontractive mappings with sequence* $\{\ell_n^i\} \subset [1, \infty)$, *where* $\ell_n^i \to 1$ *as* $n \to \infty$, *for each* $i \in R$. *Let* $k_n = \max\{\ell_n^i : i \in R\}$. *Suppose that* $\boldsymbol{F} = \bigcap_{i=1}^N F(T_i) \neq \emptyset$. *Let* $\{\alpha_n\}$ *be a sequence in* $[0, 1]$, *for each* $n \geq 1$. *Let* $\Phi(\wp) = \max\{\Phi_i(\wp) : i \in R\}$, *for each* $\wp \geq 0$. *Assume that the following conditions are satisfied:*

(i) $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(ii) $\sum_{n=1}^{\infty} \alpha_n^2 < \infty$.

Let $\{\xi_n\}$ *be a sequence generated by*

$$\begin{cases} \xi_0 \in K, \\ \xi_n = (1-\alpha_n)\xi_{n-1} + \alpha_n T_{i(n)}^{k(n)}\xi_{n-1}, \end{cases} \quad \forall n \geq 1.$$

Then the sequence $\{\xi_n\}$ *converges strongly to a point in* $\boldsymbol{F}$.

Proof. Set $\beta_n = 0$ in Corollary 3.6. □

Corollary 3.8. *Let* K *be a nonempty closed convex subset of a real Banach space* E. *Let* $N \geq 1$ *be a positive integer and* $R = \{1, 2, 3, \ldots, N\}$. *Let* $T_i : K \to K$ *a uniformly* L_t^i*–Lipschitz and asymptotically generalized* Φ*-hemicontractive mappings with sequence* $\{\ell_n^i\} \subset [1, \infty)$, *where* $\ell_n^i \to 1$ *as* $n \to \infty$, *for each* $i \in R$. *Let* $k_n = \max\{\ell_n^i : i \in R\}$. *Suppose that* $\boldsymbol{F} = \bigcap_{i=1}^N F(T_i) \neq \emptyset$. *Let* $\{u_n\}$ *be a bounded sequence in* K *and* $\{\alpha_n\}$, $\{\beta_n\}$ *be sequences in* $[0, 1]$ *such that* $\alpha_n + \beta_n \leq 1$, *for each* $n \geq 1$. *Let* $\Phi(\wp) = \max\{\Phi_i(\wp) : i \in R\}$, *for each* $\wp \geq 0$. *Assume that the following conditions are satisfied:*

(i) $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(ii) $\sum_{n=1}^{\infty} \alpha_n^2 < \infty$, $\sum_{n=1}^{\infty} \alpha_n \beta_n < \infty$;

(iii) $\sum_{n=1}^{\infty} \beta_n < \infty$;

(iv) $\alpha_n L < 1$, *where* $L = \max\{L_t^i, \ldots, L_t^N\}$.

Let $\{\xi_n\}$ *be a sequence generated by*

$$\begin{cases} \xi_0 \in K, \\ \xi_n = (1-\alpha_n-\beta_n)\xi_{n-1} + \alpha_n T_{i(n)}^{k(n)} \xi_n + \beta u_n \end{cases} \quad \forall n \geq 1.$$

Then the sequence $\{\xi_n\}$ *converges strongly to a point in* $\boldsymbol{F}$.

Proof. Put $\gamma_n = e_n = f_n = 0$, $\delta_n = 1$, $H_i = S_i = I$ in Theorem 3.1. □

