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Leila Shahbaz

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Categorical Properties of Down Closed Embeddings

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Abstract. Let $\mathcal{M}$ be a class of (mono)morphisms in a category $\mathcal{A}$. To study mathematical notions, such as injectivity, tensor products, flatness, one needs to have some categorical and algebraic information about the pair ($\mathcal{A}$,$\mathcal{M}$).

In this paper, we take $\mathcal{A}$ to be the category **Pos**-S of S-posets over a posemigroup S, and $\mathcal{M}_{dc}$ to be the class of down closed embeddings and study the categorical properties, such as limits and colimits, of the pair ($\mathcal{A}$,$\mathcal{M}$). Injectivity with respect to this class of monomorphisms have been studied by Shahbaz et al., who used it to obtain information about regular injectivity.

1. Introduction and Preliminaries

The category **Pos**-S of partially ordered sets with actions of a pomonoid S on them have been studied in various papers (see [4–6, 9, 10, 13, 15, 17, 18]). While studying injectivity with respect to embeddings one gets that the subclass of embeddings which has natural behaviour with regular injectivity is the subclass of down closed embeddings. Down closed embeddings and injectivity with respect to these embeddings were first introduced and studied by the author and Mahmoudi for S-posets over the pomonoid S. They gave a criterion for down closed injectivity (briefly, dc-injectivity) and studied such injectivity for S itself, and for its poideals (see [14]). In this paper, we take $\mathcal{A}$ to be the category **Pos**-S of S-posets over a semigroup S, and $\mathcal{M}_{dc}$ to be the class of down closed embeddings and study the categorical properties, such as limits and colimits, of the pair ($\mathcal{A}$,$\mathcal{M}$).

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First we briefly recall the definition and the categorical and algebraic ingredients of the category **Pos**-S of (right) S-posets needed in the sequel. For more information see [7, 8, 11]. Recall that a monoid (semigroup) S is said to be a pomonoid (posemigroup) if it is also a poset whose partial order $\leq$ is compatible with its binary operation (that is, for $s, s', t, t' \in S$, $s \leq t$, $s' \leq t'$ imply $ss' \leq tt'$).

A (right) S-poset is a poset A which is also an S-act whose action $\lambda : A \times S \to A$ is order-preserving, where $A \times S$ is considered as a poset with componentwise order.

An S-poset map (or morphism) is an action preserving monotone map between S-posets. Moreover, regular monomorphisms (equalizers) are exactly order embeddings (briefly, embeddings); that is, monomorphisms $f : A \to B$ for which $f(a) \leq f(a')$ if and only if $a \leq a'$, for all $a, a' \in A$.

Let A be an S-poset. A (right) S-poset congruence on A is a (right) S-act congruence θ, that is, an equivalence relation on A which is closed under S-action, with the property that the S-act A/θ can be made into an S-poset in such a way that the canonical S-act map $A \to A/\theta$ is an S-poset map. For a binary relation β on A, define a relation $\leq_\beta$ on A by $a \leq_\beta a'$ if and only if $a \leq a_1 \beta a'_1 \leq \cdots \leq a_n \beta a'_n \leq a'$ for some $a_1, a'_1, \ldots, a_n, a'_n \in A, n \in \mathbb{N}$. Then an S-act congruence θ on A is an S-poset congruence if and only if for every $a, a' \in A$, $a\theta a'$ whenever $a \leq_\theta a' \leq_\theta a$. The S-poset quotient is then the S-act quotient A/θ with the partial order given by $[a]_\theta \leq [a']_\theta$ if and only if $a \leq_\theta a'$. Clearly, $a \leq a'$ implies that $a \leq_\theta a'$.

Recall that the product of a family $\{A_i\}_{i \in I}$ of S-posets is considered as their cartesian product with the componentwise order and S-action. The coproduct of a family $\{A_i\}_{i \in I}$ of S-posets is considered as their disjoint union with the order given by $x \leq y$ in coproduct if and only if there exists an S-poset A_i of a family with $x, y \in A_i$ and $x \leq y$ in A_i, and the S-action is as follows, for $a \in \coprod_{i \in I} A_i, s \in S$ there exists $i \in I$ such that $a \in A_i$. Now, define the action of a and s as as in A_i.

