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The Operators' Theorems on Fuzzy Topological Spaces

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Abstract. Three types of fuzzy topologies defined on fuzzy normed linear spaces are considered in this paper. First, the relationship between fuzzy continuity of linear operators and fuzzy boundedness is investigated. The uniform boundedness theorem is then discussed, so too is the norm of a linear operator. Finally, the open mapping theorem is proved for each of the three defined fuzzy topologies, and the closed graph theorem is studied.

1. Introduction

Katrasas [11] was the first to introduce a fuzzy norm on a linear space in 1984. Feblin [8] then defined a fuzzy norm, which offered a fuzzy metric described by Kalva [9]. The fuzzy norm developed by Felbin was refined by Xiao and Zhu [19], who provided a linear operator's fuzzy norm between two fuzzy normed spaces. Furthermore, Cheng and Menderson [5] used the fuzzy metric concept proposed by Kramosil and Michalek [10] to create a fuzzy norm on a linear space. Bag and Samanta [2, 3] have introduced a definition of a fuzzy norm whose the associated metric is similar to the Kramosil and Michalek type metric [10], following Cheng and Menderson [5]. They also proposed the concept of boundedness of linear operators between fuzzy normed spaces, fuzzy bounded linear functionals, and fuzzy dual spaces. Also, various forms of (generalized) Hyers-Ulam-Rassias stability was investigated in this fuzzy normed linear space [1, 12]. Saadati and Vaezpour [13] were

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the first to prove the open mapping theorem and closed graph theorem on fuzzy normed linear spaces given by Bag and Samanta.

Das and Das [6] examined various features of fuzzy topological spaces described by Felbin by introducing a fuzzy topology on the fuzzy normed linear spaces given by Felbin. Fang [7] then demonstrated that this fuzzy topological space is not a fuzzy topological vector space. He also modified the fuzzy topology and looked at some of the spaces' characteristics. In addition, Xu and Fang [20] created and investigated another fuzzy topological space. On the fuzzy normed linear space established by Bag and Samanta, we proposed three fuzzy topologies, and some of the features of these fuzzy topological spaces are shown [15, 16]. Moreover, some fuzzy topological spaces and their properties are recently studied [4, 17, 18].

Preliminaries are the first section of this paper. The relationship between fuzzy continuity and fuzzy boundedness is investigated in the second section. The norm of a linear operator is then determined. Finally, three essential theorems of uniform boundedness theorem, open mapping theorem, and closed graph theorem are examined on linear operators defined on fuzzy normed linear spaces equipped with fuzzy topologies in the third section.

2. Preliminaries

We provide some basic preliminaries for this paper in this section. First, we'll go over Bag and Samanta's definition of fuzzy normed linear space.

Definition 2.1 ([2])**.** The pair (X, N) is called a fuzzy noremed linear space if X is a linear space over $\mathbb{R}$ (real number) and N is a fuzzy set on $X \times \mathbb{R}$ satisfying the following conditions, for each $x, u \in X$, $c \in \mathbb{R}$ and $s, t \in \mathbb{R}$:

$(N1)$ $N(x, t) = 0$, for all $t \leq 0$,
$(N2)$ $x = 0$ if and only if $N(x, t) = 1$, for all $t > 0$,
$(N3)$ If $c \neq 0$ then $N(cx, t) = N(x, t/|c|)$,
$(N4)$ $N(x + u, s + t) \geq \min\{N(x, s), N(u, t)\}$,
$(N5)$ $N(x, .)$ is a nondecreasing function and $\lim_{t\to\infty} N(x, t) = 1$.

Theorem 2.2 ([13])**.** *Let* (X, N_1) *and* (Y, N_2) *be two fuzzy normed linear spaces. Then the function* $N : (X \times Y) \times \mathbb{R} \to [0, 1]$ *defined by*

$$N((x, y), r) = \min\{N_1(x, r), N_2(y, r)\},$$

is a fuzzy norm on $X \times Y$.

We now declare the following additional conditions on fuzzy normed linear spaces, which we will use in this paper:

($N6$) If $N(x,t) > 0$, for all $t > 0$, then $x = 0$.
($N7$) For each $x \neq 0$, $N(x,.)$ is a continuous function and strictly increasing on the subset $\{t : 0 < N(x,t) < 1\}$ of $\mathbb{R}$.
($N8$) For each $x \neq 0$, there is $t_x > 0$ such that $N(x, t_x) = 0$.
($N9$) For each $\alpha, \beta \in (0,1)$, there is $m_{\alpha,\beta} > 0$ such that $N(x, tm_{\alpha,\beta}) \geq \beta$, where $N(x,t) \geq \alpha$.
($N10$) For each $\alpha \in (0,1)$, there is $m_\alpha > 0$ such that $N(x, tm_\alpha) \geq \alpha$, where $N(x,t) > 0$.

The descriptions and characteristics of fuzzy topologies defined on fuzzy normed linear spaces are given here.

Definition 2.3 ([6])**.** A fuzzy topology on a set X is a family τ of fuzzy subsets of X satisfying the followings:

(i) The fuzzy subsets 1 and 0 are in τ,
(ii) τ is closed under finite intersection of fuzzy subsets,
(iii) τ is closed under arbitrary union of fuzzy subsets.

The pair (X, τ) is called a fuzzy topological space.

Definition 2.4 ([7])**.** A fuzzy topology τ on a vector space X is said to be a fuzzy vector topology, if mappings

$$\begin{array}{c} f : X \times X \to X \\ (x,y) \to x+y \end{array} \quad \text{and} \quad \begin{array}{c} g : \mathbb{K} \times X \to X, \\ (t,x) \to tx \end{array}$$

are continuous. A vector space X with a fuzzy vector topology τ is called a fuzzy topological vector space (FTVS).

Theorem 2.5 ([3, 15, 16])**.** *Let (X,N) be a fuzzy normed linear space. Then*

(i) *A family $\tau_N^\dagger$ is a (classical) topology on X, which $\tau_N^\dagger$ contains all subsets G of X, if for all $x \in G$, there exist $t > 0$ and $\alpha \in (0,1)$ such that $B(x,\alpha,t) \subseteq G$, where*

$$B(x,\alpha,t) = \{y : N(x-y,t) > 1-\alpha\}.$$

(ii) *A family τ_N^* is a fuzzy topology on X, which τ_N^* contains all $\mu \in I^X$, if for all $x \in supp\mu$ and $\alpha \in (1-\mu(x), 1)$, there exists $\epsilon > 0$ such that $\mu_\alpha(x,\epsilon) \subseteq \mu$, where*

$$\mu_\alpha(x,\epsilon)(y) = \begin{cases} 1-\alpha, & N(x-y,\epsilon) > \alpha, \\ 0, & \textit{otherwise}. \end{cases}$$

(iii) *A family τ_N is a fuzzy topology on X, which τ_N contains all $\mu \in I^X$, if for all $x \in supp\mu$ and $0 < r < \mu(x)$, there exists $\epsilon > 0$ such that $x + B_\epsilon \cap \underline{r} \subseteq \mu$, where*

$$B_\epsilon(x) = \sup\{\alpha \in (0,1] : N(x,\epsilon) \geq \alpha\}.$$

Theorem 2.6 ([14, 15])**.** *Let (X, N) be a fuzzy normed linear space.*

(i) *If N satisfies (N7), then $\omega\left(\tau_N^\dagger\right) = \tau_N^*$ iff N satisfies condition (N9).*
(ii) $\tau_N \subseteq \omega\left(\tau_N^\dagger\right)$.
(iii) *If N satisfies (N7), then $\omega\left(\tau_N^\dagger\right) \subseteq \tau_N$ iff N satisfies condition (N10).*
(iv) *The topological space $\left(X, \tau_N^\dagger\right)$ is metrizable, where the metric is as in the classical case.*

Now, certain characteristics of linear operators defined on fuzzy normed linear spaces are notified.