Corollary 3.9. *Let* K *be a nonempty closed convex subset of a real Banach space* E. *Let* $N \geq 1$ *be a positive integer and* $R = \{1, 2, 3, \ldots, N\}$. *Let* $T_i : K \to K$ *a uniformly* L_t^i*–Lipschitz and asymptotically generalized* Φ*-hemicontractive mappings with sequence* $\{\ell_n^i\} \subset [1, \infty)$, *where* $\ell_n^i \to 1$ *as* $n \to \infty$, *for each* $i \in R$. *Let* $k_n = \max\{\ell_n^i : i \in R\}$. *Suppose that* $\boldsymbol{F} = \bigcap_{i=1}^N F(T_i) \neq \emptyset$. *Let* $\{\alpha_n\}$ *be a sequence in* $[0,1]$, *for each* $n \geq 1$. *Let* $\Phi(\wp) = \max\{\Phi_i(\wp) : i \in R\}$, *for each* $\wp \geq 0$. *Assume that the following conditions are satisfied:*

(i) $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(ii) $\sum_{n=1}^{\infty} \alpha_n^2 < \infty$;

(iii) $\alpha_n L < 1$, *where* $L = \max\{L_t^i, \ldots, L_t^N\}$.

Let $\{\xi_n\}$ *be a sequence generated by*

$$\begin{cases} \xi_0 \in K, \\ \xi_n = (1-\alpha_n)\xi_{n-1} + \alpha_n T_{i(n)}^{k(n)} \xi_n \quad , \end{cases} \quad \forall n \geq 1. \tag{3.12}$$

Then the sequence $\{\xi_n\}$ *converges strongly to a point in* $\boldsymbol{F}$.

Proof. Take $\beta_n = 0$ in Corollary 3.8. □

Next, we give the following example to support our main result.

Example 3.10. Let $E = (-\infty, +\infty)$ with the usual norm and $K = [0, +\infty)$. Let $\Phi : [0, \infty) \to [0, \infty)$ be a strictly increasing function with $\Phi(0) = 0$. For $N = 2$, let $\{T_i\}_{i=1}^2, \{S_i\}_{i=1}^2, \{G_i\}_{i=1}^2, \{H_i\}_{i=1}^2 : K \to K$ be defined by:

$$T_1\xi = \frac{\xi}{2(1+\xi)}, \ \xi \in [0, \infty) \ \text{ and } \ \Phi(s) = \frac{s^3}{1+s},$$

$$T_2\xi = \frac{\xi}{(1+\xi)},\ \xi \in [0,\infty) \text{ and } \Phi(s) = \frac{s^3}{1+s},$$
$$S_1\xi = \frac{2\xi^2}{1+2\xi},\ \xi \in [0,\infty) \text{ and } \Phi(s) = \frac{s^2}{1+2s},$$
$$S_2\xi = \frac{\xi}{1+\alpha\xi},\ \xi \in [0,\infty) \text{ and } \alpha \text{ is closing to zero, } \forall\, n \in \mathbb{N} \text{ and}$$
$$\Phi(s) = \frac{s^3}{1+s},$$
$$G_1\xi = \frac{\xi^3}{1+\xi^2},\ \xi \in [0,\infty) \text{ and } \Phi(s) = \frac{s^2}{1+s^2},$$
$$G_2\xi = \frac{\xi}{4},\ \xi \in [0,\infty) \text{ and } \Phi(s) = \frac{s^2}{4},$$
$$H_1\xi = \frac{\xi}{10},\ \xi \in [0,\infty),$$
$$H_2\xi = \frac{\xi}{7},\ \xi \in [0,\infty).$$

Set $\alpha_n = \frac{1}{n}, \beta_n = \frac{1}{n^2}, \delta_n = \frac{1}{(n+1)}, \gamma_n = \frac{1}{(2n+1)}, f_n = \frac{n+1}{(2n+1)}$, for all $n \geq 1$.

Clearly, $\{T_i\}_{i=1}^2, \{S_i\}_{i=1}^2$ and $\{G_i\}_{i=1}^2$ are uniformly L-Lipschitzian and asymptotically generalized Φ-hemicontractive mappings with constant sequence $\{k_n\} = \{1\}$ for all $n \geq 1$. Also, $\{H_i\}_{i=1}^2$ is a family of nonexpansive mappings.

