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New Integral Inequalities Relating to a General Integral Operators Through Monotone Functions

Bouharket Benaissa[1*] and Abdelkader Senouci[2]

Abstract. Weighted integral inequalities for general integral operators on monotone positive functions with parameters p and q are established in [4]. The aim of this work is to extend the results to different cases of these parameters, in particular for negative p and q. We give some new lemmas which will be frequently used in the proofs of the main theorems.

1. Introduction

In 1993 Shanzhong Lai [4] considered weighted norm inequalities for general integral operators of the form

$$S_\phi f(x) = \int_0^\infty \phi(x,y) f(y) dy, \quad \phi(x,.) \geq 0,\ \phi(x,.) \in L_1(0,\infty),\ x \in (0,\infty)$$

on monotone functions $f : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$. The Hardy operator

$$Hf(x) = \frac{1}{x}\int_0^x f(t)dt,$$

the Laplace transform

$$Lf(x) = \int_0^\infty e^{-xt} f(t)dt$$

and the operator

$$Sf(x) = \int_0^\infty \phi(t) f(tx) dt$$

are special cases of $S_\phi f$.

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We adopt the usual following conventions.

(i) We put $0 \times \infty = 0$, $\dfrac{\infty}{\infty} = 0$ and $\dfrac{0}{0} = 0$.

(ii) We shall write h is AC if h is locally absolutely continuous function on $(0, \infty)$ (that is h is absolutely continuous on any finite interval $(a, b) \subset (0, \infty)$).

(iii) we shall use the notation $f \uparrow$ $(f \downarrow)$ to indicate that f is positive and strictly non-increasing (non-decreasing) on $(0, \infty)$.

In [4] weight functions w, v were characterized for which the inequality

$$\|S_\phi f\|_{L_{p,w}(0,\infty)} \le C\|f\|_{L_{q,v}(0,\infty)}$$

holds for monotone functions f, where $C > 0$ is independent of f. Here and almost everywhere in the sequel w, v are positive Lebesgue measurable functions on $(0, \infty)$. Namely the following statements were proved there.

Let

$$\Phi(x,r) = \int_0^r \phi(x,y)dy, \qquad \Phi_1(x,r) = \int_r^\infty \phi(x,y)dy, \tag{1.1}$$

where $\phi : \mathbb{R}_+ \times \mathbb{R}_+ \longrightarrow \mathbb{R}_+$.

Theorem 1.1. *Let $1 \le q \le p < \infty$ and $C > 0$, then the inequality*

$$\left(\int_0^\infty f^p w\right)^{\frac{1}{p}} \le C \left[\int_0^\infty (S_\phi f)^q v\right]^{\frac{1}{q}} \tag{1.2}$$

holds for all $f \downarrow$, if and only if

$$\left(\int_0^r w\right)^{\frac{1}{p}} \le C \left[\int_0^\infty \Phi(x,r)^q v\right]^{\frac{1}{q}}, \quad \forall r > 0. \tag{1.3}$$

Inequality (1.2) holds for all $f \uparrow$, if and only if

$$\left[\int_r^\infty w\right]^{\frac{1}{p}} \le C \left(\int_0^\infty \Phi_1(x,r)^q v\right)^{\frac{1}{q}}, \quad \forall r > 0. \tag{1.4}$$

Theorem 1.2. *If $0 < q \le p \le 1$ and $C > 0$, then the inequality*

$$\left(\int_0^\infty (S_\phi f)^p w\right)^{\frac{1}{p}} \le C \left[\int_0^\infty f^q v\right]^{\frac{1}{q}} \tag{1.5}$$

holds for all $f \downarrow$, if and only if

$$\left(\int_0^\infty \Phi(x,r)^p w\right)^{\frac{1}{p}} \le C \left[\int_0^r v\right]^{\frac{1}{q}}, \quad \forall r > 0. \tag{1.6}$$

Inequality (1.5) holds for all $f\uparrow$, if and only if

$$\left(\int_0^\infty \Phi_1(x,r)^p w\right)^{\frac{1}{p}} \le C\left[\int_r^\infty v\right]^{\frac{1}{q}}, \quad \forall r>0. \tag{1.7}$$

Lemma 1.3. (i) *For $0<p\le 1$, the inequality*

$$\int_0^\infty f^p(x)x^{p-1}dx \ge \frac{1}{p}\left(\int_0^\infty f\right)^p, \quad \forall f\downarrow, \tag{1.8}$$

holds.

(ii) *For $p\ge 1$, the inequality*

$$\int_0^\infty f^p(x)x^{p-1}dx \le \frac{1}{p}\left(\int_0^\infty f\right)^p, \quad \forall f\downarrow \tag{1.9}$$

holds.