Definition 2.7 ([3])**.** Let (X, N_1) and (Y, N_2) be fuzzy normed linear spaces and $T : (X, N_1) \to (Y, N_2)$ be a linear operator.

(i) Operator T is said to be weakly fuzzy bounded, if for any $\alpha \in (0, 1)$, there exists $M_\alpha > 0$ such that $N_2(T(x), M_\alpha t) \geq \alpha$, where $N_1(x, t) \geq \alpha$, for all $x \in X$ and $t > 0$.
(ii) Operator T is said to be fuzzy continuous at $x_0 \in X$, if for each $\epsilon > 0$ and $\alpha \in (0, 1)$, there exist $\delta = \delta_{(\alpha,\epsilon)} > 0$ and $\beta = \beta_{(\alpha,\epsilon)} \in (0, 1)$ such that $N_2(T(x) - T(x_0), \epsilon) > \alpha$, where $N_1(x - x_0, \delta) > \beta$, for all $x \in X$.
(iii) Operator T is said to be weakly fuzzy continuous at $x_0 \in X$, if for each $\epsilon > 0$ and $\alpha \in (0, 1)$, there exists $\delta = \delta_{(\alpha,\epsilon)} > 0$ such that $N_2(T(x) - T(x_0), \epsilon) > \alpha$, where $N_1(x - x_0, \delta) > \alpha$, for all $x \in X$.

Theorem 2.8 ([3])**.** *Let (X, N_1) and (Y, N_2) be fuzzy normed linear spaces and $T : (X, N_1) \to (Y, N_2)$ be a linear operator. Then T is weakly fuzzy continuous iff T is weakly fuzzy bounded.*

Theorem 2.9 ([14])**.** *Let (X, N_1) and (Y, N_2) be fuzzy normed linear spaces and $T : (X, N_1) \to (Y, N_2)$ be a mapping. Then the followings are equivalent:*

(i) *T is fuzzy continuous on X.*
(ii) *For all $x \in X$ and neighborhood V of $T(x)$, there exists a neighborhood U of x such that $T(U) \subseteq V$.*
(iii) *$T : \left(X, \tau_{N_1}^\dagger\right) \to \left(Y, \tau_{N_2}^\dagger\right)$ is continuous on X.*

Theorem 2.10 ([13], Open Mapping Theorem)**.** *Let (X, N_1) and (Y, N_2) be two fuzzy Banach spaces and $T : (X, N_1) \to (Y, N_2)$ be a fuzzy continuous operator. Then $T : \left(X, \tau_{N_1}^\dagger\right) \to \left(Y, \tau_{N_2}^\dagger\right)$ is an open mapping.*

3. Fuzzy Continuity and Fuzzy Boundedness

The relationship between fuzzy boundedness and fuzzy continuity on linear operator spaces is discussed in this section. First, it is demonstrated that under fuzzy topologies defined on fuzzy normed linear spaces, every weakly fuzzy bounded linear operator is fuzzy continuous.

Theorem 3.1. *Let (X, N_1) and (Y, N_2) be fuzzy normed linear spaces such that N_2 satisfying (N7) and $T : (X, N_1) \to (Y, N_2)$ be a weakly fuzzy bounded linear operator. Then $T : \left(X, \tau^*_{N_1}\right) \to \left(Y, \tau^*_{N_2}\right)$ is fuzzy continuous.*

Proof. Let T be a weakly fuzzy bounded linear operator. Then for any $\alpha \in (0,1)$, there exists $M_\alpha > 0$ such that $N_2(T(x), M_\alpha t) \geq \alpha$, where $N_1(x,t) \geq \alpha$, for all $x \in X$ and $t \in \mathbb{R}$.

Assume that $\mu \in \tau^*_{N_2}$, $T^{-1}(\mu)(x) > 0$ and $\alpha \in (1 - T^{-1}(\mu)(x), 1)$. Hence $\alpha \in (1 - \mu(T(x)), 1)$. Since $\mu \in \tau^*_{N_2}$, there exists $\epsilon > 0$ such that $\mu_\alpha(T(x), \epsilon) \subseteq \mu$.

Suppose that $\mu_\alpha(x, \epsilon/2M_\alpha)(y) = 1 - \alpha$. Thus $N_1(x - y, \epsilon/2M_\alpha) > \alpha$. Therefore $N_2(T(x) - T(y), \epsilon/2) \geq \alpha$. By (N7), we have

$$\begin{aligned} N_2\left(T(x) - T(y), \epsilon\right) &> N_2\left(T(x) - T(y), \epsilon/2\right) \\ &\geq \alpha. \end{aligned}$$

Therefore $\mu_\alpha(T(x), 2\epsilon)(T(y)) = 1 - \alpha \leq \mu(T(y)) = T^{-1}(\mu)(y)$. This implies that $\mu_\alpha(x, \epsilon/2M_\alpha) \subseteq T^{-1}(\mu)$. Hence $T^{-1}(\mu) \in \tau^*_{N_1}$. Thus T is fuzzy continuous. □

Theorem 3.2. *Let (X, N_1) and (Y, N_2) be fuzzy normed linear spaces and $T : (X, N_1) \to (Y, N_2)$ be a weakly fuzzy bounded linear operator. Then $T : (X, \tau_{N_1}) \to (Y, \tau_{N_2})$ is fuzzy continuous.*

Proof. Let $\mu \in \tau_{N_2}$, $x_0 \in suppT^{-1}(\mu)$ and $0 < r < T^{-1}(\mu)(x_0)$. Hence $T(x_0) \in supp\mu$ and $0 < r < \mu(T(x_0))$. Since $\mu \in \tau_{N_2}$, there exits $\epsilon > 0$ such that $T(x_0) + B_\epsilon \cap \underline{r} \subseteq \mu$.

Since T is a weakly fuzzy bounded linear operator, for any $\alpha \in (0,1)$, there exists $M_\alpha > 0$ such that $N_2(T(x), M_\alpha t) \geq \alpha$, where $N_1(x,t) \geq \alpha$, for all $x \in X$ and $t \in \mathbb{R}$.

Assume that $x \in X$, $\alpha \in (0,1]$ and $N(x, \epsilon/M_\alpha) \geq \alpha$. Therefore $N(T(x), \epsilon) \geq \alpha$. So $B_\epsilon(T(x)) \geq \alpha$. This implies that $B_{\epsilon/M_\alpha}(x) \leq B_\epsilon(T(x))$. Hence

$$\begin{aligned} (x_0 + B_{\epsilon/M_\alpha} \cap \underline{r})(x) &= (B_{\epsilon/M_\alpha} \cap \underline{r})(x - x_0) \\ &\leq (B_\epsilon \cap \underline{r})(T(x - x_0)) \\ &= (B_\epsilon \cap \underline{r})(T(x) - T(x_0)) \\ &= (T(x_0) + B_\epsilon \cap \underline{r})(T(x)) \end{aligned}$$

$$\le \mu(T(x))$$
$$= T^{-1}(\mu)(x).$$

Then $x_0 + B_{\epsilon/M_\alpha} \cap \underline{r} \subseteq T^{-1}(\mu)$. Thus $T^{-1}(\mu) \in \tau_{N_1}$. Hence $T : (X, \tau_{N_1}) \to (Y, \tau_{N_2})$ is fuzzy continuous. □

Theorem 3.3. *Let (X, N_1) and (Y, N_2) be fuzzy normed linear spaces and $T : (X, N_1) \to (Y, N_2)$ be a weakly fuzzy continuous linear operator. Then $T : \left(X, \tau^{\dagger}_{N_1}\right) \to \left(Y, \tau^{\dagger}_{N_2}\right)$ is continuous.*

Proof. Let $V \in \tau^{\dagger}_{N_2}$, $x_0 \in T^{-1}(V)$ and $T : (X, N_1) \to (Y, N_2)$ be a weakly fuzzy continuous linear operator. Since $V \in \tau^{\dagger}_{N_2}$ and $T(x_0) \in V$, there exist $\epsilon > 0$ and $\alpha \in (0, 1)$ such that $B(T(x_0), \alpha, \epsilon) \subseteq V$.

