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Some Results on Cesàro summability in Intuitionistic Fuzzy n-normed linear Spaces

Pradip Debnath

Abstract. The concept of summability plays a central role in finding formal solutions of partial differential equations. In this paper, we introduce the concept of Cesàro summability in an intuitionistic fuzzy n-normed linear space (IFnNLS). We show that Cesàro summability method is regular in an IFnNLS, but Cesàro summability does not imply usual convergence in general. Further, we search for additional conditions under which the converse holds.

1. Introduction

Zadeh's concept of fuzzy set theory [29] found applications in a variety of areas of mathematics such as approximation theory [1], theory of functions [14, 26], metric and topological spaces [8, 10, 15]. This theory is also extensively used in quantum physics [18], computer programming [12], population dynamics [4], nonlinear dynamical systems [13].

Katsaras [16] introduced the notion of fuzzy norm. Later it was improved by different authors from diverse points of views [3, 9, 15, 18, 27]. The concept of an IFnNLS [19] naturally generalized an intuitionistic fuzzy normed space that was introduced by Saadati and Park [20].

In the present paper, our aim is to introduce summability theory in an IFnNLS. In this connection, we put forward the notion of Cesàro and establish the associated Tauberian theorems. Our results depend on the definition of convergence of a sequence in IFnNLS. A new and unambiguous definition of the same was given in [21, 22]. Our current results are developed on the basis of this new definition. For the classical

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analogues of the results discussed in this paper, we refer to [2, 7, 24] and the references therein.

2. Preliminaries

A fuzzy metric and an intuitionistic fuzzy metric induce the same topology [11]. Thus, to obtain significant and original results, the definition of an intuitionistic fuzzy norm was slightly refined by authors [17, 28]. Following the same direction, a modified definition of an IFnNLS was given by Debnath and Sen [5, 6] as follows:

Definition 2.1. The five-tuple $(V, \eta, \gamma, *, \circ)$ is called an IFnNLS, where V is a vector space of dimension $d \geq n$ over the field F, $*$ is a continuous t-norm, $\circ$ is a continuous t-conorm, η, γ are fuzzy sets on $V^n \times (0, \infty)$, η signifies the degree of membership and γ signifies the degree of non-membership of $(\mathfrak{g}_1, \mathfrak{g}_2, \ldots, \mathfrak{g}_n, t) \in V^n \times (0, 1)$. The following conditions are satisfied for every $(\mathfrak{g}_1, \mathfrak{g}_2, \ldots, \mathfrak{g}_n) \in V^n$ and $s, t > 0$:

(i) $\eta(\mathfrak{g}_1, \mathfrak{g}_2, \ldots, \mathfrak{g}_n, r) = 0$ and $\gamma(\mathfrak{g}_1, \mathfrak{g}_2, \ldots, \mathfrak{g}_n, r) = 1$ for all non-positive real numbers t,

(ii) $\eta(\mathfrak{g}_1, \mathfrak{g}_2, \ldots, \mathfrak{g}_n, r) = 1$ and $\gamma(\mathfrak{g}_1, \mathfrak{g}_2, \ldots, \mathfrak{g}_n, r) = 0$ for all positive r if and only if $\mathfrak{g}_1, \mathfrak{g}_2, \ldots, \mathfrak{g}_n$ are linearly dependent,

(iii) $\eta(\mathfrak{g}_1, \mathfrak{g}_2, \ldots, \mathfrak{g}_n, r)$ and $\gamma(\mathfrak{g}_1, \mathfrak{g}_2, \ldots, \mathfrak{g}_n, r)$ are invariant under any permutation of $\mathfrak{g}_1, \mathfrak{g}_2, \ldots, \mathfrak{g}_n$,

(iv) $\eta(\mathfrak{g}_1, \mathfrak{g}_2, \ldots, c\mathfrak{g}_n, r) = \eta\left(\mathfrak{g}_1, \mathfrak{g}_2, \ldots, \mathfrak{g}_n, \frac{t}{|c|}\right)$ and $\gamma(\mathfrak{g}_1, \mathfrak{g}_2, \ldots, c\mathfrak{g}_n, r)$ $= \gamma\left(\mathfrak{g}_1, \mathfrak{g}_2, \ldots, \mathfrak{g}_n, \frac{r}{|c|}\right)$ if $c \neq 0, c \in F$,