For a family $\{A_i : i \in I\}$ of S-posets with a unique fixed element 0, the direct sum $\bigoplus_{i \in I} A_i$ is defined to be the sub S-poset of the product $\prod_{i \in I} A_i$ consisting of all $(a_i)_{i \in I}$ such that $a_i = 0$ for all $i \in I$ except a finite number. Recall from [14] that for a family $\{A_i\}_{i \in I}$ of S-posets where I is a chain, the order sum, denoted by $\biguplus_{i \in I} A_i$ is defined to be the disjoint union $\dot{\bigcup}_{i \in I} A_i$ with the order given by $x \leq y$ if and only if either $x, y \in A_i$ for some $i \in I$ and $x \leq y$ in A_i, or $x \in A_i, y \in A_j$ and $i < j$.

The pullback of S-poset maps $f : A \to C$ and $g : B \to C$ is the sub S-poset $P = \{(a, b) : f(a) = g(b)\}$ of $A \times B$, together with the restricted projection maps. The pushout of S-poset maps $f : A \to B$ and $g : A \to C$ is the quotient of the coproduct $B \sqcup C = (\{1\} \times B) \cup (\{2\} \times C)$ by the congruence $\theta(H)$ generated by $H = \{((1, f(a)), (2, g(a))) : a \in A\}$.

Now, we introduce the class of down closed embeddings needed to define down closed injectivity.

Definition 1.1. A possibly empty sub S-poset A of an S-poset B is said to be down closed in B if for each $a \in A$ and $b \in B$ with $b \le a$ we have $b \in A$. By a down closed embedding (or briefly dc-embedding), we mean an embedding $f : A \to B$ such that $f(A)$ is a down closed sub S-poset of B.

2. Categorical Properties of Down Closed Embeddings

In this section we study some categorical and algebraic properties of the category **Pos**-S with respect to the class $\mathcal{M}_{dc}$ of down closed embeddings, mostly to do with the composition, limit, and colimit properties.

First, we investigate the composition properties of down closed embeddings of S-posets.

Lemma 2.1. (i) *The composition of down closed embeddings is a down closed embedding.*
(ii) *The class of down closed embeddings is isomorphism closed.*

Proof. (i) Let $f : A \to B$ and $g : B \to C$ be two down closed embeddings and $c \le gf(a)$. Since g is down closed embedding there exists $b \in B$ such that $c = g(b)$ and so $g(b) \le gf(a)$. Then $b \le f(a)$ since g is regular. Now, $b = f(a')$ for some $a' \in A$ since f is down closed embedding. Hence $c = g(b) = g(f(a'))$.
(ii) Since each isomorphism is a down closed embedding, (i) gives the result. □

Proposition 2.2. *The class of all down closed embeddings is left cancellable.*

Proof. Let gf be a down closed embeddings for monomorphisms $f : A \to B$, $g : B \to C$. First we show that f is an embedding. For, let $f(a) \le f(a')$ for $a, a' \in A$. Since g is order preserving one gets $gf(a) \le gf(a')$, which implies that $a \le a'$ by regularity of gf. This means that f is an embedding. To show that f is down closed, take $b \in B, a \in A$ with $b \le f(a)$. Since g is order preserving one gets $g(b) \le gf(a)$, which implies that there exists $a' \in A$ such that $g(b) = (gf)(a')$ since gf is down closed. Since g is a monomorphism one gets $b = f(a')$ as required. □

Remark 2.3. For every pomonoid S, the class $\mathcal{M}_{dc}$ is not right cancellable. For example, consider the inclusions $\mathbf{2} \overset{\mathbf{f}}{\hookrightarrow} \mathbf{2}\dot{\cup}\mathbf{1} \overset{\mathbf{g}}{\hookrightarrow} \mathbf{1} \uplus \mathbf{3}$ with trivial actions of a pomonoid S on $\mathbf{2}$, $\mathbf{2}\dot\cup\mathbf{1}$ and $\mathbf{1}\uplus\mathbf{3}$. Then $gf\in\mathcal{M}_{dc}$ but g is not in $\mathcal{M}_{dc}$. Also, for every pomonoid S, there always exists a monomorphism that is not down closed and regular. To see this, it suffices to take the inclusion $i:\mathbf{1}\dot\cup\mathbf{1}\hookrightarrow\mathbf{1}\uplus\mathbf{2}$ with trivial actions of a pomonoid S on $\mathbf{1}\dot\cup\mathbf{1}$ and $\mathbf{1}\uplus\mathbf{2}$.