Lemma 1.4. 1. *Let $C_1>0$ $g:\mathbb{R}_+\longrightarrow\mathbb{R}_+$, be Lebesgue measurable function on $\mathbb{R}_+$, h be AC and $h'\le 0$ almost everywhere on $(0,\infty)$, $h(+\infty)=0$, then*

(i) *For $p\ge 1$, the inequality*

$$\int_0^\infty f^p g \le C_1\left[-\int_0^\infty fh'\right]^p, \quad \forall f\uparrow, \tag{1.10}$$

holds if and only if

$$\int_r^\infty g \le C_1 h(r)^p, \quad \forall r>0. \tag{1.11}$$

(ii) *For $0<p\le 1$, the inequality*

$$\int_0^\infty f^p g \ge C_1\left[-\int_0^\infty fh'\right]^p, \quad \forall f\uparrow, \tag{1.12}$$

holds if and only if

$$\int_r^\infty g \ge C_1 h(r)^p, \quad \forall r>0. \tag{1.13}$$

2. *Let $C_2>0$ $g:\mathbb{R}_+\longrightarrow\mathbb{R}_+$, be Lebesgue measurable function on $\mathbb{R}_+$, h be AC and $h'\ge 0$ almost everywhere on $(0,\infty)$, $h(+0)=0$, then*

(iii) *For $p\ge 1$, the inequality*

$$\int_0^\infty f^p g \le C_2\left[\int_0^\infty fh'\right]^p, \quad \forall f\downarrow, \tag{1.14}$$

holds if and only if

$$\int_0^r g \le C_2 h(r)^p, \quad \forall r>0.$$

(iv) *For* $0 < p \leq 1$,*the inequality*

$$\int_0^\infty f^p g \geq C_2 \left[\int_0^\infty f h'\right]^p, \quad \forall f \downarrow, \tag{1.15}$$

holds if and only if

$$\int_0^r g \geq C_2 h(r)^p, \quad \forall r > 0. \tag{1.16}$$

In [2] the following theorem was proved.

Theorem 1.5. (i) *Let* $0 < p < 1$, $-\infty \leq a < b \leq +\infty$ *and* $-\infty \leq c < d \leq +\infty$. *Suppose that* f *is measurable non-negative (non-positive) on* $(a, b) \times (c, d)$ *and* $f(., y) \in L_p(a, b)$ *for almost all* $y \in (c, d)$. *Then*

$$\left\|\int_c^d f(x,y)dy\right\|_{L_p(a,b)} \geq \int_c^d \|f(x,y)\|_{L_p(a,b)} dy, \tag{1.17}$$

if the left-hand side is finite.

(ii) *Let* $p < 0$, $-\infty \leq a < b \leq \infty$ *and* $-\infty \leq c < d \leq \infty$. *Suppose that* f *is measurable non-negative (non-positive) on* $(a, b) \times (c, d)$ *and* $f(., y) \in L_p(a, b)$ *for almost all* $y \in (c, d)$. *Then*

$$\left\|\int_c^d f(x,y)dy\right\|_{L_p(a,b)} \geq \int_c^d \|f(x,y)\|_{L_p(a,b)} dy, \tag{1.18}$$

if the left-hand side is finite.

2. Preliminaries

In this section, we prove some lemmas which will be frequently used in the proofs of the main theorems.

Lemma 2.1. (i) *If* $0 < p \leq 1$, *then*

$$p \int_0^\infty f^p(x) x^{-p-1} dx \geq \left(\int_0^\infty \frac{1}{x^2} f(x) dx\right)^p, \quad \forall f \uparrow. \tag{2.1}$$

(ii) *If* $p \geq 1$, *then*

$$\int_0^\infty f^p(x) x^{-p-1} dx \leq \frac{1}{p} \left(\int_0^\infty \frac{1}{x^2} f(x) dx\right)^p, \quad \forall f \uparrow. \tag{2.2}$$

Proof. This is a particular case of Lemma 1.4, which can be derived by taking

$$g(x) = x^{-1-p}, \qquad h(r) = \frac{1}{r}, \qquad C_1 = \frac{1}{p}.$$

□

Remark 2.2. If, for $t \in (0,\infty)$, we put $f(x) = \begin{cases} 0, & x \in (0,t) \\ 1, & x \in (t,\infty) \end{cases}$ we obtain equality in (2.1) and (2.2). Consequently, p and $\frac{1}{p}$ are sharp constants in (2.1), (2.2) respectively.

Lemma 2.3. *Let $p < 0$. If f is non-negative and non-increasing on $(0,+\infty)$, then*

$$\int_0^\infty f^p(x)x^{p-1}dx \geq 2^{-p}\left(\frac{p}{p-1}\right)^{p-1}\left(\int_0^\infty f\right)^p. \tag{2.3}$$

If f is non-negative and non-decreasing on $(0,+\infty)$, then

$$\int_0^\infty f^p(x)x^{-p-1}dx \geq 2^{-p}\left(\frac{p}{p-1}\right)^{p-1}\left(\int_0^\infty \frac{1}{x^2}f(x)dx\right)^p. \tag{2.4}$$

Proof. 1) Let f be non-increasing.
From Lemma 1.3 (i), with $0 < q' \leq 1$, we have

$$\left(\int_0^\infty f\right)^{q'} \leq q'\int_0^\infty f^{q'}(x)x^{q'-1}dx.$$

Let $p' = q' - 1$, then for $-1 < p' < 0$, we get

$$\left(\int_0^\infty f\right)^{\frac{p'+1}{p'}} \geq (p'+1)^{\frac{1}{p'}}\left(\int_0^\infty f^{p'+1}(x)x^{p'}dx\right)^{\frac{1}{p'}}.$$