(v) $\eta(\mathfrak{g}_1, \mathfrak{g}_2, \ldots, \mathfrak{g}_n, s) * \eta\left(\mathfrak{g}_1, \mathfrak{g}_2, \ldots, \mathfrak{g}_n^{'}, r\right) \leq \eta\left(\mathfrak{g}_1, \mathfrak{g}_2, \ldots, \mathfrak{g}_n + \mathfrak{g}_n^{'}, s + r\right)$,

(vi) $\eta(\mathfrak{g}_1, \mathfrak{g}_2, \ldots, \mathfrak{g}_n, s) \circ \gamma\left(\mathfrak{g}_1, \mathfrak{g}_2, \ldots, \mathfrak{g}_n^{'}, r\right) \geq \gamma\left(\mathfrak{g}_1, \mathfrak{g}_2, \ldots, \mathfrak{g}_n + \mathfrak{g}_n^{'}, s + r\right)$,

(vii) $\eta(\mathfrak{g}_1, \mathfrak{g}_2, \ldots, \mathfrak{g}_n, r) : (0, \infty) \to [0, 1]$ and $\gamma(\mathfrak{g}_1, \mathfrak{g}_2, \ldots, \mathfrak{g}_n, r) : (0, \infty)$ $\to [0, 1]$ are continuous in r,

(viii) $\lim_{r \to \infty} \eta(\mathfrak{g}_1, \mathfrak{g}_2, \ldots, \mathfrak{g}_n, r) = 1$ and $\lim_{r \to 0} \eta(\mathfrak{g}_1, \mathfrak{g}_2, \ldots, \mathfrak{g}_n, r) = 0$,

(ix) $\lim_{r \to \infty} \gamma(\mathfrak{g}_1, \mathfrak{g}_2, \ldots, \mathfrak{g}_n, r) = 0$ and $\lim_{r \to 0} \gamma(\mathfrak{g}_1, \mathfrak{g}_2, \ldots, \mathfrak{g}_n, r) = 1$.

Definition 2.2 ([21, 22])**.** Let $(V, \eta, \gamma, *, \circ)$ be an IFnNLS. A sequence $v = \{v_k\}$ in V is called convergent to $\varsigma \in V$ with respect to the intuitionistic fuzzy n-norm (IFnN) $(\eta, \gamma)^n$ if, for every $\epsilon \in (0, 1)$, $r > 0$ and $\mathfrak{g}_1, \mathfrak{g}_2, \ldots, \mathfrak{g}_{n-1} \in V$, there exists $k_0 \in \mathbb{N}$ such that $\eta(\mathfrak{g}_1, \mathfrak{g}_2, \ldots, \mathfrak{g}_{n-1}, v_k - \varsigma, r) > 1 - \epsilon$ and $\gamma(\mathfrak{g}_1, \mathfrak{g}_2, \ldots, \mathfrak{g}_{n-1}, v_k - \varsigma, r) < \epsilon$ for all $k \geq k_0$. We denote it by $(\eta, \gamma)^n - \lim v = \varsigma$ or $v_k \overset{(\eta,\gamma)^n}{\to} \varsigma$ as $k \to \infty$.

Proposition 2.3 ([25])**.** *In an IFnNLS V, $(\eta, \gamma)^n - \lim v = \varsigma$ if and only if for every $r > 0$ and $\mathfrak{g}_1, \mathfrak{g}_2, \ldots, \mathfrak{g}_{n-1} \in V$, $\eta(\mathfrak{g}_1, \ldots, \mathfrak{g}_{n-1}, v_k - \varsigma, r) \to 1$ and $\gamma(\mathfrak{g}_1, \ldots, \mathfrak{g}_{n-1}, v_k - \varsigma, r) \to 0$ as $k \to \infty$.*

Definition 2.4 ([21, 22])**.** Let $(V, \eta, \gamma, *, \circ)$ be an IFnNLS. A sequence $v = \{v_k\}$ in V is said to be Cauchy with respect to the IFnN $(\eta, \gamma)^n$ if, for every $\epsilon \in (0,1)$, $r > 0$ and $\mathfrak{g}_1, \mathfrak{g}_2, \ldots, \mathfrak{g}_{n-1} \in V$, there exists $k_0 \in \mathbb{N}$ such that $\eta(\mathfrak{g}_1, \mathfrak{g}_2, \ldots, \mathfrak{g}_{n-1}, v_k - v_m, r) > 1 - \epsilon$ and $\gamma(\mathfrak{g}_1, \mathfrak{g}_2, \ldots, \mathfrak{g}_{n-1}, v_k - v_m, r) < \epsilon$ for all $k, m \geq k_0$.