2.1. **Limits of Down Closed Embeddings.** In this subsection some of the categorical properties of down closed embeddings related to limits such as products and pullbacks are studied.

Proposition 2.4. (i) *The class $\mathcal{M}_{dc}$ is closed under products.*

(ii) *Let $\{f_i : A\to B_i \mid i\in I\}$ be a family of down closed embeddings. Then their product homomorphism $f:A\to\prod B_i$ is also a down closed embedding.*

Proof. (i) Let $(f_i : A_i\to B_i)_{i\in I}$ be a family of down closed embeddings. Consider the commutative diagram

$$\begin{array}{ccc} \prod_{i\in I}A_i & \xrightarrow{f} & \prod_{i\in I}B_i \\ {\scriptstyle p_{A_i}}\downarrow & & \downarrow{\scriptstyle p_{B_i}} \\ A_i & \xrightarrow[f_i]{} & B_i \end{array}$$

which f exists by the universal property of products. It is easily proved that f is an embedding. So we show that f is down closed. For, let $b=(b_i)_{i\in I}\in\prod_{i\in I}B_i, a=(a_i)_{i\in I}\in\prod_{i\in I}A_i$ and $(b_i)_{i\in I}=b\leq f(a)=(f_i(a_i))_{i\in I}$. Then $b_i\leq f_i(a_i)$ for each $i\in I$, and since each f_i is down closed, there exists an element $a'_i\in A_i$ with $b_i=f_i(a'_i)$ for each $i\in I$. Therefore, $b=(b_i)_{i\in I}=(f_i(a'_i))_{i\in I}=f(a')$ which proves that f is down closed.

(ii) It is similar to the proof of (i). □

Recall that a class of morphisms of a category is called pullback stable if pullbacks transfer those morphisms. In the next result, we study this property for down closed embeddings of S-posets.

Proposition 2.5. *The class of down closed embeddings is pullback stable.*

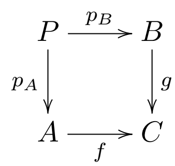
Proof. Consider the pullback diagram

$$\begin{array}{ccc} P & \xrightarrow{p_B} & B \\ {\scriptstyle p_A}\downarrow & & \downarrow{\scriptstyle g} \\ A & \xrightarrow[f]{} & C \end{array}$$

where P is the sub S-poset $\{(a,b) : f(a) = g(b)\}$ of $A \times B$, and pullback maps $p_A : P \to A$, $p_B : P \to B$ are restrictions of the projection maps and f is a down closed embedding. We want to show that p_B is a down closed embedding. By [12], it is known that p_B is an embedding so it is enough to show that it is down closed. For, let $b' \leq p_B(a,b) = b, a \in A, b, b' \in B$. Then $g(b') \leq g(b) = f(a)$ since $(a,b) \in P$. Now, since f is down closed one gets $g(b') \in f(A)$ which means that $g(b') = f(a')$ for some $a' \in A$. Hence $(a', b') \in P$ and $b' = p_B(a', b') \in p_B(P)$. □

Proposition 2.6. *The class of down closed embeddings is closed under $\mathcal{M}_{dc}$-pullbacks.*

Proof. Consider the pullback diagram



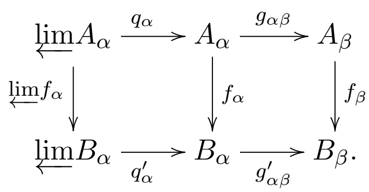
where g, f are down closed embeddings. Then by the above proposition p_A, p_B are down closed embeddings and thus since the class of all down closed embeddings is closed under composition, one gets that fp_A and gp_B are down closed embeddings. □

Now, we mention the construction of the general limits which we need in the following proposition.