If we put $\frac{p'+1}{p'} = p = 1 + \frac{1}{p'}$, then

$$\left(\int_0^\infty f\right)^p \geq \left(\frac{p}{p-1}\right)^{p-1}\left(\int_0^\infty f^{\frac{p}{p-1}}(x)x^{\frac{1}{p-1}}dx\right)^{p-1}.$$

Let

$$R = \int_0^\infty f^{\frac{p}{p-1}}(x)x^{\frac{1}{p-1}}dx,$$

by Hölder's inequality with $r = \frac{p-1}{p} > 1$ and $r' = 1 - p$, we obtain

$$\begin{aligned} R &= \int_0^\infty \left(xf^{\frac{p}{p-1}}(x)\right)^{-1}\left(xf^2(x)\right)^{\frac{p}{p-1}}dx \\ &\leq \left(\int_0^\infty x^{p-1}f^p(x)dx\right)^{\frac{-1}{p-1}}\left(\int_0^\infty xf^2(x)dx\right)^{\frac{p}{p-1}}, \end{aligned}$$

consequently

$$R^{p-1} \geq \left(\int_0^\infty x^{p-1}f^p(x)dx\right)^{-1}\left(\int_0^\infty xf^2(x)dx\right)^p$$

and

$$\left(\int_0^\infty f\right)^{-p} \leq \left(\frac{p}{p-1}\right)^{1-p} \left(\int_0^\infty x^{p-1} f^p(x)dx\right) \left(\int_0^\infty x f^2(x)dx\right)^{-p}. \tag{2.5}$$

By using Lemma 1.3 (ii), for $p = 2$, we have

$$\left(\int_0^\infty f\right)^2 \geq 2 \int_0^\infty x f^2(x)dx,$$

then

$$\left(\int_0^\infty f\right) \geq 2 \left(\int_0^\infty x f^2(x)dx\right) \left(\int_0^\infty f\right)^{-1},$$

for $p < 0$

$$\left(\int_0^\infty f\right)^p \leq 2^p \left(\int_0^\infty x f^2(x)dx\right)^p \left(\int_0^\infty f\right)^{-p}. \tag{2.6}$$

By (2.5) and (2.6) we get (2.3).

2) Let f be non decreasing. By applying Lemma 2.1, (for $p = 2$) the proof of (2.4) is similar that of (2.3).

□

Lemma 2.4. *Suppose that $w \in L_1(0,\infty)$ is weight function and $\varphi : (0,\infty) \longrightarrow (0,\infty)$ is continues and strictly increasing function such that $\varphi(0,\infty) = (0,\infty)$, $\varphi \in C^1(0,+\infty)$. Then for all $p < 0$ and $y \in (0,+\infty)$*

$$\int_0^\infty \int_{\varphi(y)}^\infty w(x)dxdy = \int_0^\infty \varphi^{-1}(x)w(x)dx \tag{2.7}$$

$$\int_0^\infty \Phi(x,\varphi(y))y^{\frac{1}{p}-1}dy = -p \int_0^\infty \left(\varphi^{-1}(t)\right)^{\frac{1}{p}} \phi(x,t)dt. \tag{2.8}$$

$$\int_0^\infty \frac{1}{y^2} \int_0^{\varphi(y)} w(x)dxdy = \int_0^\infty \left(\varphi^{-1}(x)\right)^{-1} w(x)dx \tag{2.9}$$

$$\int_0^\infty \Phi_1(x,\varphi(y))y^{-\frac{1}{p}-1}dy = -p \int_0^\infty \left(\varphi^{-1}(t)\right)^{-\frac{1}{p}} \phi(x,t)dt. \tag{2.10}$$

Proof. From (1.1) it follows that for almost all $r > 0$, $\Phi'_r = \phi(x,r)$ and $(\Phi_1)'_r = -\phi(x,r)$. We set $W(t) = \int_0^t w(x)dx$, $W_1(t) = \int_t^\infty w(x)dx$, then $W'(t) = w(t)$ and $(W_1)'(t) = -w(t)$. If we put $\varphi(y) = t$ and integrating by parts, we obtain

$$\begin{aligned}\int_0^\infty \int_{\varphi(y)}^\infty w(x)dxdy &= \int_0^\infty \int_t^\infty w(x)dxd\varphi^{-1}(t)\\ &= \int_0^\infty W_1(t)d\varphi^{-1}(t)\end{aligned}$$

$$= \lim_{\tau\to\infty, s\to 0^+} \Big[W_1(t)\varphi^{-1}(t)\Big]_s^\tau + \int_0^\infty \varphi^{-1}(t)w(t)dt$$
$$= \int_0^\infty \varphi^{-1}(t)w(t)dt$$

and we get equality (2.7).

The proof of equality (2.8) is similar.