Definition 2.5. An IFnNLS V is complete with respect to the IFnN $(\eta, \gamma)^n$ if every Cauchy sequence is convergent.

Proposition 2.6 ([25])**.** *If every Cauchy sequence in an IFnNLS V has a convergent subsequence, then V is complete.*

The following lemmas will be used in the sequel.

Lemma 2.7 ([23])**.** *For every $\delta > 0$, we define $< \delta >= \delta - [\delta]$, where $[\cdot]$ denotes the greatest integer function. The following are true:*

(i) If $\delta > 1$, then $\delta_n > n$ for each $n \in \mathbb{N} \setminus \{0\}$ with $n \geq \frac{1}{\langle \delta \rangle}$,

(ii) If $0 < \delta < 1$, then $\delta_n < n$ for each $n \in \mathbb{N} \setminus \{0\}$, where $\delta_n = [n\delta]$.

Lemma 2.8 ([23])**.** *The following statements are true:*

(i) If $\delta > 1$, then for each $n \in \mathbb{N} \setminus \{0\}$ with $n \geq \frac{3\delta - 1}{\delta(\delta - 1)}$, we have $\frac{\delta}{\delta - 1} < \frac{\delta_n + 1}{\delta_n - n} < \frac{2\delta}{\delta - 1}$,

(ii) If $0 < \delta < 1$, then for each $n \in \mathbb{N} \setminus \{0\}$ with $n > \frac{1}{\delta}$, we have $0 < \frac{\delta_n + 1}{n - \delta_n} < \frac{2\delta}{1 - \delta}$.

3. Cesàro Summability in IFnNLS

At the first, we define the concept of Cesàro summability in an IFnNLS as given below.

Definition 3.1. Let $\{v_n\}$ be a sequence in an IFnNLS $(V, \eta, \gamma, *, \circ)$. By χ_n, we denote the Arithmetic means of $\{v_n\}$, which are defined as

$$\chi_n = \frac{1}{n+1} \sum_{k=0}^{n} v_k.$$

$\{v_n\}$ is said to be Cesàro summable to $v \in V$ if $\lim_{n \to \infty} \chi_n = v$.

Theorem 3.2. *Let $\{v_n\}$ be a sequence in an IFnNLS $(V, \eta, \gamma, *, \circ)$. If $\{v_n\}$ converges to $v \in V$, then $\{v_n\}$ is Cesàro summable to v.*

Proof. Let $\{v_n\}$ converges to $v \in V$. Fix $r > 0$ and $\mathfrak{g}_1, \mathfrak{g}_2, \ldots, \mathfrak{g}_{n-1} \in V$. Then for given $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$\eta\left(\mathfrak{g}_1, \mathfrak{g}_2, \ldots, \mathfrak{g}_{n-1}, v_n - v, \frac{r}{2}\right) > 1 - \epsilon,$$

$$\gamma\left(\mathfrak{g}_1, \mathfrak{g}_2, \ldots, \mathfrak{g}_{n-1}, v_n - v, \frac{r}{2}\right) < \epsilon, \quad \text{for all } n > n_0.$$

Also, from Definition 2.1, we have that

$$\lim_{n\to\infty} \eta\left(\mathfrak{g}_1, \mathfrak{g}_2, \ldots, \mathfrak{g}_{n-1}, \sum_{k=0}^{n_0}(v_k - v), \frac{(n+1)r}{2}\right) = 1,$$

$$\lim_{n\to\infty} \gamma\left(\mathfrak{g}_1, \mathfrak{g}_2, \ldots, \mathfrak{g}_{n-1}, \sum_{k=0}^{n_0}(v_k - v), \frac{(n+1)r}{2}\right) = 0.$$