Obviously,

$$\begin{aligned}\mathbf{F} &= \left(\bigcap_{i=1}^{N} F(H_i)\right) \bigcap \left(\bigcap_{i=1}^{N} F(T_i)\right) \bigcap \left(\bigcap_{i=1}^{N} F(S_i)\right) \bigcap \left(\bigcap_{i=1}^{N} F(G_i)\right) \\ &= \{0\} \\ &\neq \emptyset.\end{aligned}$$

For arbitrary $\xi_0 \in K$, the sequence $\{\xi_n\}_{n=1}^{\infty} \in K$ defined by (1.18) converges strongly to the common fixed point of T_i and S_i G_i and H_i ($i = 1, 2$) which is $\{0\}$, satisfying Theorem 3.1. This means that Theorem 3.1 is applicable.

4. Conclusion

Theorem 3.1 generalizes, extends, improves and unifies the corresponding results of Chang [4], Chidume [5–7], Cho et al. [9], Kim [24], Lv et al.[25], Ofoedu [26], Rafiq and Imdad [33], Schu [34], Thakur [43], Yang [47], Zeng [48] and several other results in the existing literature.

Acknowledgment. The authors wish to thank the reviewer(s) and editor(s) for their useful comments and suggestions.

References

1. Y.I. Alber, C.E. Chidume and H. Zegeye, *Regularization of nonlinear ill-posed equations with accretive operators*, Fixed Point Theory and Appl., 1 (2005), pp. 11-33.
2. R.P. Agarwal, Y.J. Cho, J. Li and N.J. Huang, *Stability of iterative procedures with errors approximating common fixed points for a couple of quasi-contractive mappings in q-uniformly smooth Banach spaces*, J. Math. Anal. Appl., 272 (2002), pp. 435-447.
3. A. Bnouhachem, M.A. Noor and T.M. Rassias, *Three-steps iterative algorithms for mixed variational inequalities*, Appl. Math. Comput., 183 (2006), pp. 436-446.
4. S.S. Chang, *Some results for asymptotically pseudocontractive mappings and asymptotically nonexpansive mappings*, Proc. Amer. Math. Soc., 129 (2001), pp. 845-853.
5. C.E. Chidume, A.U. Bello, M.E. Okpala and P. Ndambomve, *Strong convergence theorem for fixed points of nearly nniformly L-Lipschitzian Asymptotically Generalized Φ-hemicontractive mappings*, Int. J. Math. Anal., 9 (2015), pp. 2555-2569.
6. C.E. Chidume and C.O. Chidume, *Convergence theorems for fixed points of uniformly continuous generalized Φ-hemi-contractive mappings*, J. Math. Anal. Appl., 303 (2005), pp. 545-554.
7. C.E. Chidume and C.O. Chidume, *Convergence theorem for zeros of generalized Lipschitz generalized phi-quasi-accretive operators*, Proc. Amer. Math. Soc., 134 (2006), pp. 243-251.
8. R.C. Chen, Y.S. Song and H. Zhou, *Convergence theorems for implicit iteration process for a finite family continuous pseudocontractive mappings*, J. Math. Anal. Appl., 314 (2006), pp. 701-706.
9. Y.J. Cho, H.Y. Zhou and G. Guo, *Weak and strong convergence theorems for three-step iterations with errors for asymptotically nonexpansive mappings*, Comput. Math. Appl., 47 (2004), pp. 707-717.
10. P. Chuadchawna, A. Farajzadeh and A. Kaewcharoen, *On convergence theorems for two generalized nonexpansive multivalued mappings in hyperbolic spaces*, Thai J. Math., 17 (2019), pp. 445-461.
11. P. Chuadchawna, A. Farajzadeh and A. Kaewcharoen, *Fixed-point approximations of generalized nonexpansive mappings via generalized M-iteration process in hyperbolic spaces*, Int. J. Math. Math. Sci., (2020), pp. 1-8, article ID 6435043.