Next we prove equality (2.9):

$$\begin{aligned}\int_0^\infty \frac{1}{y^2}\int_0^{\varphi(y)} w(x)dxdy &= \int_0^\infty \frac{1}{(\varphi^{-1}(t))^2}\left(\int_0^t w(x)dx\right)d\varphi^{-1}(t)\\ &= -\int_0^\infty W(t)d\left(\frac{1}{\varphi^{-1}(t)}\right)\\ &= \lim_{\tau\to\infty, s\to 0^+} -\left[W(t)\frac{1}{\varphi^{-1}(t)}\right]_s^\tau\\ &\quad+\int_0^\infty \frac{1}{\varphi^{-1}(t)}w(t)dt\\ &= \int_0^\infty \frac{1}{\varphi^{-1}(t)}w(t)dt.\end{aligned}$$

The proof of equality (2.10) is similar to that of (2.8). □

Lemma 2.5. *Suppose that $w \in L_1(0,\infty)$ is a weight function, and $\psi : (0,\infty) \longrightarrow (0,\infty)$ is is continues and strictly decreasing function such that $\psi((0,\infty)) = (0,\infty)$, $\psi \in C^1(0,+\infty)$. Then for all $p > 0$ and $y \in (0,+\infty)$*

$$\int_0^\infty \int_0^{\psi(y)} w(x)dxdy = \int_0^\infty \psi^{-1}(x)w(x)dx. \tag{2.11}$$

$$\int_0^\infty \Phi(x,\psi(y))y^{\frac{1}{p}-1}dy = p\int_0^\infty \left(\psi^{-1}(t)\right)^{\frac{1}{p}}\phi(x,t)dt. \tag{2.12}$$

$$\int_0^\infty \frac{1}{y^2}\int_{\psi(y)}^\infty w(x)dxdy = \int_0^\infty \left(\psi^{-1}(x)\right)^{-1}w(x)dx. \tag{2.13}$$

$$\int_0^\infty \Phi_1(x,\psi(y))y^{-\frac{1}{p}-1}dy = p\int_0^\infty \left(\psi^{-1}(t)\right)^{-\frac{1}{p}}\phi(x,t)dt. \tag{2.14}$$

Proof. Let $\psi(y) = t$ and integrating by parts, we get

$$\begin{aligned}\int_0^\infty \int_0^{\psi(y)} w(x)dxdy &= -\int_0^\infty \int_0^t w(x)dxd\psi^{-1}(t)\\ &= \int_0^\infty \psi^{-1}(t)w(t)dt.\end{aligned}$$

The proof of equality (2.12) is similar,

$$\int_0^\infty \Phi(x,\psi(y))y^{\frac{1}{p}-1}dy = -\int_0^\infty \Phi(x,t)\left(\psi^{-1}(t)\right)^{\frac{1}{p}-1}d\psi^{-1}(t)$$
$$= p\int_0^\infty \left(\psi^{-1}(t)\right)^{\frac{1}{p}}\phi(x,t)dt.$$

Next we prove equality (2.13)

$$\int_0^\infty \frac{1}{y^2}\int_{\psi(y)}^\infty w(x)dxdy = -\int_0^\infty \frac{1}{(\psi^{-1}(t))^2}\left(\int_t^\infty w(x)dx\right)d\psi^{-1}(t)$$
$$= \lim_{\tau\to\infty, s\to 0^+}\left[W_1(t)\frac{1}{\psi^{-1}(t)}\right]_s^\tau + \int_0^\infty \frac{1}{\psi^{-1}(t)}w(t)dt$$
$$= \int_0^\infty \frac{1}{\psi^{-1}(t)}w(t)dt.$$

The proof of equality (2.14) is similar to that of (2.12). □

3. Mains Results

In this section we prove theorems similar to Theorems 1 and 2 but for a different range of the parameters p and q.

Theorem 3.1. *Let $1 \le p < \infty$, $0 < q < 1$, and $C_3 > 0$, then the inequality*

$$\left(\int_0^\infty f^p w\right)^{\frac{1}{p}} \le C_3\left[\int_0^\infty (S_\phi f)^q v\right]^{\frac{1}{q}} \tag{3.1}$$

holds for all $f\downarrow$ if and only if

$$\left(\int_0^r w\right)^{\frac{1}{p}} \le C_3\left[\int_0^\infty \Phi(x,r)^q v\right]^{\frac{1}{q}}, \quad \forall r>0. \tag{3.2}$$

Inequality (3.1) holds for all $f\uparrow$ if and only if

$$\left(\int_r^\infty w\right)^{\frac{1}{p}} \le C_3\left[\int_0^\infty \Phi_1(x,r)^q v\right]^{\frac{1}{q}}, \quad \forall r>0. \tag{3.3}$$

Proof. (3.1) $\longrightarrow$ (3.2). It suffices to take $f = \mathbf{1}_{(0,r)}$, (3.2) $\longrightarrow$ (3.1). First, let $f\in C^1(0,\infty)$ and be positive and strictly decreasing on $(0,\infty)$. $\psi(t) = (f^p)^{-1}(t)$ then $\psi(t)$ satisfies the assumptions of Lemma 6. Now, let $r=\psi(y)$ in (3.2) and integrate in y over $(0,\infty)$, then

$$\int_0^\infty\left(\int_0^{\psi(y)} w(x)dx\right)^{\frac{1}{p}} y^{\frac{1}{p}-1}dy \le C_3\int_0^\infty\left[\int_0^\infty \Phi(x,\psi(y))^q v(x)dx\right]^{\frac{1}{q}} y^{\frac{1}{p}-1}dy. \tag{3.4}$$