Thus, there exists $n_1 \in \mathbb{N}$ such that

$$\eta\left(\mathfrak{g}_1, \mathfrak{g}_2, \ldots, \mathfrak{g}_{n-1}, \sum_{k=0}^{n_0}(v_k - v), \frac{(n+1)r}{2}\right) > 1 - \epsilon,$$

$$\gamma\left(\mathfrak{g}_1, \mathfrak{g}_2, \ldots, \mathfrak{g}_{n-1}, \sum_{k=0}^{n_0}(v_k - v), \frac{(n+1)r}{2}\right) < \epsilon, \quad \text{for all } n > n_1.$$

Now we have that

$$\begin{aligned}
&\eta\left(\mathfrak{g}_1, \mathfrak{g}_2, \ldots, \mathfrak{g}_{n-1}, \frac{1}{n+1}\sum_{k=0}^{n} v_k - v, r\right) \\
&\quad = \eta\left(\mathfrak{g}_1, \mathfrak{g}_2, \ldots, \mathfrak{g}_{n-1}, \frac{1}{n+1}\sum_{k=0}^{n}(v_k - v), r\right) \\
&\quad \geq \min\left\{\eta\left(\mathfrak{g}_1, \mathfrak{g}_2, \ldots, \mathfrak{g}_{n-1}, \sum_{k=0}^{n_0}(v_k - v), \frac{(n+1)r}{2}\right),\right. \\
&\qquad \left.\eta\left(\mathfrak{g}_1, \mathfrak{g}_2, \ldots, \mathfrak{g}_{n-1}, \sum_{k=n_0+1}^{n}(v_k - v), \frac{(n+1)r}{2}\right)\right\} \\
&\quad \geq \min\left\{\eta\left(\mathfrak{g}_1, \mathfrak{g}_2, \ldots, \mathfrak{g}_{n-1}, \sum_{k=0}^{n_0}(v_k - v), \frac{(n+1)r}{2}\right),\right. \\
&\qquad \left.\eta\left(\mathfrak{g}_1, \mathfrak{g}_2, \ldots, \mathfrak{g}_{n-1}, \sum_{k=n_0+1}^{n}(v_k - v), \frac{(n-n_0)r}{2}\right)\right\} \\
&\quad \geq \min\left\{\eta\left(\mathfrak{g}_1, \mathfrak{g}_2, \ldots, \mathfrak{g}_{n-1}, \sum_{k=0}^{n_0}(v_k - v), \frac{(n+1)r}{2}\right),\right. \\
&\qquad \eta\left(\mathfrak{g}_1, \mathfrak{g}_2, \ldots, \mathfrak{g}_{n-1}, v_{n_0+1} - v, \frac{r}{2}\right), \\
&\qquad \left.\eta\left(\mathfrak{g}_1, \mathfrak{g}_2, \ldots, \mathfrak{g}_{n-1}, v_{n_0+2} - v, \frac{r}{2}\right), \ldots, \eta\left(\mathfrak{g}_1, \mathfrak{g}_2, \ldots, \mathfrak{g}_{n-1}, v_n - v, \frac{r}{2}\right)\right\}
\end{aligned}$$

$$> 1 - \epsilon,$$

and in a similar manner, we also have that

$$\begin{aligned}
&\gamma\left(\mathfrak{g}_1, \mathfrak{g}_2, \ldots, \mathfrak{g}_{n-1}, \frac{1}{n+1}\sum_{k=0}^{n} v_k - v, r\right) \\
&= \gamma\left(\mathfrak{g}_1, \mathfrak{g}_2, \ldots, \mathfrak{g}_{n-1}, \frac{1}{n+1}\sum_{k=0}^{n} (v_k - v), r\right) \\
&\leq \max\left\{\gamma\left(\mathfrak{g}_1, \mathfrak{g}_2, \ldots, \mathfrak{g}_{n-1}, \sum_{k=0}^{n_0} (v_k - v), \frac{(n+1)r}{2}\right),\right. \\
&\quad \left.\gamma\left(\mathfrak{g}_1, \mathfrak{g}_2, \ldots, \mathfrak{g}_{n-1}, \sum_{k=n_0+1}^{n} (v_k - v), \frac{(n+1)r}{2}\right)\right\} \\
&\leq \max\left\{\gamma\left(\mathfrak{g}_1, \mathfrak{g}_2, \ldots, \mathfrak{g}_{n-1}, \sum_{k=0}^{n_0} (v_k - v), \frac{(n+1)r}{2}\right),\right. \\
&\quad \left.\gamma\left(\mathfrak{g}_1, \mathfrak{g}_2, \ldots, \mathfrak{g}_{n-1}, \sum_{k=n_0+1}^{n} (v_k - v), \frac{(n-n_0)r}{2}\right)\right\} \\
&\leq \max\left\{\gamma\left(\mathfrak{g}_1, \mathfrak{g}_2, \ldots, \mathfrak{g}_{n-1}, \sum_{k=0}^{n_0} (v_k - v), \frac{(n+1)r}{2}\right),\right. \\
&\quad \gamma\left(\mathfrak{g}_1, \mathfrak{g}_2, \ldots, \mathfrak{g}_{n-1}, v_{n_0+1} - v, \frac{r}{2}\right), \gamma\left(\mathfrak{g}_1, \mathfrak{g}_2, \ldots, \mathfrak{g}_{n-1}, v_{n_0+2} - v, \frac{r}{2}\right), \\
&\quad \left.\ldots, \gamma\left(\mathfrak{g}_1, \mathfrak{g}_2, \ldots, \mathfrak{g}_{n-1}, v_n - v, \frac{r}{2}\right)\right\} \\
&< \epsilon,
\end{aligned}$$