Since T is weakly fuzzy continuous at x_0, there existss $\delta > 0$ such that $N_2(T(x) - T(x_0), \epsilon) > 1 - \alpha$, where $N_1(x - x_0, \delta) > 1 - \alpha$, for all $x \in X$. Hence $T\left(B(x_0, \alpha, \delta)\right) \subseteq B\left(T(x_0), \alpha, \epsilon\right) \subseteq V$. Thus $B\left(x_0, \alpha, \delta\right) \subseteq T^{-1}(V)$. So $T^{-1}(V) \in \tau^{\dagger}_{N_1}$. This implies that $T : \left(X, \tau^{\dagger}_{N_1}\right) \to \left(Y, \tau^{\dagger}_{N_2}\right)$ is continuous. □

Theorem 3.4. *Let (X, N_1) and (Y, N_2) be fuzzy normed linear spaces and $T : (X, N_1) \to (Y, N_2)$ be a weakly fuzzy bounded linear operator. Then $T : \left(X, \omega\left(\tau^{\dagger}_{N_1}\right)\right) \to \left(Y, \omega\left(\tau^{\dagger}_{N_2}\right)\right)$ is fuzzy continuous.*

Proof. Let $\mu \in \omega\left(\tau^{\dagger}_{N_2}\right)$. Hence $\sigma_\alpha(\mu) \in \tau^{\dagger}_{N_2}$, for all $\alpha \in [0, 1)$. By Theorem 3.3, $T : \left(X, \tau^{\dagger}_{N_1}\right) \to \left(Y, \tau^{\dagger}_{N_2}\right)$ is continuous. Thus $\sigma_\alpha(T^{-1}(\mu)) = T^{-1}(\sigma_\alpha(\mu)) \in \tau^{\dagger}_{N_1}$, for all $\alpha \in [0, 1)$. Therefore $T^{-1}(\mu) \in \omega\left(\tau^{\dagger}_{N_1}\right)$. So $T : \left(X, \omega\left(\tau^{\dagger}_{N_1}\right)\right) \to \left(Y, \omega\left(\tau^{\dagger}_{N_2}\right)\right)$ is fuzzy continuous. □

Now, we decide to discuss the converse of Theorems 3.1-3.4. It is shown that any fuzzy continuous linear operator on fuzzy topological spaces is weakly fuzzy bounded.

Theorem 3.5. *Let (X, N_1) and (Y, N_2) be fuzzy normed linear spaces such that N_1 satisfying (N9) and $T : \left(X, \tau^{*}_{N_1}\right) \to \left(Y, \tau^{*}_{N_2}\right)$ be a fuzzy continuous linear operator. Then $T : (X, N_1) \to (Y, N_2)$ is weakly fuzzy bounded.*

Proof. Let T be a fuzzy continuous linear operator and $\alpha \in (0, 1)$. Assume that $x \in X$, $t \in \mathbb{R}$, $N_1(x, t) \geq \alpha$ and $\alpha < \beta < 1$. Since T is fuzzy continuous, there exists $\epsilon > 0$ such that $\mu_\beta(0, \epsilon) \subseteq T^{-1}(\mu_\alpha(0, 1))$. By (N9), there exists $m_{\alpha,\beta} > 0$ such that $N_1(x, m_{\alpha,\beta} t) > \beta$. Hence

$N_1(\epsilon x/m_{\alpha,\beta}t, \epsilon) > \beta$. Thus

$$\begin{aligned} 1-\beta &= \mu_\beta(0,\epsilon)(\epsilon x/m_{\alpha,\beta}t) \\ &\leq T^{-1}(\mu_\alpha(0,1))(\epsilon x/m_{\alpha,\beta}t) \\ &= \mu_\alpha(0,1)(T(\epsilon x/m_{\alpha,\beta}t)). \end{aligned}$$

Therefore $\mu_\alpha(0,1)(T(\epsilon x/m_{\alpha,\beta}t)) = 1-\alpha$. This implies that

$$N_2\left(T(\epsilon x/m_{\alpha,\beta}t), 1\right) > \alpha.$$

So $N_2\left((\epsilon/m_{\alpha,\beta}t)T(x), 1\right) > \alpha$. Then $N_2\left(T(x), m_{\alpha,\beta}t/\epsilon\right) > \alpha$.

Assume that $M_\alpha = m_{\alpha,\beta}/\epsilon$. Hence $N_2(T(x), M_\alpha t) \geq \alpha$. Thus T is weakly fuzzy bounded. ☐

Theorem 3.6. *Let (X, N_1) and (Y, N_2) be fuzzy normed linear spaces and $T : (X, \tau_{N_1}) \to (Y, \tau_{N_2})$ be a fuzzy continuous linear operator. Then $T : (X, N_1) \to (Y, N_2)$ is weakly fuzzy bounded.*

Proof. Let $T : (X, \tau_{N_1}) \to (Y, \tau_{N_2})$ be a fuzzy continuous linear operator and $\alpha \in (0,1)$. We have $B_1 \in \tau_{N_2}$. Hence $T^{-1}(B_1) \in \tau_{N_1}$. Since $T^{-1}(B_1) \in \tau_{N_1}$ and $T^{-1}(B_1)(0) = B_1(T(0)) = B_1(0) = 1 > \alpha > 0$, there exists $\epsilon_\alpha > 0$ such that $B_{\epsilon_\alpha} \cap \underline{\alpha} \subseteq T^{-1}(B_1)$.

Assume that $N(x,t) \geq \alpha$. Hence $N((\epsilon_\alpha/t)x, \epsilon_\alpha) = N(x,t) \geq \alpha$. So

$$\begin{aligned} B_1(T((\epsilon_\alpha/t)x)) &= T^{-1}(B_1)((\epsilon_\alpha/t)x) \\ &\geq B_{\epsilon_\alpha} \cap \underline{\alpha}((\epsilon_\alpha/t)x) \\ &= \alpha. \end{aligned}$$

Therefore $\sup\{\beta \in (0,1] : N(T((\epsilon_\alpha/t)x), 1) \geq \beta\} \geq \alpha$. This implies that

$$\begin{aligned} N(T(x), t/\epsilon_\alpha) &= N((\epsilon_\alpha/t)T(x), 1) \\ &= N(T((\epsilon_\alpha/t)x), 1) \\ &\geq \alpha. \end{aligned}$$

Let $M_\alpha = 1/\epsilon_\alpha > 0$. Thus $N(T(x), M_\alpha t) \geq \alpha$. Hence $T : (X, N_1) \to (Y, N_2)$ is weakly fuzzy bounded. ☐

Theorem 3.7. *Let (X, N_1) and (Y, N_2) be fuzzy normed linear spaces such that N_1 satisfying (N9) and $T : (X, N_1) \to (Y, N_2)$ be a fuzzy continuous linear operator. Then $T : (X, N_1) \to (Y, N_2)$ is weakly fuzzy continuous.*

Proof. Let $T : (X, N_1) \to (Y, N_2)$ be fuzzy continuous at $x_0 \in X$. Suppose that $\epsilon > 0$ and $\alpha \in (0,1)$. Hence there exist $\delta > 0$ and $\beta \in (0,1)$ such that $N_2(T(x) - T(x_0), \epsilon) > \alpha$, where $N_1(x - x_0, \delta) > \beta$, for all $x \in X$.