12. P. Chuadchawna, A. Farajzadeh and A. Kaewcharoen, *Convergence theorems for total asymptotically nonexpansive single-valued and quasi nonexpansive multi-valued mappings in hyperbolic spaces*, J. Appl. Anal., 27 (2021), pp. 129-142.
13. G. Das and J.P. Debata, *Fixed points of Quasi-nonexpansive mappings*, Indian J. Pure. Appl. Math., 17 (1986), pp. 1263-1269.
14. L.C. Deng, P. Cubiotti and J.C. Yao, *Approximation of common fixed points of families of nonexpansive mappings*, Tai. J. Math., 12 (2008), pp. 487-500.
15. L.C. Deng, P. Cubiotti and J.C. Yao, *An implicit iteration scheme for monotone variational inequalities and fixed point problems*, Nonlinear Anal., 69 (2008), pp. 2445-2457.
16. L.C. Deng, S. Schaible and J.C. Yao, *Implicit iteration scheme with perturbed mapping for equilibrium problems and fixed point problems of finitely many nonexpansive mappings*, J. Optim. Theory Appl., 139 (2008), pp. 403-418.
17. R. Glowinski and P. Le-Tallec, *Augmented Lagrangian and operator-splitting methods in nonlinear mechanics*, SIAM, Philadelphia, 1989.
18. S. Haubruge, V.H. Nguyen and J.J. Strodiot, *Convergence analysis and applications of the Glowinski-Le-Tallec splitting method for finding a zero of the sum of two maximal monotone operators*, J. Optim. Theory Appl., 97 (1998), pp. 645-673.
19. F.Gu, *Convergence theorems for ϕ-pseudocontractive type mappings in normed linear spaces*, Northeast Math. J., 17 (2001), pp. 340-346.
20. F. Gu, *Strong convergence of an implicit iteration process for a finite family of uniformly L-Lipschitzian mapping in Banach spaces*, J. Ineq. and Appl., doi:10.1155/2010/801961.
21. S. Ishikawa, *Fixed points by a new iteration method, Proceeding of the America Mathematical society*, 4 (1974), pp. 157-150.
22. S.H. Khan and W. Takahashi, *Approximating common fixed points of two asymptotically nonexpansive mappings*, Sci. Math. Jpn., 53 (2001), pp. 143-148.
23. S.H. Khan, I. Yildirim and M. Ozdemir, *Some results for finite families of fniformly L-Lipschitzian mappings in Banach paces*, Thai J. Math., 9 (2011), pp. 319-331.
24. J.K. Kim, D.R. Sahu and Y.M. Nam, *Convergence theorems for fixed points of nearly uniformly L-Lipschitzian asymptotically generalized Φ- hemicontractive mappings*, Nonlinear Anal., 71 (2009), pp. 2833-2838.