for all $n > \max\{n_0, n_1\}$. This completes the proof. □

Our next example shows that usual convergence implies Cesàro summability but the converse is not necessarily true.

Example 3.3. Let $V = \mathbb{R}^n$ with

$$\|\mathfrak{g}_1, \mathfrak{g}_2, \ldots, \mathfrak{g}_n\| = abs\left(\begin{vmatrix} \mathfrak{g}_{11} & \cdots & \mathfrak{g}_{1n} \\ \vdots & \ddots & \vdots \\ \mathfrak{g}_{n1} & \cdots & \mathfrak{g}_{nn} \end{vmatrix}\right),$$

where $\mathfrak{g}_i = (\mathfrak{g}_{i1}, \mathfrak{g}_{i2}, \ldots, \mathfrak{g}_{in}) \in \mathbb{R}^n$ for each $i = 1, 2, \ldots, n$ and let $a * b = ab$, $a \circ b = \min\{a + b, 1\}$ for all $a, b \in [0, 1]$. Now for all $w_1, w_2, \ldots, w_n \in$

$\mathbb{R}^n$ and $r > 0$, let us define $\mu(w_1, w_2, \ldots, w_n, r) = \frac{r}{r+\|w_1,w_2,\ldots,w_n\|}$ and $\nu(w_1, w_2, \ldots, w_n, r) = \frac{\|w_1,w_2,\ldots,w_n\|}{r+\|w_1,w_2,\ldots,w_n\|}$. Then $(\mathbb{R}^n, \eta, \gamma, *, \circ)$ is an IFnNLS.

Consider the sequence $\{v_k\} = \big((-1)^{k+1}, 0, 0, \ldots, 0\big) \in \mathbb{R}^n$. Then

$$\begin{aligned}\lim_{n\to\infty} \eta(\mathfrak{g}_1, \mathfrak{g}_2, \ldots, \chi_{2n}, r) &= \lim_{n\to\infty} \frac{r}{r + \left\|\mathfrak{g}_1, \mathfrak{g}_2, \ldots, \mathfrak{g}_{n-1}, -\frac{1}{2n+1}\right\|}\\ &= \lim_{n\to\infty} \frac{r}{r + \left|-\frac{1}{2n+1}\right| A}\\ &= 1,\end{aligned}$$

where A is always a finite number whose value depends on the choice of $\mathfrak{g}_1, \mathfrak{g}_2, \ldots, \mathfrak{g}_{n-1}$.

And,

$$\begin{aligned}\lim_{n\to\infty} \gamma(\mathfrak{g}_1, \mathfrak{g}_2, \ldots, \chi_{2n}, r) &= \lim_{n\to\infty} \frac{\left\|\mathfrak{g}_1, \mathfrak{g}_2, \ldots, \mathfrak{g}_{n-1}, -\frac{1}{2n+1}\right\|}{r + \left\|\mathfrak{g}_1, \mathfrak{g}_2, \ldots, \mathfrak{g}_{n-1}, -\frac{1}{2n+1}\right\|}\\ &= \lim_{n\to\infty} \frac{\left|-\frac{1}{2n+1}\right| B}{r + \left|-\frac{1}{2n+1}\right| B}\\ &= 1,\end{aligned}$$

where B is always a finite number whose value depends on the choice of $\mathfrak{g}_1, \mathfrak{g}_2, \ldots, \mathfrak{g}_{n-1}$.