Suppose that $N_1(x-x_0, \delta/m_{\alpha,\beta}) \geq \alpha$. By (N9), we get $N_1(x-x_0, \delta) \geq \beta$. Therefore $N_2(T(x) - T(x_0), \epsilon) > \alpha$. Thus $T : (X, N_1) \to (Y, N_2)$ is weakly fuzzy continuous at $x_0 \in X$. □

Corollary 3.8. *Let (X, N_1) and (Y, N_2) be fuzzy normed linear spaces such that N_1 satisfying (N9). Moreover, let $T : (X, N_1) \to (Y, N_2)$ or $T : \left(X, \tau^{\dagger}_{N_1}\right) \to \left(Y, \tau^{\dagger}_{N_2}\right)$ be a fuzzy continuous linear operator. Then $T : (X, N_1) \to (Y, N_2)$ is weakly fuzzy bounded.*

Theorem 3.9. *Let (X, N_1) and (Y, N_2) be fuzzy normed linear spaces such that N_1 satisfying (N9) and $T : \left(X, \omega\left(\tau^{\dagger}_{N_1}\right)\right) \to \left(Y, \omega\left(\tau^{\dagger}_{N_2}\right)\right)$ be a fuzzy continuous linear operator. Then $T : (X, N_1) \to (Y, N_2)$ is weakly fuzzy bounded.*

Proof. Let $T : \left(X, \omega\left(\tau^{\dagger}_{N_1}\right)\right) \to \left(Y, \omega\left(\tau^{\dagger}_{N_2}\right)\right)$ be a fuzzy continuous linear operator and $O \in \tau^{\dagger}_{N_2}$. Hence $\chi_O \in \omega\left(\tau^{\dagger}_{N_2}\right)$.

Since $T : \left(X, \omega\left(\tau^{\dagger}_{N_1}\right)\right) \to \left(Y, \omega\left(\tau^{\dagger}_{N_2}\right)\right)$ is a fuzzy continuous linear operator, it follows that $T^{-1}(\chi_O) \in \omega\left(\tau^{\dagger}_{N_1}\right)$. Thus

$$\chi_{T^{-1}(O)} = T^{-1}(\chi_O) \in \omega\left(\tau^{\dagger}_{N_1}\right).$$

Therefore $T^{-1}(O) \in \tau^{\dagger}_{N_1}$. This implies that $T : \left(X, \tau^{\dagger}_{N_1}\right) \to \left(Y, \tau^{\dagger}_{N_2}\right)$ is a continuous linear operator. By Corollary 3.8, $T : (X, N_1) \to (Y, N_2)$ is weakly fuzzy bounded. □

4. Uniform Boundedness Theorem, Open Mapping Theorem and Closed Graph Theorem

The Uniform Boundedness Theorem, Open Mapping Theorem, and Closed Graph Theorem are discussed in this section. The Uniform Boundedness Theorem will be studied first. As a result, the linear operator's norm is required. As a result, the linear operator's norm will be defined next.

Notation 4.1. The set of all weakly fuzzy bounded linear operators $T : X \to Y$, is denoted by $B(X, Y)$.

Definition 4.2. Let (X, N_1) and (Y, N_2) be two fuzzy normed linear spaces, $T \in B(X, Y)$ and $\alpha \in (0, 1)$. Moreover, let $K_{\alpha,T} = \inf A$, which $A = \{M_\alpha : N(T(x), M_\alpha t) \geq \alpha, \text{ where } N(x, t) \geq \alpha, \forall x \in X, t \in \mathbb{R}\}$ and $S_\alpha = \sup\{K_{\beta,T} : \beta \leq \alpha\}$. Define

$$B_F(X, Y) = \{T \in B(X, Y) : S_\alpha < \infty, \text{ for all } \alpha \in (0, 1)\}.$$

Example 4.3. Let $(X, \|.\|)$ be a normed space and $T : X \to X$ be a bounded linear operator. Then is there exists $M > 0$ such that

$$\|Tx\| \leq M\|x\|, \text{ for all } x \in X.$$

We define fuzzy norm N on X as follows:

$$N(x,t) = \begin{cases} 1, & t > \|x\|, \ x \in X \\ (2t - \|x\|)/\|x\|, & \|x\|/2 < t \leq \|x\|, \ x \in X, \\ 0, & t \leq \|x\|/2, \ x \in X. \end{cases}$$

Let $\alpha \in (0,1)$, $x \in X$, $t \in \mathbb{R}$ and $N(x,t) \geq \alpha$.

Case1: If $\|Tx\| < Mt$, then $N(Tx, Mt) = 1 \geq \alpha$.

Case2: If $Mt \leq \|Tx\|$, then $Mt \leq \|Tx\| \leq M\|x\|$. Hence $t \leq \|x\|$. Since $N(x,t) \geq \alpha$, it follows that $\|x\|/2 < t \leq \|x\|$. So $\|Tx\|/2 \leq M\|x\|/2 < Mt$.

We have $N(x,t) = (2t - \|x\|)/\|x\| \geq \alpha$. Thus we obtain that $(1 + \alpha)\|x\| \leq 2t$. Therefore $(1+\alpha)\|Tx\| \leq (1+\alpha)M\|x\| \leq 2Mt$. Hence $N(Tx, Mt) = (2Mt - \|Tx\|)/\|Tx\| \geq \alpha$.

These imply that $K_{\alpha,T} \leq M$, for all $\alpha \in (0,1)$. Then $S_\alpha \leq M$, for all $\alpha \in (0,1)$. So $T \in B_F(X,X)$.

Theorem 4.4. *Let (X, N_1) and (Y, N_2) be two fuzzy normed linear spaces and $T \in B_F(X,Y)$. Then $N : B_F(X,Y) \times \mathbb{R} \to [0,1]$ defined by,*

$$N(T,s) = \begin{cases} \sup\{\alpha \in (0,1) : K_{\beta,T} \leq s, \text{ for all } \beta \leq \alpha\}, & s > 0, \\ 0, & s \leq 0, \end{cases}$$

is a fuzzy norm on $B_F(X,Y)$.

Proof. We show that N satisfies the conditions of Definition 2.1.

(N1) Let $T \in B_F(X,Y)$. It is clear that $N(T,s) = 0$, for all $s \leq 0$.

(N2) Let $T = 0$, $\alpha \in (0,1)$ and $M > 0$. Assume that $N(x,t) \geq \alpha$. Hence $N(T(x), Mt) = N(0, Mt) = 1 \geq \alpha$. Thus $K_{\alpha,T} = 0$, for all $\alpha \in (0,1)$. So $N(T,s) = 1$, for all $s > 0$.

Conversely, let $N(T,s) = 1$, for all $s > 0$. Therefore $K_{\alpha,T} \leq s$, for all $s > 0$ and $\alpha \in (0,1)$. Hence $K_{\alpha,T} = 0$, for all $\alpha \in (0,1)$. Thus $N(T(x), t) = 1$, for all $t > 0$ and $x \in X$. So $T(x) = 0$, for all $x \in X$. This implies that $T = 0$.