Definition 2.7. Let $\mathcal{A} : \mathbf{I} \to \mathbf{Pos} - S$ be a diagram in $\mathbf{Pos}$-S determining the S-posets A_α, for $\alpha \in I = Obj\mathbf{I}$, and S-poset maps $g_{\alpha\beta} : A_\alpha \to A_\beta$, for $\alpha \to \beta$ in $Mor\mathbf{I}$. Recall that the limit of this diagram is $A = \varprojlim_\alpha A_\alpha := \bigcap_{\alpha \in I} E_\alpha$, where $E_\alpha = \{a = (a_\alpha)_{\alpha \in I} \in \prod_\alpha A_\alpha : g_{\alpha\beta} p_\alpha(a) = p_\beta(a)\}$ and p_α, p_β are the α, βth projection maps of the product. Also, the limit S-poset maps are $q_\alpha =: p_\alpha \mid_A : \varprojlim_\alpha A_\alpha \to A_\alpha$.

Proposition 2.8. *The class of down closed embeddings is closed under limits.*

Proof. Let $\mathcal{A}, \mathcal{B} : \mathbf{I} \to \mathbf{Pos} - S$ be diagrams in $\mathbf{Pos}$-S determining the S-posets A_α, B_α, for $\alpha \in I = Obj(\mathbf{I})$, and S-poset maps $g_{\alpha\beta} : A_\alpha \to A_\beta$, $g'_{\alpha\beta} : B_\alpha \to B_\beta$, for $\alpha \to \beta$ in $Mor(\mathbf{I})$. Consider limits of these diagrams with limit maps $q_\alpha : \varprojlim A_\alpha \to A_\alpha$, $q'_\alpha : \varprojlim B_\alpha \to B_\alpha$. Let $\{f_\alpha : A_\alpha \to B_\alpha : \alpha \in I\}$ be a family of down closed embeddings such that $g'_{\alpha\beta} f_\alpha = f_\beta g_{\alpha\beta}$. Let f denote $\varprojlim f_\alpha : \varprojlim A_\alpha \to \varprojlim B_\alpha$ which exists by the universal property of limits. We show that f is a down closed embedding. Consider the diagram

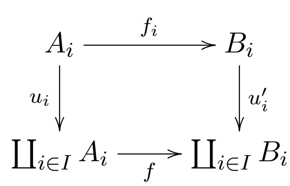


First, we show that f is an embedding. For, let $f(a) \leq f(a')$ for $a = (a_\alpha)_\alpha, a' = (a'_\alpha)_\alpha \in \varprojlim A_\alpha$. For each $\alpha \in I$, we have $f_\alpha(a) = p_\alpha f(a) \leq p_\alpha f(a') = f_\alpha(a')$. Since f_α is an embedding by the assumption, $a \leq a'$. Thus f is an embedding. To show that f is down closed, let $b = (b_\alpha)_\alpha \leq f((a_\alpha)_\alpha) = (f_\alpha(a_\alpha))_\alpha, a = (a_\alpha)_\alpha \in (a_\alpha)_\alpha \in \varprojlim A_\alpha, b = (b_\alpha)_\alpha \in \varprojlim B_\alpha$. Thus For each $\alpha \in I$, we have $b_\alpha \leq f_\alpha(a_\alpha)$. Now, since each f_α is down closed, for each $\alpha \in I$ there exists $a'_\alpha \in A_\alpha$ such that $b_\alpha = f_\alpha(a'_\alpha)$. If we show that $a' = (a'_\alpha)_\alpha$ belongs to $\varprojlim A_\alpha$ then $b = (b_\alpha)_\alpha = f((a'_\alpha)_\alpha)$ which completes the proof. It is enough to show that $g_{\alpha\beta} p_\alpha(a') = p_\beta(a')$. One knows that $g'_{\alpha\beta} f_\alpha(a'_\alpha) = g'_{\alpha\beta} p'_\alpha((f_\alpha(a'_\alpha))_\alpha) = g'_{\alpha\beta} p'_\alpha((b_\alpha)_\alpha) = p'_\beta((f_\alpha(a'_\alpha))_\alpha) = f_\beta(a'_\beta)$. Thus $f_\beta g_{\alpha\beta}(a'_\alpha) = g'_{\alpha\beta} f_\alpha(a'_\alpha) = f_\beta(a'_\beta)$, which implies that $g_{\alpha\beta} p_\alpha(a') = g_{\alpha\beta}(a'_\alpha) = a'_\beta = p_\beta((a'_\alpha)_\alpha) = p_\beta(a')$ since f_β is an embedding. Therefore, f is an order embedding. □