Therefore, we have that $\chi_{2n} \to \overline{0} = (0, 0, \ldots, 0) \in \mathbb{R}^n$.

Also,

$$\begin{aligned}\lim_{n\to\infty} \eta(\mathfrak{g}_1, \mathfrak{g}_2, \ldots, \chi_{2n+1}, r) &= \lim_{n\to\infty} \eta(\mathfrak{g}_1, \mathfrak{g}_2, \ldots, \overline{0}, r)\\ &= 1,\end{aligned}$$

and,

$$\begin{aligned}\lim_{n\to\infty} \gamma(\mathfrak{g}_1, \mathfrak{g}_2, \ldots, \mathfrak{g}_{n-1}, \chi_{2n+1}, r) &= \lim_{n\to\infty} \gamma(\mathfrak{g}_1, \mathfrak{g}_2, \ldots, \overline{0}, r)\\ &= 0.\end{aligned}$$

Thus, we have $\chi_{2n+1} \to \overline{0}$.

From the above arguments, we conclude that $\chi_n \to \overline{0}$, i.e., the sequence $\{v_k\}$ is Cesàro summable to $\overline{0}$.

But it is easy to see that $\{v_k\}$ is not convergent because $\{v_{2k}\} \to (-1, 0, 0, \ldots, 0)$ and $\{v_{2k+1}\} \to (1, 0, 0, \ldots, 0)$.

In the next result, we establish additional conditions under which a Cesàro summable sequence becomes convergent.

Theorem 3.4. *Let $\{v_n\}$ be a sequence in an IFnNLS $(V, \eta, \gamma, *, \circ)$. If $\{v_n\}$ is Cesàro summable to v, then it is convergent to v if and only if for any $\mathfrak{g}_1, \mathfrak{g}_2, \ldots, \mathfrak{g}_{n-1} \in V$ and $r > 0$ the following are satisfied:*

$$\sup_{\lambda>1} \liminf_{n\to\infty} \eta\big(\mathfrak{g}_1, \mathfrak{g}_2, \ldots, \mathfrak{g}_{n-1}, \frac{1}{\lambda_n - n} \sum_{k=n+1}^{\lambda_n} (v_k - v_n), r\big) = 1 \tag{3.1}$$

and

$$\inf_{\lambda>1} \limsup_{n\to\infty} \gamma\big(\mathfrak{g}_1, \mathfrak{g}_2, \ldots, \mathfrak{g}_{n-1}, \frac{1}{\lambda_n - n} \sum_{k=n+1}^{\lambda_n} (v_k - v_n), r\big) = 0. \tag{3.2}$$

Proof. Let $\{v_n\}$ be Cesàro summable to v. Also suppose that $\{v_n\}$ converges to v. Fix $\mathfrak{g}_1, \mathfrak{g}_2, \ldots, \mathfrak{g}_{n-1} \in V$ and $r > 0$. For any $\lambda > 1$, using Lemma 2.7, for each $n \in \mathbb{N} \setminus \{0\}$ with $n \geq < \lambda >^{-1}$, we have

$$v_n - \chi_n = \frac{\lambda_n + 1}{\lambda_n - n}(\chi_{\lambda_n} - \chi_n) - \frac{1}{\lambda_n - n} \sum_{k=n+1}^{\lambda_n} (v_k - v_n). \tag{3.3}$$