(N3) Let $T \in B_F(X,Y)$ and $c \neq 0$ and $s > 0$. Since $N(cT(x), M_\alpha t) = N(T(x), M_\alpha t/|c|)$, it follows that $K_{\alpha,cT} = |c|K_{\alpha,T}$. Hence

$$\begin{aligned} N(cT,s) &= \sup\{\alpha \in (0,1) : K_{\alpha,cT} \leq s\} \\ &= \sup\{\alpha \in (0,1) : |c|K_{\alpha,T} \leq s\} \\ &= \sup\{\alpha \in (0,1) : K_{\alpha,T} \leq s/|c|\} \\ &= N(T, s/|c|). \end{aligned}$$

(N4) Let $T_1, T_2 \in B_F(X, Y)$, $s_1, s_2 > 0$ and $\alpha \in (0, 1)$. Assume that $N(T_1(x), M_{1\alpha}t) \geq \alpha$ and $N(T_2(x), M_{2\alpha}t) \geq \alpha$, where $N(x, t) \geq \alpha$, for all $x \in X$ and $t \in \mathbb{R}$. Hence

$$\begin{aligned} N\left((T_1 + T_2)(x), (M_{1\alpha} + M_{2\alpha})t\right) \\ &\geq \min\{N(T_1(x), M_{1\alpha}t), N(T_2(x), M_{2\alpha}t)\} \\ &\geq \alpha, \end{aligned}$$

where $N(x, t) \geq \alpha$. Hence $K_{\alpha,T_1+T_2} \leq K_{\alpha,T_1} + K_{\alpha,T_2}$.

Suppose that $N(T_1, s_1) \leq N(T_2, s_2)$ and $\beta < N(T_1, s_1)$. Then there exist $\beta_1, \beta_2 > \beta$ such that $K_{\gamma,T_1} \leq s_1$, for all $\gamma \leq \beta_1$ and $K_{\gamma,T_2} \leq s_2$, for all $\gamma \leq \beta_2$. Therefore $K_{\gamma,T_1} \leq s_1$ and $K_{\gamma,T_2} \leq s_2$, for all $\gamma \leq \beta$. So $K_{\gamma,T_1+T_2} \leq s_1 + s_2$, for all $\gamma \leq \beta$. Thus $N(T_1 + T_2, s_1 + s_2) \geq \beta$. As $\beta \to N(T_1, s_1)$, we obtain that

$$\begin{aligned} N(T_1 + T_2, s_1 + s_2) &\geq N(T_1, s_1) \\ &= \min\{N(T_1, s_1), N(T_2, s_2)\}. \end{aligned}$$

(N5) Let $T \in B_F(X, Y)$. It is clear that $N(T, .)$ is a nondecreasing function of $\mathbb{R}$. Since $\sup\{K_{\beta,T} : \beta \leq \alpha\} < \infty$, for all $\alpha \in (0, 1)$, it follows that $\lim_{s\to\infty} N(T, s) = 1$.

□

Example 4.5. Let $X = l^2$ and N_1, N_2 be fuzzy norms defined by

$$N_1(x, t) = \begin{cases} \sup\{\alpha \in (0, 1) : \|x\| \leq t/\alpha\}, & (x, t) \neq (0, 0), \\ 0, & (x, t) = (0, 0), \end{cases}$$

and

$$N_2(x, t) = \begin{cases} t/(t + \|x\|), & t > 0, \\ 0, & t \leq 0. \end{cases}$$

We introduce $T : (l^2, N_1) \to (l^2, N_2)$ defined by $T(a_1, a_2, \dots) = (a_2, a_3, \dots)$. First, we show that $K_{\alpha,T} = (1 - \alpha)^{-1}$, for all $\alpha \in (0, 1)$.

Let $\alpha \in (0, 1)$. Assume that $x \in X$, $t \in \mathbb{R}$ and $N_1(x, t) \geq \alpha$. Then $\|x\| \leq t/\alpha$. Hence $\|Tx\| \leq \|x\| \leq t/\alpha$. Thus $\|Tx\| \leq ((1 - \alpha)/\alpha)M_\alpha t$, where $M_\alpha = (1 - \alpha)^{-1}$. So $N_2(Tx, M_\alpha t) = M_\alpha t/(M_\alpha t + \|Tx\|) \geq \alpha$. Therefore $K_{\alpha,T} \leq M_\alpha = (1 - \alpha)^{-1}$.

Suppose that $M'_\alpha < (1 - \alpha)^{-1}$, $t > 0$ and $\|Tx\| = \|x\| = t/\alpha$. Then $N_1(x, t) \geq \alpha$. But $\|Tx\| = t/\alpha > ((1-\alpha)/\alpha)M'_\alpha t$. So $N_2(Tx, M'_\alpha t) < \alpha$. Hence $K_{\alpha,T} \geq (1 - \alpha)^{-1}$.

These imply that $K_{\alpha,T} = (1 - \alpha)^{-1}$. Thus

$$\begin{aligned} S_\alpha &= \sup\{K_{\beta,T} : \beta \leq \alpha\} \\ &= \sup\left\{(1 - \beta)^{-1} : \beta \leq \alpha\right\} \end{aligned}$$

$$= (1-\alpha)^{-1}.$$

Therefore

$$N(T,s) = \begin{cases} \sup\{\alpha \in (0,1) : K_{\beta,T} \leq s, \text{ for all } \beta \leq \alpha\}, & s > 0, \\ 0, & s \leq 0. \end{cases}$$
$$= \begin{cases} \sup\{\alpha \in (0,1) : (1-\beta)^{-1} \leq s, \text{ for all } \beta \leq \alpha\}, & s > 0, \\ 0, & s \leq 0. \end{cases}$$
$$= \begin{cases} \sup\{\alpha \in (0,1) : (1-\alpha)^{-1} \leq s\}, & s > 0, \\ 0, & s \leq 0. \end{cases}$$
$$= \begin{cases} 1 - 1/s, & s > 1, \\ 0, & s \leq 1. \end{cases}$$

Theorem 4.6. *Let (X, N_1) and (Y, N_2) be two fuzzy normed linear spaces such that N_1 satisfying (N10) and $T \in B(X,Y)$. Then $T \in B_F(X,Y)$.*

Proof. Let $T \in B(X,Y)$, $\alpha, \beta \in (0,1)$ and $\beta \leq \alpha$. Assume that $N(T(x), M_\alpha t) \geq \alpha$, where $N(x,t) \geq \alpha$, for all $x \in X$ and $t \in \mathbb{R}$.

Suppose that $N(x,t) \geq \beta$. Then $N(x,t) > 0$. By (N10), there exists $m_\alpha > 0$ such that $N(x, m_\alpha t) \geq \alpha$. Hence $N(T(x), M_\alpha m_\alpha t) \geq \alpha \geq \beta$. Thus $N(T(x), M_\alpha m_\alpha t) \geq \beta$, where $N(x,t) \geq \beta$, for all $x \in X$, and $t \in \mathbb{R}$. Therefore $K_{\beta,T} \leq m_\alpha K_{\alpha,T}$. This implies that

$$\sup\{K_{\beta,T} : \beta \leq \alpha\} \leq m_\alpha K_{\alpha,T}.$$

So $T \in B_F(X,Y)$. □

Theorem 4.7. *Let (X, N) be a fuzzy normed linear space and consider the sequence $\{x_n\} \subseteq X$. Then $\{x_n\}$ converges to x in (X, N) if and only if $\{x_n\}$ converges to x in $\left(X, \tau_N^\dagger\right)$.*

Theorem 4.8 (Baire Category Theorem)**.** *Let (X, N) be a complete fuzzy normed linear space. Then $\left(X, \tau_N^\dagger\right)$ is a Baire space.*