2.2. **Colimits of Down Closed Embeddings.** This subsection is devoted to the study of the categorical properties of down closed embeddings related to colimits such as coproducts, direct sums, pushouts and directed colimits.

Proposition 2.9. *The class of down closed embeddings is closed under coproducts and direct sums.*

Proof. Consider the diagram

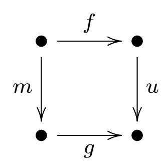


in which $\{f_i : A_i \to B_i : i \in I\}$ is a family of down closed embeddings. Let $f = \coprod_{i\in I} f_i : \coprod_{i\in I} A_i \to \coprod_{i\in I} B_i$ be the S-poset map satisfying $f(u_i(a_i)) = u'_i f_i(a_i)$, for $a_i \in A_i$, which exists by the universal property of coproducts; in fact, $f(a_i, i) = (f_i(a_i), i)$. We have to show that f is a down closed embedding. By [12], it is obvious that f is an embedding so it is enough to show that f is down closed. For, let

$(a,i) \in \coprod_{i\in I} A_i, (b,i) \in \coprod_{i\in I} B_i$ and $(b,i) \leq f(a,i) = (f_i(a),i)$. Then $b \leq f_i(a)$ and hence there exists $a' \in A_i$ such that $b = f_i(a')$, since for each $i \in I, f_i$ is down closed. Therefore, $(b,i) = f(a',i)$ and the proof is complete.

Now, let $\{f_i : A_i \to B_i : i \in I\}$ be a family of down closed embeddings. Then, by Proposition 2.4, one gets that $f = \bigoplus_{i\in I} f_i = \prod_{i\in I} f_i : \bigoplus_{i\in I} A_i \to \bigoplus_{i\in I} B_i$ is a down closed embedding. More precisely, if $(a_i)_{i\in I} \in \bigoplus_{i\in I} A_i, (b_i)_{i\in I} \in \bigoplus_{i\in I} B_i$ and $(b_i)_{i\in I} \leq f((a_i)_{i\in I}) = \bigoplus_{i\in I} f_i((a_i)_{i\in I}) = (f_i(a_i))_{i\in I}$, then, for each $i \in I, b_i \leq f_i(a_i)$ and since each f_i is down closed we get that there exists $a'_i \in A_i$ with $b_i = f_i(a'_i)$. Hence $(b_i)_{i\in I} = f((a'_i)_{i\in I})$ and the result holds. □

The category $\mathcal{A}$ is said to satisfy the $\mathcal{M}$-transferability property, for a subclass $\mathcal{M}$ of monomorphisms, if for all $f \in \mathcal{A}$ and $m \in \mathcal{M}$ with common domain there is a commutative diagram



with $u \in \mathcal{M}$.

The notion of $\mathcal{M}$-transferability property is used by universal algebrists whereas category theorists prefer "$\mathcal{M}$'s are preserved by pushouts" or "pushouts transfer $\mathcal{M}$'s" which is the same, provided pushouts exist and $\mathcal{M}$ is left cancellable.

Recall the following lemma from [14].

Lemma 2.10. *In the category of S-posets, pushouts transfer down closed embeddings.*