Next let $g(y) = \displaystyle\int_0^{\psi(y)} w(x)dx,$ since $w(x) > 0$ a .e. , then $g \downarrow$, $0 < \dfrac{1}{p} \leq 1$ and by (1.8) we obtain

$$\int_0^\infty g^{\frac{1}{p}}(y)y^{\frac{1}{p}-1}dy \geq p\left(\int_0^\infty g(y)dy\right)^{\frac{1}{p}}.$$

If we denote by *Lhs* and *Rhs* respectively the left-hand side, right-hand side of inequality (3.4), using (2.11)we get

$$\begin{aligned} Lhs &\geq p\left(\int_0^\infty \left(\int_0^{\psi(y)} w(x)dx\right)dy\right)^{\frac{1}{p}} \\ &= p\left(\int_0^\infty \psi^{-1}(x)w(x)dx\right)^{\frac{1}{p}}. \end{aligned}$$

and

$$\begin{aligned} Rhs &= C_3\int_0^\infty \left[\int_0^\infty \Phi(x,\psi(y))^q v(x)dx\right]^{\frac{1}{q}} y^{\frac{1}{p}-1}dy \\ &= C_3\int_0^\infty \left[\int_0^\infty \Phi(x,\psi(y))^q y^{q(\frac{1}{p}-1)}v(x)dx\right]^{\frac{1}{q}} dy \\ &= C_3\int_0^\infty \left[\int_0^\infty \left(\Phi(x,\psi(y))y^{\frac{1}{p}-1}v^{\frac{1}{q}}(x)\right)^q dx\right]^{\frac{1}{q}} dy \\ &= C_3\int_0^\infty \parallel \Phi(x,\psi(y))y^{\frac{1}{p}-1}v^{\frac{1}{q}}(x) \parallel_{L_{q,x}} dy \end{aligned}$$

and by the reverse integral Minkowsky inequality (see[2])and by (2.12), we get

$$\begin{aligned} Rhs &\leq C_3 \parallel \int_0^\infty \Phi(x,\psi(y))y^{\frac{1}{p}-1}v^{\frac{1}{q}}(x)dy \parallel_{L_{q,x}} \\ &= C_3\left[\int_0^\infty v(x)\left(\int_0^\infty \Phi(x,\psi(y))y^{\frac{1}{p}-1}dy\right)^q dx\right]^{\frac{1}{q}} \\ &= C_3\left[\int_0^\infty v(x)\left(p\int_0^\infty \left(\psi^{-1}(t)\right)^{\frac{1}{p}}\phi(x,t)dt\right)^q dx\right]^{\frac{1}{q}} \\ &= C_3 p\left[\int_0^\infty v(x)\left(\int_0^\infty \left(\psi^{-1}(t)\right)^{\frac{1}{p}}\phi(x,t)dt\right)^q dx\right]^{\frac{1}{q}}. \end{aligned}$$

By setting $f(t) = (\psi^{-1}(t))^{\frac{1}{p}}$ we get inequality (3.1).

(3.1) $\longrightarrow$ (3.3), let $f = 1_{(r,\infty)}$. (3.3) $\longrightarrow$ (3.1). Let $r = \psi(y)$ and by integrating (3.3), we conclude that

(3.5)
$$\int_0^\infty \left(\int_{\psi(y)}^\infty w(x)dx\right)^{\frac{1}{p}} y^{-\frac{1}{p}-1}dy \le C_3 \int_0^\infty \left[\int_0^\infty \Phi_1(x,\psi(y))^q v(x)dx\right]^{\frac{1}{q}} y^{-\frac{1}{p}-1}dy.$$

Let $g(y) = \int_{\psi(y)}^\infty w(x)dx$, $g \uparrow$, $0 < \frac{1}{p} \le 1$ and by (2.1) we get

$$\int_0^\infty g^{\frac{1}{p}}(y) y^{-\frac{1}{p}-1}dy \ge p\left(\int_0^\infty \frac{1}{y^2} g(y)dy\right)^{\frac{1}{p}}.$$

By inequality (3.5) and (2.13) we have

$$\begin{aligned} Lhs &\ge p\left(\int_0^\infty \left(\frac{1}{y^2}\int_{\psi(y)}^\infty w(x)dx\right)dy\right)^{\frac{1}{p}} \\ &= p\left(\int_0^\infty (\psi^{-1}(x))^{-1} w(x)dx\right)^{\frac{1}{p}} \end{aligned}$$

and

$$\begin{aligned} Rhs &= C_3 \int_0^\infty \left[\int_0^\infty \Phi_1(x,\psi(y))^q v(x)dx\right]^{\frac{1}{q}} y^{-\frac{1}{p}-1}dy \\ &= C_3 \int_0^\infty \left[\int_0^\infty \left(\Phi_1(x,\psi(y)) y^{-\frac{1}{p}-1} v^{\frac{1}{q}}(x)\right)^q dx\right]^{\frac{1}{q}} dy \\ &= C_3 \int_0^\infty \parallel \Phi_1(x,\psi(y)) y^{-\frac{1}{p}-1} v^{\frac{1}{q}}(x) \parallel_{L_{q,x}} dy \end{aligned}$$