Proof. The proof is clear by Theorems 2.6 and 4.7. □

Theorem 4.9 (Uniform Boundedness Theorem)**.** *Let (X, N_1) and (Y, N_2) be two fuzzy normed linear spaces such that N_1 satisfying (N7), (N10) and $\{T_n\} \subseteq B(X,Y)$. Moreover, let for every $x \in X$ and $\alpha \in (0,1)$, there exists $M_{\alpha,x} > 0$ such that $N_2(T_n(x), M_{\alpha,x}) \geq \alpha$, for all $n \in \mathbb{N}$. Then for every $\alpha \in (0,1)$, there exists $M_\alpha > 0$ such that $N(T_n, M_\alpha) \geq \alpha$, for all $n \in \mathbb{N}$.*

Proof. Let $\alpha \in (0,1)$, $0 < \beta \leq \alpha$ and

$$A_k = \{x \in X : N_2(T_n(x), k) \geq \alpha, \text{ for all } n \in \mathbb{N}\}.$$

We have $X = \bigcup_{k=1}^{\infty} A_k$. Assume that $k \in \mathbb{N}$ and $x \in \overline{A_k}$. So there exists a sequence $\{x_m\}$ in A_k converging to x. Thus $N_2(T_n(x_m), k) \geq \alpha$, for all $m, n \in \mathbb{N}$. Since $T_n \in B(X, Y)$, it follows that $\lim_{m\to\infty} T_n(x_m) = T_n(x)$, for all $n \in \mathbb{N}$.

Let $\varepsilon > 0$ and $n \in \mathbb{N}$. We have $\lim_{m\to\infty} N_2(T_n(x_m - x), \varepsilon) = 1$. Therefore there exists $N \in \mathbb{N}$ such that $N_2(T_n(x_m - x), \varepsilon) \geq \alpha$, for all $m \geq N$. Now, we get

$$\begin{aligned} N_2(T_n(x), k+\varepsilon) &\geq \min\{N_2(T_n(x_N - x), \varepsilon), N_2(T_n(x_N), k)\} \\ &\geq \alpha. \end{aligned}$$

So $N_2(T_n(x), k+\varepsilon) \geq \alpha$, for all $\varepsilon > 0$ and all $n \in \mathbb{N}$. By (N7), we obtain that $N_2(T_n(x), k) \geq \alpha$, for all $n \in \mathbb{N}$. Thus $x \in A_k$. Hence A_k is closed. Baire Category Theorem implies that some A_k contains an open set. Thus there exist $k_0 \in \mathbb{N}$, $x_0 \in X$, $\gamma \in (0,1)$ and $\epsilon > 0$ such that $B(x_0, \gamma, \epsilon) \subseteq A_{k_0}$.

Let $N_1(x,t) \geq \beta$. Then $N_1(x,t) > 0$. By (N10), there exists $M_{1-\gamma} > 0$ such that $N_1(x, M_{1-\gamma}t) \geq 1-\gamma$. By (N7), we obtain that $N_1(x, 2M_{1-\gamma}t) > 1-\gamma$.

Suppose that $r = \epsilon/(2M_{1-\gamma}t)$ and $z = x_0 + rx$. We have

$$\begin{aligned} N_1(z - x_0, \epsilon) &= N_1(rx, \epsilon) \\ &= N_1(x, \epsilon/r) \\ &= N_1(x, 2M_{1-\gamma}t) \\ &> 1-\gamma. \end{aligned}$$

So $z \in B(x_0, \gamma, \epsilon)$. Therefore $z \in A_{k_0}$. Hence $N_2(T_n(z), k_0) \geq \alpha \geq \beta$, for all $n \in \mathbb{N}$. On the other hand, we have $x_0 \in B(x_0, \gamma, \epsilon)$. Thus $x_0 \in A_{k_0}$. Hence

$$N_2(T_n(x_0), k_0) \geq \alpha \geq \beta, \quad \text{for all } n \in \mathbb{N}.$$

These imply that

$$\begin{aligned} N_2(T_n(x), 2k_0/r) &= N_2(T_n(z - x_0)/r, 2k_0/r) \\ &= N_2(T_n(z) - T_n(x_0), 2k_0) \\ &\geq \min\{N_2(T_n(z), k_0), N_2(T_n(z), k_0)\} \\ &\geq \alpha \\ &\geq \beta, \quad \text{for all } n \in \mathbb{N}. \end{aligned}$$

Therefore $N_2(T_n(x), (4M_{1-\gamma}k_0/\epsilon)t) \geq \beta$, for all $n \in \mathbb{N}$. Hence $4M_{1-\gamma}k_0/\epsilon \in \{M_\beta : N(x,t) \geq \beta \text{ implies that } N_2(T_n(x), M_\beta t) \geq \beta,\ \forall x \in X,\ t \in \mathbb{R}\}$, for all $n \in \mathbb{N}$. Thus $K_{\beta,T_n} \leq 4M_{1-\gamma}k_0/\epsilon$, for all $n \in \mathbb{N}$. This implies that

$$N(T_n, 4M_{1-\gamma}k_0/\epsilon) \geq \alpha, \quad \text{for all } n \in \mathbb{N}.$$

□

Example 4.10. Let (l^2, N) be a normed space defined by

$$N(x,t) = \begin{cases} 1, & t > \|x\|,\ x \in l^2, \\ (2t - \|x\|)/\|x\|, & \|x\|/2 < t \leq \|x\|,\ x \in l^2, \\ 0, & t \leq \|x\|/2,\ x \in l^2. \end{cases}$$

Now, we introduce $T_n = S^n$, where $S : l^2 \to l^2$ is defined by $S(a_1, a_2, \dots) = (a_2, a_3, \dots)$.

Let $x = (a_1, a_2, \dots) \in l^2$ and $\alpha \in (0,1)$. Since $\sum_{n=1}^{\infty} |a_n|^2 < \infty$, there exists $N > 0$ such that $\sum_{n=k}^{\infty} |a_n|^2 < 1$, for all $k \geq N$. Hence $N(T(x), 1) = 1 \geq \alpha$, for all $k \geq N$. Assume that

$$M_{\alpha,x} = \max\{\|T_n(x)\| + 2 : n < N\}.$$

This implies that $N(T_n(x), M_{\alpha,x}) = 1 \geq \alpha$, for all $n \in \mathbb{N}$. By Uniform Boundedness Theorem, there exists $M_\alpha > 0$ such that $N(T_n, M_\alpha) \geq \alpha$, for all $n \in \mathbb{N}$.

Corollary 4.11. *Let (X, N_1) and (Y, N_2) be two fuzzy normed linear spaces such that N_1 satisfying (N7), (N10) and $T \in B(X,Y)$. Moreover, let for every $x \in X$ and $\alpha \in (0,1)$, there exists $M_{\alpha,x} > 0$ such that $N_2(T_n(x), M_{\alpha,x}) \geq \alpha$, for all $n \in \mathbb{N}$. Then there exists $M > 0$ such that $N(T_n, M) \geq \alpha$, for all $n \in \mathbb{N}$.*

The Open Mapping Theorem is already being considered to a variety of fuzzy topologies.