Proof. It is known that in the category **Pos-S**, pushouts transfer regular monomorphisms. To show that pushouts transfer down closed embeddings, we should show that if f is a down closed embedding, then so is q_C. Let $[x]_\theta \in (B \sqcup C)/\theta, [(2,c)]_\theta \in q_C(C)$ for some $c \in C$ and $[x]_\theta \leq [(2,c)]_\theta$, then either $[x]_\theta = [(1,b)]_\theta$ for some $b \in B$, or $[x]_\theta = [(2,c')]_\theta$ for some $c' \in C$. In the latter case, we have $[x]_\theta = [(2,c')]_\theta \in q_C(C)$ and so the result holds. In the former case, we have $[(1,b)]_\theta \leq [(2,c)]_\theta$. Now, $(1,b) \leq (2,c)$ (which is not possible by the definition of order on coproducts), or there exist $s_1, s_2, \ldots, s_n \in S$ such that $(1,b) \leq c_1 s_1$, $d_1 s_1 \leq c_2 s_2$, $d_2 s_2 \leq c_3 s_3$, ..., $d_n s_n \leq (2,c)$ where $(c_i,d_i) \in H \cup H^{-1}, i = 1,2,\ldots,n$. Thus there exist $a_1,\ldots,a_n \in A$ such that $(1,b) \leq (1,f(a_1))s_1$, $(2,g(a_1))s_1 \leq (2,g(a_2))s_2$, $(1,f(a_2))s_2 \leq (1,f(a_3))s_3$, ..., $(1,f(a_{n-1}))s_{n-1} \leq (1,f(a_n))s_n$, $(2,g(a_n))s_n \leq (2,c)$.

So $b \leq f(a_1 s_1)$. Since f is down closed, $b \in f(A)$. So there exists $a' \in A$ such that $b = f(a')$. Thus $[(1,b)] = q_B(b) = q_B(f(a')) = q_C(g(a'))$, and then $[(1,b)] \in q_C(C)$. Therefore, q_C is down closed. □

Note that since the composition of down closed embeddings is a down closed embedding, we have the direct corollary of the above proposition.

Proposition 2.11. *The pushout of down closed embeddings belongs to* $\mathcal{M}_{dc}$.

Proof. Applying the notations of Lemma 2.10, with a similar argument to its proof, one gets that when f and g in the pushout diagram are down closed embeddings, then so is $kf = hg$. □

For a class $\mathcal{E}$ of morphisms of a category, we say that multiple pushouts transfer $\mathcal{E}$-morphisms if in the multiple pushout $(Q, (A_i \stackrel{q_i}{\to} Q)_{i\in I})$ of a family $\{ f_i : A \to A_i | \, i \in I\}$ of $\mathcal{E}$-morphisms, $q_i \in \mathcal{E}$, for every $i \in I$.

Analogously to the pushouts, the following theorem is obtained.

Theorem 2.12. *Multiple pushouts transfer down closed embeddings.*

Proof. Assume that $(Q, (A_i \stackrel{q_i}{\to} Q)_{i\in I})$ is the multiple pushout of a family $\{ f_i : A \to A_i | \, i \in I\}$ of down closed embeddings. Recall that $Q = (\coprod A_i)/\theta(H)$, where $\theta = \theta(H)$ is the S-poset congruence on $\coprod A_i$ generated by $H = \{ (u_i(f_i(a)), u_j(f_j(a))) | \, a \in A, i, j \in I\}$, and $q_i = \pi u_i$, where $\pi : \coprod A_i \to Q$ is the natural map and $u_i : A_i \to \coprod A_i, i \in I$ are the coproduct injections. We prove that for each $i \in I, q_i$ is a down closed embedding. In [12], it is shown that each q_i is an embedding. So we show that for each $i \in I, q_i$ is down closed. Let $[x]_\theta \leq q_i(a_i) = [(i, a_i)]_\theta$ where $[x]_\theta \in (\coprod A_i)/\theta, [(i, a_i)]_\theta \in q_i(A_i)$ for some $a_i \in A_i$, then either $[x]_\theta = [(j, a_j)]_\theta$ for some $a_j \in A_j, j \neq i$, or $[x]_\theta = [(i, a'_i)]_\theta$ for some $a'_i \in A_i$. In the latter case, we have $[x]_\theta = [(i, a'_i)]_\theta \in q_i(A_i)$ and so the result holds. In the former case, we have $[(j, a_j)]_\theta \leq [(i, a_i)]_\theta$. Now, $(j, a_j) \leq [(i, a_i)]$ (which is not possible by the definition of order on coproducts), or there exist $s_1, s_2, \ldots, s_n \in S$ such that $(j, a_j) \leq c_1 s_1$, $d_1 s_1 \leq c_2 s_2$, $d_2 s_2 \leq c_3 s_3$, ..., $d_n s_n \leq (i, a_i)$ where $(c_i, d_i) \in H \cup H^{-1}, i = 1, 2, \ldots, n$.