by the reverse integral Minkowski inequality and (2.14), we obtain

$$\begin{aligned} Rhs &\le C_3 \left\| \int_0^\infty \Phi_1(x,\psi(y)) y^{-\frac{1}{p}-1} v^{\frac{1}{q}}(x)dy \right\|_{L_{q,x}} \\ &= C_3 \left[\int_0^\infty v(x)\left(\int_0^\infty \Phi_1(x,\psi(y)) y^{-\frac{1}{p}-1}dy\right)^q dx\right]^{\frac{1}{q}} \\ &= C_3 \left[\int_0^\infty v(x)\left(p\int_0^\infty \left(\psi^{-1}(t)\right)^{-\frac{1}{p}} \phi(x,t)dt\right)^q dx\right]^{\frac{1}{q}} \\ &= C_3 p \left[\int_0^\infty v(x)\left(\int_0^\infty \left(\psi^{-1}(t)\right)^{-\frac{1}{p}} \phi(x,t)dt\right)^q dx\right]^{\frac{1}{q}}. \end{aligned}$$

To complete the proof it suffices to take $f(t) = (\psi^{-1}(t))^{-\frac{1}{p}}$. □

Theorem 3.2. *Let $p \geq 1$, $q < 0$ and $C_4 > 0$, then the inequality*

$$\left(\int_0^\infty f^p w\right)^{\frac{1}{p}} \leq C_4 \left[\int_0^\infty (S_\phi f)^q v\right]^{\frac{1}{q}} \tag{3.6}$$

holds for all $f \downarrow$ if and only if

$$\left(\int_0^r w\right)^{\frac{1}{p}} \leq C_4 \left[\int_0^\infty \Phi(x,r)^q v\right]^{\frac{1}{q}}, \quad \forall r > 0. \tag{3.7}$$

Inequality (3.6) holds for all $f \uparrow$ if and only if

$$\left(\int_r^\infty w\right)^{\frac{1}{p}} \leq C_4 \left[\int_0^\infty \Phi_1(x,r)^q v\right]^{\frac{1}{q}}, \quad \forall r > 0. \tag{3.8}$$

Proof. Since the reverse integral Minkowski inequality for $q < 0$ is similar to the case $0 < q < 1$, proof of Theorem 3.2 is similar to the proof of Theorem 3.1. □

Theorem 3.3. *Let $0 < p < 1$, $q \geq 1$ and $C_5 > 0$, then the inequality*

$$\left(\int_0^\infty f^p w\right)^{\frac{1}{p}} \geq C_5 \left[\int_0^\infty (S_\phi f)^q v\right]^{\frac{1}{q}} \tag{3.9}$$

holds for all $f \downarrow$ if and only if

$$\left(\int_0^r w\right)^{\frac{1}{p}} \geq C_5 \left[\int_0^\infty \Phi(x,r)^q v\right]^{\frac{1}{q}}, \quad \forall r > 0. \tag{3.10}$$

Inequality (3.9) holds for all $f \uparrow$ if and only if

$$\left(\int_r^\infty w\right)^{\frac{1}{p}} \geq C_5 \left[\int_0^\infty \Phi_1(x,r)^q v\right]^{\frac{1}{q}}, \quad \forall r > 0. \tag{3.11}$$

Proof. The proof of this Theorem is similar to that of Theorem 3.1. □

Theorem 3.4. *Let $p < 0$, $q < 0$ and $C_6 > 0$. If*

$$\left(\int_0^r w\right)^{\frac{1}{p}} \leq C_6 \left[\int_0^\infty \Phi_1(x,r)^q v\right]^{\frac{1}{q}}, \quad \forall r > 0, \tag{3.12}$$

then

$$\left(\int_0^\infty f^p w\right)^{\frac{1}{p}} \leq C_7 \left[\int_0^\infty (S_\phi f)^q v\right]^{\frac{1}{q}} \tag{3.13}$$

holds for all $f \uparrow$.

If

$$\left(\int_r^\infty w\right)^{\frac{1}{p}} \leq C_8 \left[\int_0^\infty \Phi(x,r)^q v\right]^{\frac{1}{q}}, \quad \forall r > 0 \tag{3.14}$$

then (3.13) holds for all $f \downarrow$.