Theorem 4.12 (Open Mapping Theorem)**.** *Let (X, N_1) and (Y, N_2) be two fuzzy Banach spaces such that N_1 (N9). Moreover, let $T : (X, N_1) \to (Y, N_2)$ be a weakly fuzzy bounded operator and surjective. Then $T : \left(X, , \tau^{\dagger}_{N_1}\right) \to \left(Y, , \tau^{\dagger}_{N_2}\right)$ is an open mapping.*

Proof. The proof is clear by Theorem 3.7 and 4.12. □

Corollary 4.13. *Let (X, N_1) and (Y, N_2) be two fuzzy Banach spaces such that N_1 and N_2 satisfying (N7), (N9). Moreover, let the operator $T : (X, N_1) \to (Y, N_2)$ be weakly fuzzy bounded and bijective. Then operator $T^{-1} : (Y, N_2) \to (X, N_1)$ is weakly fuzzy bounded.*

Theorem 4.14 (Open Mapping Theorem)**.** *Let (X, N_1) and (Y, N_2) be two fuzzy Banach spaces such that N_1 and N_2 satisfying (N7), (N9). Moreover, let $T : (X, N_1) \to (Y, N_2)$ be a weakly fuzzy bounded operator which is surjective. Then $T : \left(X, \omega\left(\tau^{\dagger}_{N_1}\right)\right) \to \left(Y, \omega\left(\tau^{\dagger}_{N_2}\right)\right)$ is an open mapping.*

Proof. Let $\mu \in \omega(\tau^{\dagger}_{N_1})$. This implies that $\sigma_\alpha(\mu) \in \tau^{\dagger}_{N_1}$, for all $\alpha \in [0, 1)$. By Theorem 4.12, $T : \left(X, \tau^{\dagger}_{N_1}\right) \to \left(Y, \tau^{\dagger}_{N_2}\right)$ is an open mapping. Thus $T(\sigma_\alpha(\mu)) \in \tau^{\dagger}_{N_2}$, for all $\alpha \in [0, 1)$. Now we have

$$\begin{aligned}\sigma_\alpha(T(\mu)) &= \{y : T(\mu)(y) > \alpha\}\\ &= \{y : \sup\{\mu(x) : T(x) = y\} > \alpha\}\\ &= \{y : \exists\ T(x) = y \text{ such that } \mu(x) > \alpha\}\\ &= \{T(x) : \mu(x) > \alpha\}\\ &= \{T(x) : x \in \sigma_\alpha(\mu)\}\\ &= T(\sigma_\alpha(\mu)), \quad \text{for all } \alpha \in [0, 1).\end{aligned}$$

Therefore $\sigma_\alpha(T(\mu)) \in \tau^{\dagger}_{N_2}$, for all $\alpha \in [0, 1)$. Hence $T(\mu) \in \omega\left(\tau^{\dagger}_{N_2}\right)$. □

Theorem 4.15 (Open Mapping Theorem)**.** *Let (X, N_1) and (Y, N_2) be two fuzzy Banach spaces such that N_1 and N_2 satisfying (N7), (N9). Moreover, let $T : (X, N_1) \to (Y, N_2)$ be a weakly fuzzy bounded operator which is surjective. Then $T : (X, \tau^*_{N_1}) \to (Y, \tau^*_{N_2})$ is an open mapping.*

Proof. Let $\mu \in \tau^*_{N_1}$. By Theorem 2.6, we get $\mu \in \omega(\tau^{\dagger}_{N_1})$. By Theorem 4.14, we obtain that $T(\mu) \in \omega\left(\tau^{\dagger}_{N_2}\right)$. By Theorem 2.6, we have $T(\mu) \in \tau^*_{N_2}$. □

Theorem 4.16 (Open Mapping Theorem)**.** *Let (X, N_1) and (Y, N_2) be two fuzzy Banach spaces such that N_1 satisfying (N7), (N9) and N_2 satisfying (N7), (N10). Moreover, let $T : (X, N_1) \to (Y, N_2)$ be a weakly fuzzy bounded operator which is surjective. Then $T : (X, \tau_{N_1}) \to (Y, \tau_{N_2})$ is an open mapping.*

Proof. Let $\mu \in \tau_{N_1}$. By Theorem 2.6, we get $\mu \in \omega(\tau^{\dagger}_{N_1})$. By Theorem 4.14, we obtain that $T(\mu) \in \omega\left(\tau^{\dagger}_{N_2}\right)$. By Theorem 2.6, we have $T(\mu) \in \tau_{N_2}$. □

We'll finish up with a look at the closed graph theorem. As a consequence, we must present a Saadati and Vaezpour [13] fuzzy norm on the linear space $X \times Y$.

Definition 4.17. Let (X, N) be a fuzzy normed linear space and $E \subseteq X$. The set of all $x \in X$ such that there exists a sequence $\{x_n\}$ of elements E with $\lim_{n\to\infty} x_n = x$ is defined as the fuzzy closure of E and denoted by $\overline{E}$. If $E = \overline{E}$, the set E is said to be fuzzy closed.

Definition 4.18 (Closed linear operator)**.** Let (X, N_1) and (Y, N_2) be fuzzy normed spaces, $D(T) \subseteq X$ and $T : D(T) \to Y$ be a linear operator. If T's graph $G(T) = \{(x, y) : \; x \in D(T), y = Tx\}$ is closed in the fuzzy normed linear space $X \times Y$, where the fuzzy norm on $X \times Y$ is determined in Theorem 4.19, then T is called a closed linear operator.

Theorem 4.19 (Closed Graph Theorem)**.** *Let (X, N_1) and (Y, N_2) be fuzzy Banach spaces, $D(T) \subseteq X$ be closed and $T : D(T) \to Y$ be a closed linear operator. Then T is weakly fuzzy bounded.*

Proof. Let $\{(x_n, y_n)\}$ be Cauchy in $X \times Y$. Assume that $t, \epsilon > 0$. Hence there exists $N > 0$ such that $N((x_n, y_n) - (x_m, y_m), t) \geq 1 - \epsilon$, for all $m, n \geq N$. So

$$N((x_n - x_m, y_n - y_m), t) \geq 1 - \epsilon, \quad \text{for all } m, n \geq N.$$

Thus

$$\min\{N_1(x_n - x_m, t), N_2(y_n - y_m, t)\} \geq 1 - \epsilon, \quad \text{for all } m, n \geq N.$$

Then $N_1(x_n - x_m, t) \geq 1 - \epsilon$ and $N_2(y_n - y_m, t) \geq 1 - \epsilon$, for all $m, n \geq N$. Hence $\{x_n\}$ and $\{y_n\}$ are Cauchy in X and Y, respectively. Since X and Y are complete, there exist $x \in X$ and $y \in Y$ such that $x_n \to x$ and $y_n \to y$. Then there exists $N' > 0$ such that $N_1(x_n - x, t) \geq 1 - \epsilon$ and $N_2(y_n - y, t) \geq 1 - \epsilon$, for all $n \geq N'$. Hence

$$\begin{aligned} N((x_n, y_n) - (x, y), t) &= N((x_n - x, y_n - y), t) \\ &= \min\{N_1(x_n - x, t), N_2(y_n - y, t)\} \\ &\geq 1 - \epsilon, \quad \text{for all } n \geq N'. \end{aligned}$$

Therefore $(x_n, y_n) \to (x, y)$. Thus $X \times Y$ is complete.