25. G. Lv, A. Rafiq and Z. Xue, *Implicit iteration scheme for two phi-hemicontractive operators in arbitrary Banach spaces*, Journal of Ineq. and Appl., 2013, 2013:521.
26. E.U. Ofoedu, *Strong convergence theorem for uniformly L-Lipschitzian asymptotically pseudocontractive mapping in a real Banach space*, J. Math. Anal. Appl., 321 (2006), pp. 722-728.
27. M.A. Noor, T.M. Kassias and Z. Huang, *Three-step iterations for nonlinear accretive operator equations*, J. Math. Anal. Appl., 274 (2001), pp. 59-68.
28. M.O. Osilike, *Iterative solution of nonlinear equations of the ϕ-strongly accretive type*, J. Math. Anal. Appl., 200 (1996), pp. 259-271.
29. M.O. Osilike and B.G. Akuchu, *Common fixed points of finite family of asymptotically pseudocontractive mappings*, Fixed Point Theory and Appl., 2004 (2004), pp. 81-88.
30. M.O. Osilike, S.C. Aniagbosor and B.G. Akuchu, *Fixed points of asymptotically demicontractive mappings in arbitrary Banach spaces*, Panam. Math. J., 12 (2002), pp. 77-88.
31. W.R. Mann, *Mean value methods in iteration*, Proc. Amer. Math. Sci., 4 (2003), pp. 506-510.
32. A. Rafiq, *On convergence of implicit iteration scheme for two hemicontractive mappins*, Sci. Int. (Lahore), 24 (2012), pp. 431-434.
33. A. Rafiq and M. Imdad, *Implicit Mann Iteration Scheme for hemicontractive mapping*, J. Indian Math. Soc., 81 (2014), pp. 147-153.
34. J. Schu, *Weak and strong convergence to fixed points of asymptotically nonexpansive mappings*, Bull. Austr. Math. Soc., 43 (1991), pp. 153-159.
35. J. Schu, *Iterative construction of fixed point of asymptotically nonexpansive mappings*, J. Math. Anal. Appl., 158 (1991), pp. 407-413.
36. H.F. Senter and W.G. Dotson, *Approximating fixed points of nonexpansive mappings*, Proc. Amer. Math. Soc., 44 (1974), pp. 375-380.
37. N. Shahzad and A. Udomene, *Approximating common fixed points of two asymptotically quasinonexpansive mappings in Banach spaces*, Fixed Point Theory Appl., (2006), pp. 1-10, article ID 18909.
38. Z.H. Sun, *Strong convergence of an implicit iteration process for a finite family of asymptotically quasi-nonexpansive mappings*, J. Math. Anal. Appl., 286 (2003), pp. 351-358.
39. S. Suantai, *Weak and strong convergence criteria of Noor iterations for asymptotically nonexpansive mappings*, J. Math. Anal. Appl., 311 (2005), pp. 506-517.

40. W. Takahashi, *Iterative methods for approximation of fixed points and their applications*, J. Oper. Res. Soc. Jpn., 43 (2000), pp. 87-108.
41. W. Takahashi and T. Tamura, *Limit theorems of operators by convex combinations of nonexpansive retractions in Banach spaces*, J. Approx. Theory, 91 (1997), pp. 386-397.
42. B.S. Thakur, *Weak and strong convergence of composite implicit iteration process*, Appl. Math. Comput., 190 (2007), pp. 965-973.
43. B.S. Thakur, *Strong Convergence for Asymptotically generalized Φ-hemicontractive mappings*, ROMAI J., 8 (2012), pp. 165-171.
44. H.K. Xu and R.G. Ori, *An implicit iteration process for nonexpansive mapping*, Num. Fun. Anal. Optim., 22 (2001), pp. 767-773.
45. Y. Yao, *Convergence of three-step iterations for asymptotically nonexpansive mappings*, Appl. Math. Comput., 187 (2007), pp. 883-892.
46. L. P. Yang, *Convergence theorem of an implicit iteration process for asymptotically pseudocontractive mappings*, Bull. of the Iran. Math. Soc., 38 (2012), pp. 699-713.
47. L.P. Yang and G. Hu, *Convergence of implicit iteration process with random errors*, Acta Math. Sinica (Chin. Ser.), 51 (2008), pp. 11-22.
48. L.C. Zeng, *On the approximation of fixed points for asymptotically nonexpansive mappings in Banach spaces*, Acta Math. Sci., 23 (2003), pp. 31-37.
49. L.C. Zeng, *On the iterative approximation for asymptotically pseudocontractive mappings in uniformly smooth Banach spaces*, Chinese Math. Ann., 26 (2005), pp. 283-290.

[1] Department of Mathematics, University of Uyo, Uyo, Nigeria.
Email address: ofemaustine@gmail.com

[2] Department of Mathematics, Michael Okpara University of Agriculture, Umudike, Nigeria.
Email address: igbokwedi@yahoo.com