Again, by Lemma 2.8, for $n \geq \frac{3\lambda-1}{\lambda(\lambda-1)}$, we have

$$\begin{aligned}
&\eta(\mathfrak{g}_1, \mathfrak{g}_2, \ldots, \mathfrak{g}_{n-1}, \frac{\lambda_n + 1}{\lambda_n - n}(\chi_{\lambda_n} - \chi_n), r) \\
&\qquad = \eta(\mathfrak{g}_1, \mathfrak{g}_2, \ldots, \mathfrak{g}_{n-1}, (\chi_{\lambda_n} - \chi_n), \frac{r}{\frac{\lambda_n+1}{\lambda_n-n}}) \\
&\qquad \geq \eta(\mathfrak{g}_1, \mathfrak{g}_2, \ldots, \mathfrak{g}_{n-1}, (\chi_{\lambda_n} - \chi_n), \frac{r}{\frac{2\lambda}{\lambda-1}}),
\end{aligned}$$

and

$$\begin{aligned}
&\gamma(\mathfrak{g}_1, \mathfrak{g}_2, \ldots, \mathfrak{g}_{n-1}, \frac{\lambda_n + 1}{\lambda_n - n}(\chi_{\lambda_n} - \chi_n), r) \\
&\qquad = \gamma(\mathfrak{g}_1, \mathfrak{g}_2, \ldots, \mathfrak{g}_{n-1}, (\chi_{\lambda_n} - \chi_n), \frac{r}{\frac{\lambda_n+1}{\lambda_n-n}}) \\
&\qquad \leq \gamma(\mathfrak{g}_1, \mathfrak{g}_2, \ldots, \mathfrak{g}_{n-1}, (\chi_{\lambda_n} - \chi_n), \frac{r}{\frac{2\lambda}{\lambda-1}}).
\end{aligned}$$

Since $\{\chi_n\}$ is a Cauchy sequence, we have

$$\lim_{n\to\infty} \eta(\mathfrak{g}_1, \mathfrak{g}_2, \ldots, \mathfrak{g}_{n-1}, \frac{\lambda_n + 1}{\lambda_n - n}(\chi_{\lambda_n} - \chi_n), r) = 1$$

and

$$\lim_{n\to\infty} \gamma(\mathfrak{g}_1, \mathfrak{g}_2, \ldots, \mathfrak{g}_{n-1}, \frac{\lambda_n + 1}{\lambda_n - n}(\chi_{\lambda_n} - \chi_n), r) = 0,$$

and therefore $\lim\limits_{n\to\infty} \frac{\lambda_n+1}{\lambda_n-n}(\chi_{\lambda_n} - \chi_n) = 0$.

Hence using (3.3), we have

$$\lim_{n\to\infty} \eta(\mathfrak{g}_1, \mathfrak{g}_2, \ldots, \mathfrak{g}_{n-1}, \frac{1}{\lambda_n - n} \sum_{k=n+1}^{\lambda_n} (v_k - v_n), r) = 1$$

and

$$\lim_{n\to\infty} \gamma(\mathfrak{g}_1, \mathfrak{g}_2, \ldots, \mathfrak{g}_{n-1}, \frac{1}{\lambda_n - n} \sum_{k=n+1}^{\lambda_n} (v_k - v_n), r) = 0.$$

This proves (3.1) and (3.2).

To prove the converse, we assume that (3.1) and (3.2) are true. Fix $\mathfrak{g}_1, \mathfrak{g}_2, \ldots, \mathfrak{g}_{n-1} \in V$ and $r > 0$. Then for given $\epsilon > 0$, we have the following:

(i) There exist $\lambda > 1$ and $n_0 \in \mathbb{N}$ such that

$$\eta(\mathfrak{g}_1, \mathfrak{g}_2, \ldots, \mathfrak{g}_{n-1}, \frac{1}{\lambda_n - n} \sum_{k=n+1}^{\lambda_n} (v_k - v_n), \frac{r}{3}) > 1 - \epsilon$$

and

$$\gamma(\mathfrak{g}_1, \mathfrak{g}_2, \ldots, \mathfrak{g}_{n-1}, \frac{1}{\lambda_n - n} \sum_{k=n+1}^{\lambda_n} (v_k - v_n), \frac{r}{3}) < \epsilon$$

for all $n > n_0$.

(ii) There exists $n_1 \in \mathbb{N}$ such that $\eta(\mathfrak{g}_1, \mathfrak{g}_2, \ldots, \mathfrak{g}_{n-1}, \chi_n - v, \frac{r}{3}) > 1 - \epsilon$ and $\gamma(\mathfrak{g}_1, \mathfrak{g}_2, \ldots, \mathfrak{g}_{n-1}, \chi_n - v, \frac{r}{3}) < \epsilon$ for all $n > n_1$.