Moreover $C_7 = (1-p)^{\frac{1}{p}} \left(\frac{p}{p-1}\right) C_6.$

Proof. (3.12) $\longrightarrow$ (3.13). Let $r = \varphi(y) \uparrow$, then

(3.15)
$$\int_0^\infty \left(\int_0^{\varphi(y)} w(x)dx\right)^{\frac{1}{p}} y^{-\frac{1}{p}-1}dy \le C_6 \int_0^\infty \left[\int_0^\infty \Phi_1(x,\varphi(y))^q v(x)dx\right]^{\frac{1}{q}} y^{-\frac{1}{p}-1}dy.$$

Let $g(y) = \int_0^{\varphi(y)} w(x)dx,\ g\uparrow,\ \frac{1}{p}<0$, by (2.4) we obtain

$$\int_0^\infty g^{\frac{1}{p}}(y)y^{-\frac{1}{p}-1}dy \ge \left(\frac{1}{1-p}\right)^{\frac{1}{p}-1} \left(\int_0^\infty \frac{1}{y^2}g(y)dy\right)^{\frac{1}{p}}.$$

By using inequality (3.15) and (2.9), we get

$$\begin{aligned} Lhs &\ge \left(\frac{1}{1-p}\right)^{\frac{1}{p}-1} \left(\int_0^\infty \left(\frac{1}{y^2}\int_0^{\varphi(y)} w(x)dx\right)dy\right)^{\frac{1}{p}} \\ &= \left(\frac{1}{1-p}\right)^{\frac{1}{p}-1} \left(\int_0^\infty \left(\varphi^{-1}(x)\right)^{-1} w(x)dx\right)^{\frac{1}{p}}. \end{aligned}$$

Let

$$\begin{aligned} R &= \int_0^\infty \left[\int_0^\infty \Phi_1(x,\varphi(y))^q v(x)dx\right]^{\frac{1}{q}} y^{-\frac{1}{p}-1}dy \\ &= \int_0^\infty \left[y^{q\left(-\frac{1}{p}-1\right)} \int_0^\infty \Phi_1(x,\varphi(y))^q v(x)dx\right]^{\frac{1}{q}} dy \\ &= \int_0^\infty \left\|\Phi_1(x,\varphi(y))y^{-\frac{1}{p}-1}v^{\frac{1}{q}}(x)\right\|_{L_{q,x}} dy. \end{aligned}$$

By the reverse integral Minkowski inequality and (2.10), we obtain

$$\begin{aligned} R &\le \left\|\int_0^\infty \Phi_1(x,\varphi(y))y^{-\frac{1}{p}-1}v^{\frac{1}{q}}(x)dy\right\|_{L_{q,x}} \\ &= \left\{\int_0^\infty v(x)\left(\int_0^\infty \Phi_1(x,\varphi(y))y^{-\frac{1}{p}-1}dy\right)^q d(x)\right\}^{\frac{1}{q}} \\ &= \left\{\int_0^\infty v(x)\left(-p\int_0^\infty \left(\varphi^{-1}(t)\right)^{-\frac{1}{p}}\phi(x,t)dt\right)^q dx\right\}^{\frac{1}{q}} \\ &= -p\left\{\int_0^\infty v(x)\left(\int_0^\infty \left(\varphi^{-1}(t)\right)^{-\frac{1}{p}}\phi(x,t)dt\right)^q d(x)\right\}^{\frac{1}{q}}. \end{aligned}$$

Finally by taking $f(t) = \left(\varphi^{-1}(t)\right)^{-\frac{1}{p}}$ and $C_7 = (1-p)^{\frac{1}{p}} \left(\frac{p}{p-1}\right) C_6$ we obtain (3.13). $\square$

The proof of the second implication (3.14) $\longrightarrow$ (3.13) is similar to the first one.

Theorem 3.5. *Let* $p < 0$, $0 < q < 1$ *and* $C_9 > 0$. *If*

$$\left(\int_r^\infty w\right)^{\frac{1}{p}} \leq C_9 \left[\int_0^\infty \Phi(x,r)^q v\right]^{\frac{1}{q}}, \quad \forall r > 0, \tag{3.16}$$

then

$$\left(\int_0^\infty f^p w\right)^{\frac{1}{p}} \leq C_{10} \left[\int_0^\infty (S_\phi f)^q v\right]^{\frac{1}{q}} \tag{3.17}$$

holds for all $f \downarrow$.

If

$$\left(\int_0^r w\right)^{\frac{1}{p}} \leq C_{11} \left[\int_0^\infty \Phi_1(x,r)^q v\right]^{\frac{1}{q}}, \quad \forall r > 0, \tag{3.18}$$

then (3.17) holds for all $f \uparrow$.

Moreover $C_{10} = (1-p)^{\frac{1}{p}} \left(\frac{p}{p-1}\right) C_9$.

Proof. The proof of Theorem 3.5 is similar to that of Theorem 3.4. $\square$

Theorem 3.6. *Let* $p < 0$, $q \geq 1$ *and* $C_{12} > 0$. *If*

$$\left(\int_r^\infty w\right)^{\frac{1}{p}} \leq C_{12} \left[\int_0^\infty \Phi(x,r)^q v\right]^{\frac{1}{q}}, \quad \forall r > 0, \tag{3.19}$$

then

$$\left(\int_0^\infty f^p w\right)^{\frac{1}{p}} \leq C_{13} \left[\int_0^\infty (S_\phi f)^q v\right]^{\frac{1}{q}} \tag{3.20}$$

holds for all $f \downarrow$.

If

$$\left(\int_0^r w\right)^{\frac{1}{p}} \leq C_{14} \left[\int_0^\infty \Phi(x,r)^q v\right]^{\frac{1}{q}}, \quad \forall r > 0, \tag{3.21}$$

then (3.20) holds for all $f \uparrow$.