By assumption, $G(T)$ is closed in $X \times Y$ and $D(T)$ is closed in X. Hence $G(T)$ and $D(T)$ are complete. We define the linear operator $P : G(T) \to D(T)$ by $P(x, Tx) = x$. Now, we show that P is bounded. We Have

$$\begin{aligned} N_1(P(x, Tx), t) &= N_1(x, t) \\ &\geq \min\{N_1(x, t), N_2(Tx, t)\} \\ &= N((x, Tx), t), \end{aligned}$$

for all $x \in X$ and all $t \in \mathbb{R}$. So P is weakly fuzzy bounded. It is clear that P is bijective. By open mapping theorem, $P^{-1} : D(T) \to G(T)$ is a bounded linear operator. Hence for any $\alpha \in (0, 1)$, there exists $M_\alpha > 0$

such that if $N_1(x, t/M_\alpha) \geq \alpha$, then $N(P^{-1}(x), t) \geq \alpha$, for all $x \in X$ and $t \in \mathbb{R}$. Suppose that $x \in X$, $t > 0$ and $N_1(x, t/M_\alpha) \geq \alpha$. Hence

$$\begin{aligned} N_2(Tx, t) &\geq \min\{N_1(x, t), N_2(Tx, t)\} \\ &= N((x, Tx), t) \\ &= N(P^{-1}(x), t) \\ &\geq \alpha. \end{aligned}$$

This implies that T is weakly fuzzy bounded. □

Theorem 4.20 (Closed linear operator)**.** *Let (X, N_1) and (Y, N_2) be fuzzy normed linear spaces, $D(T) \subseteq X$ and $T : D(T) \to Y$ be a linear operator. Then the followings are equivalent:*

(i) *Linear operator T is closed.*

(ii) *If sequence $\{x_n\} \subseteq D(T)$, $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} T(x_n) = y$, then $x \in D(T)$ and $Tx = y$.*

Proof. $(i) \Rightarrow (ii)$ Let T be a closed linear operator. Suppose that $\{x_n\} \subseteq D(T)$, $\lim_{n\to\infty} x_n = x$, $\lim_{n\to\infty} T(x_n) = y$ and $t > 0$. Then there exists $N' > 0$ such that $N_1(x_n - x, t) \geq 1 - \epsilon$ and $N_2(T(x_n) - y, t) \geq 1 - \epsilon$, for all $n \geq N'$. Hence

$$\begin{aligned} N((x_n, T(x_n)) - (x, y), t) &= N((x_n - x, T(x_n) - y), t) \\ &= \min\{N_1(x_n - x, t), N_2(T(x_n) - y, t)\} \\ &\geq 1 - \epsilon, \quad \text{for all } n \geq N'. \end{aligned}$$

Therefore $(x_n, T(x_n)) \to (x, y)$. Since T is closed, $(x, y) \in G(T)$. Thus $x \in D(T)$ and $T(x) = y$.

$(ii) \Rightarrow (i)$ Let $(x_n, T(x_n)) \to (x, y)$ and $t > 0$. Then there exists $N' > 0$ such that

$$\min\{N_1(x_n - x, t), N_2(T(x_n) - y, t)\} \geq 1 - \epsilon, \quad \text{for all } n \geq N'.$$

Hence $N_1(x_n - x, t) \geq 1 - \epsilon$ and $N_2(T(x_n) - y, t)\} \geq 1 - \epsilon$, for all $n \geq N'$. Therefore $x_n \to x$ and $T(x_n) \to y$. Thus $x \in D(T)$ and $T(x) = y$. This implies that T is a closed linear operator. □

Theorem 4.21 (Closed operator)**.** *Let (X, N_1) and (Y, N_2) be fuzzy normed linear spaces, $D(T) \subseteq X$ and $T : D(T) \to Y$ be a weakly fuzzy bounded linear operator. Then*

(i) *If $D(T)$ is a closed subset of X, then T is closed.*

(ii) *If T is closed and Y is complete, then $D(T)$ is a closed subset of X.*

Proof. Let $T : D(T) \to Y$ be a weakly fuzzy bounded linear operator.

(i) Let $\{x_n\} \subseteq D(T)$, $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} T(x_n) = y$. Since $D(T)$ is closed, it follows that $x \in D(T)$. Also, since T is a weakly fuzzy bounded, $\lim_{n\to\infty} T(x_n) = T(x)$. By Theorem 4.20, T is closed.

(ii) Let $\{x_n\} \subseteq D(T)$ and $\lim_{n\to\infty} x_n = x$. Hence $\{x_n\}$ is Cauchy. Since T is weakly fuzzy bounded, $\{T(x_n)\}$ is Cauchy. Also, since Y is complete, there exists $y \in Y$ such that $\lim_{n\to\infty} T(x_n) = y$. $x \in D(T)$ and $T(x) = y$ because T is closed. Hence $D(T)$ is closed.

□

According to theorem 4.21, any weakly fuzzy bounded linear operator defined on a closed subset of fuzzy normed linear spaces is closed. In specifically, any weakly fuzzy bounded linear operator defined on fuzzy normed linear spaces is closed. Also, closedness does not guarantee the boundedness of a linear operator, as shown in the following examples. Boundedness, on the other hand, does not imply closedness.

Example 4.22. Let (X, N) be a fuzzy normed linear space and $I : D(I) \to X$ be the identity operator, where $D(I)$ is a proper dense subspace of X. We know that I is a bounded linear operator. Suppose that $x \in X - D(I)$ and $\{x_n\} \subseteq D(I)$ is a sequence converges to x. By Theorem 4.20, operator I is not closed.

Example 4.23. Let $X = C[0,1]$. We define the fuzzy norm N on X as

$$N(x,t) = \begin{cases} t/(t + \|x\|), & t > 0, \\ 0, & t \leq 0. \end{cases}$$

Suppose that $D(T) = \{x \in X : \ x \text{ has a continuous derivative}\}$. Now, we introduce operator $T : D(T) \to X$ by $T(x) = x'$, where the prime denotes differentiation.

First, we show that T is not weakly fuzzy bounded.

Let $\alpha = 1/3$, $t = 1/2$, $n \in \mathbb{N}$ and $x_n(z) = z^n$, for all $z \in [0,1]$. We have $\|x_n\| = 1$. Then and $N(x_n, t) = 1/3 \geq \alpha$. On the other hand, we know that $x_n'(z) = nz^{n-1}$, for all $z \in [0,1]$. Hence $\|T(x_n)\| = n$. Thus $N(T(x_n), s) = s/(s+n)$, for all $s > 0$. Therefore $\lim_{n\to\infty} N(T(x_n), s) = 0$, for all $s > 0$. This implies that T is not bounded.

In the sequel, we show that T is closed. Let $\{x_n\} \subseteq D(T)$, $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} T(x_n) = \lim_{n\to\infty} x_n' = y$. Then $\lim_{n\to\infty} N(x_n - x, t) = 1$ and $\lim_{n\to\infty} N(x_n' - y, t) = 1$, for all $t > 0$. So $\lim_{n\to\infty} t/(t + \|x_n - x\|) = 1$ and $\lim_{n\to\infty} t/(t + \|x_n' - y\|) = 1$, for all $t > 0$. Thus $\lim_{n\to\infty} \|x_n - x\| = 0$ and

$\lim_{n\to\infty} \|x'_n - y\| = 0$. Therefore

$$\begin{aligned}\int_0^z y(w)dw &= \int_0^z (\lim_{n\to\infty} x'_n(w))dw \\ &= \lim_{n\to\infty} \int_0^z x'_n(w)dw \\ &= x(z) - x(0).\end{aligned}$$

Then $x(z) = x(0) \int_0^z y(w)dw$. This shows that $x \in D(T)$ and $x' = y$. By Theorem 4.20, T is closed.

5. Conclusion

Linear operators'continuity and bondedness are known to be equivalent in classical functional analysis. As a result, the relation between the fuzzy continuity and fuzzy boundedness of linear operators is investigated. Moreover, the uniform boundedness theorem specifies the requirements that must be met in order to a family of bounded linear operators be bounded. It has a variety of (primarily and more complex) applications in analysis, such as with Fourier series, weak convergence, sequence summability, numerical integration, etc. According to the open mapping theorem, a bounded linear operator is an open mapping. So, the linear operator T^{-1} is continuous if the bounded linear operator T is bijective ("bounded inverse theorem"). The closed graph theorem is one of the most notable applications of the open mapping theorem. The closed graph theorem states that a closed linear operator is bounded under certain circumstances. Physical and other applications rely heavily on closed linear operators. Therefore, these theorems are studied on fuzzy normed linear spaces.

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