(iii) Also, since $\lim\limits_{n\to\infty} \frac{\lambda_n + 1}{\lambda_n - n}(\chi_{\lambda_n} - \chi_n) = 0$, there exists $n_2 \in \mathbb{N}$ such that

$$\eta(\mathfrak{g}_1, \mathfrak{g}_2, \ldots, \mathfrak{g}_{n-1}, \frac{\lambda_n + 1}{\lambda_n - n}(\chi_{\lambda_n} - \chi_n), \frac{r}{3}) > 1 - \epsilon$$

and

$$\gamma(\mathfrak{g}_1, \mathfrak{g}_2, \ldots, \mathfrak{g}_{n-1}, \frac{\lambda_n + 1}{\lambda_n - n}(\chi_{\lambda_n} - \chi_n), \frac{r}{3}) < \epsilon,$$

for all $n > n_2$.

Therefore, we have

$$\begin{aligned}
&\eta(\mathfrak{g}_1, \mathfrak{g}_2, \ldots, \mathfrak{g}_{n-1}, v_n - v, r) = \eta(\mathfrak{g}_1, \mathfrak{g}_2, \ldots, \mathfrak{g}_{n-1}, v_n - \chi_n + \chi_n - v, r) \\
&= \eta(\mathfrak{g}_1, \mathfrak{g}_2, \ldots, \mathfrak{g}_{n-1}, \frac{\lambda_n + 1}{\lambda_n - n}(\chi_{\lambda_n} - \chi_n) \\
&- \frac{1}{\lambda_n - n} \sum_{k=n+1}^{\lambda_n} (v_k - v_n) + \chi_n - v, r)
\end{aligned}$$

$$\geq \min\{\eta(\mathfrak{g}_1, \mathfrak{g}_2, \ldots, \mathfrak{g}_{n-1}, \frac{\lambda_n + 1}{\lambda_n - n}(\chi_{\lambda_n} - \chi_n, \frac{r}{3}),$$

$$\eta(\mathfrak{g}_1, \mathfrak{g}_2, \ldots, \mathfrak{g}_{n-1}, \frac{1}{\lambda_n - n}\sum_{k=n+1}^{\lambda_n}(v_k - v_n), \frac{r}{3}),$$

$$\eta(\mathfrak{g}_1, \mathfrak{g}_2, \ldots, \mathfrak{g}_{n-1}, \chi_n - v, \frac{r}{3})\}$$

$$> 1 - \epsilon$$

and

$$\gamma(\mathfrak{g}_1, \mathfrak{g}_2, \ldots, \mathfrak{g}_{n-1}, v_n - v, r) = \eta(\mathfrak{g}_1, \mathfrak{g}_2, \ldots, \mathfrak{g}_{n-1}, v_n - \chi_n + \chi_n - v, r)$$

$$= \gamma(\mathfrak{g}_1, \mathfrak{g}_2, \ldots, \mathfrak{g}_{n-1}, \frac{\lambda_n + 1}{\lambda_n - n}(\chi_{\lambda_n} - \chi_n)$$

$$- \frac{1}{\lambda_n - n}\sum_{k=n+1}^{\lambda_n}(v_k - v_n) + \chi_n - v, r)$$

$$\leq \max\{\gamma(\mathfrak{g}_1, \mathfrak{g}_2, \ldots, \mathfrak{g}_{n-1}, \frac{\lambda_n + 1}{\lambda_n - n}(\chi_{\lambda_n} - \chi_n, \frac{r}{3}),$$

$$\gamma(\mathfrak{g}_1, \mathfrak{g}_2, \ldots, \mathfrak{g}_{n-1}, \frac{1}{\lambda_n - n}\sum_{k=n+1}^{\lambda_n}(v_k - v_n), \frac{r}{3}),$$

$$\gamma(\mathfrak{g}_1, \mathfrak{g}_2, \ldots, \mathfrak{g}_{n-1}, \chi_n - v, \frac{r}{3})\}$$

$$< \epsilon,$$

for all $n > \max\{n_0, n_1, n_2\}$. This completes the proof.

□

4. Conclusion and future work

Summability theory occupies a very significant area in the study of partial differential equations. In the current work, we introduced the concept of Cesàro summability in an IFnNLS, which is one of the most general mathematical structures having both algebraic and analytic properties. As such, the current results in Cesàro summability extend and generalize many existing theorems. Proving the Tauberian theorems in the present context is a suggested future work.

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