Moreover $C_{13} = \left(1 - \frac{p}{q}\right)^{\frac{1}{p} - \frac{1}{q}} \left(-\frac{p}{q}\right)^{\frac{1}{q}} C_{12}$.

Proof. (3.19) $\longrightarrow$ (3.20). Let $r=\varphi(y)\uparrow$, then we have

(3.22)

$$\int_0^\infty\left(\int_{\varphi(y)}^\infty w(x)dx\right)^{\frac{q}{p}}y^{\frac{q}{p}-1}dy\le C_{12}^q\int_0^\infty\left(\int_0^\infty\Phi(x,\varphi(y))^q v(x)dx\right)y^{\frac{q}{p}-1}dy.$$

By (2.3) we get

$$\int_0^\infty\left(\int_{\varphi(y)}^\infty w(x)dx\right)^{\frac{q}{p}}y^{\frac{q}{p}-1}dy\ge(\frac{q}{q-p})^{\frac{q}{p}-1}\left(\int_0^\infty\left(\int_{\varphi(y)}^\infty w(x)dx\right)dy\right)^{\frac{q}{p}}.$$

By (2.7) we deduce that

$$Lhs\ge(\frac{q}{q-p})^{\frac{q}{p}-1}\left(\int_0^\infty\varphi^{-1}(x)w(x)dx\right)^{\frac{q}{p}}.$$

$$Rhs=C_{12}\int_0^\infty I(x)v(x)dx,$$

where

$$I(x)\equiv\int_0^\infty\Phi(x,\varphi(y))^q y^{\frac{q}{p}-1}dy.$$

If we fix $x>0$ with $t=\varphi(y)$ in I(x) and integrating by parts, we have

$$I(x)=-p\int_0^\infty\varphi^{-1}(t)^{\frac{q}{p}}\Phi(x,t)^{q-1}\phi(x,t)dt.$$

Now we use Lemma 1.4 (iii).

Let

$$g(t)=\Phi(x,t)^{q-1}\phi(x,t),\qquad h(t)=\Phi(x,t),$$
$$f(t)=\varphi^{-1}(t)^{\frac{1}{p}},\downarrow\qquad C_2=\tfrac{1}{q}$$

and

$$\begin{aligned}\int_0^r g(t)dt&=\int_0^r\Phi(x,t)^{q-1}\phi(x,t)dt\\&=\frac{1}{q}\Phi(x,r)^q\\&\le C_2h(r)^q.\end{aligned}$$

Thus

$$\int_0^\infty f^q(t)g(t)dt\le C_2\left[\int_0^\infty f(t)h'(t)dt\right]^q$$

and

$$\int_0^\infty\varphi^{-1}(t)^{\frac{q}{p}}\Phi(x,t)^{q-1}\phi(x,t)dt\le\frac{1}{q}\left[\int_0^\infty\varphi^{-1}(t)^{\frac{1}{p}}\phi(x,t)dt\right]^q$$

$$I(x)\le-\frac{p}{q}\left[\int_0^\infty\varphi^{-1}(t)^{\frac{1}{p}}\phi(x,t)dt\right]^q.$$

Then

$$Rhs \leq C_{12}^q - \frac{p}{q}\int_0^\infty \left[\int_0^\infty \varphi^{-1}(t)^{\frac{1}{p}}\phi(x,t)dt\right]^q v(x)dx,$$

consequently

$$\left(\frac{q}{q-p}\right)^{\frac{q}{p}-1}\left(\int_0^\infty \varphi^{-1}(x)(x)dx\right)^{\frac{q}{p}} \leq C_{12}^q\left(-\frac{p}{q}\right)\left[\int_0^\infty \varphi^{-1}(t)^{\frac{1}{p}}\phi(x,t)dt\right]^q v(x)dx$$

and

$$\left(\int_0^\infty \varphi^{-1}(x)w(x)dx\right)^{\frac{q}{p}} \leq \left(\frac{q}{q-p}\right)^{\frac{1}{q}-\frac{1}{p}} C_{12}\left(-\frac{p}{q}\right)^{\frac{1}{q}} \times\left\{\int_0^\infty \left[\int_0^\infty \varphi^{-1}(t)^{\frac{1}{p}}\phi(x,t)dt\right]^q v(x)dx\right\}^{\frac{1}{q}}.$$

It suffices to take $f = (\varphi^{-1}(t))^{\frac{1}{p}}$, then

$$\left(\int_0^\infty f^p(x)w(x)dx\right)^{\frac{1}{p}} \leq C_{12}\left(1-\frac{p}{q}\right)^{\frac{1}{p}-\frac{1}{q}}\left(-\frac{p}{q}\right)^{\frac{1}{q}}\left(\int_0^\infty (S_\phi f)^q(x)v(x)dx\right)^{\frac{1}{q}}.$$

The proof of the second implication (3.21) $\longrightarrow$ (3.20) is similar to the first one. □

Remark 3.7. The case $0 < p < 1$, $q < 0$ is similar to the case $0 < p < 1$, $0 < q < 1$ (Theorem 1.2).

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