Thus there exist $a_1, \ldots, a_n \in A$ such that $(j, a_j) \leq (j, f_j(a_1))s_1$, $(j_1, f_{j_1}(a_1))s_1 \leq (j_1, f_{j_1}(a_2))s_2$, $(j_2, f_{j_2}(a_2))s_2 \leq (j_2, f_{j_2}(a_3))s_3$, ..., $(j_{n-1}, f_{n-1}(a_{n-1}))s_{n-1} \leq (j_{n-1}, f_{n-1}(a_n))s_n$, $(i, f_i(a_n))s_n \leq (i, a_i)$.

So $a_j \leq f_j(a_1 s_1)$. Since f_j is down closed, there exists $a' \in A$ such that $a_j = f_j(a')$. Thus $[(j, a_j)] = q_j(a_j) = q_j(f_j(a')) = q_i(f_i(a'))$, and then $(j, a_j) \in q_i(A_i)$. Therefore, q_i is down closed. □

Corollary 2.13. *Every multiple pushout of down closed embeddings* (the diagonal maps on the multiple pushout diagram) is a down closed embedding.

Proof. Since the composition of down closed embeddings is a down closed embedding, the proof is clear by the above theorem. □

Definition 2.14. Let $\mathcal{M}$ be a class of morphisms of a category $\mathcal{C}$. We say that $\mathcal{C}$ has:

(i) $\mathcal{M}$-bounds if for every small and non-empty family $\{h_i : A \to B_i\}_{i\in I}$ of $\mathcal{M}$-morphisms, there is an $\mathcal{M}$-morphism $h : A \to B$ which factorizes through all h_i's.
(ii) $\mathcal{M}$-amalgamation property if in (i), h factorizes through all h_i's by $\mathcal{M}$-morphisms.

Since, by Proposition 2.12, multiple pushouts transfer down closed embeddings, the following is immediate:

Proposition 2.15. *The category of S-posets has $\mathcal{M}_{dc}$-amalgamation property and so also has $\mathcal{M}_{dc}$-bounds.*

Recall that a directed system of S-posets and S-poset maps is a family $(A_i)_{i\in I}$ of S-posets indexed by an up-directed set I endowed by a family $(\psi_{ij} : A_i \to A_j)_{i\leq j\in I}$ of S-poset maps such that given $i \leq j \leq k \in I$, $\psi_{ik} = \psi_{jk}\psi_{ij}$ and $\psi_{ii} = \mathrm{id}$. Also the pair $(\varinjlim A_i, \{\alpha_i : A_i \to \varinjlim A_i\})$ or in abbreviation, $\varinjlim A_i$ is called the directed colimit (or direct limit) of the directed system $((A_i)_{i\in I}, (\psi_{ij})_{i\leq j})$ if for every $i \leq j \in I$, $\alpha_j\psi_{ij} = \alpha_i$, and for every $(B, f_i : A_i \to B)$ with $f_j\psi_{ij} = f_i$, $i \leq j \in I$, there exists a unique S-poset map $\nu : \varinjlim A_i \to B$ such that $\nu\alpha_i = f_i$, for every $i \in I$.

Recall from [6] that the directed colimit of any up-directed system $((A_i)_{i\in I}, (\psi_{ij})_{i\leq j})$ of S-posets exists, and may be represented as $(A/\theta, (\psi_i : A_i \to A/\theta)_{i\in I})$, where

(1) $A = \coprod A_i$;
(2) $a\theta a' (a \in A_i, a' \in A_j)$ if and only if $\exists k \geq i, j : \psi_{ik}(a) = \psi_{jk}(a')$;
(3) $[a]_\theta \leq [a']_\theta (a \in A_i, a' \in A_j)$ if and only if $\exists k \geq i, j : \psi_{ik}(a) \leq \psi_{jk}(a')$;
(4) for each $i \in I$ and $a \in A_i$, $\psi_i(a) = [a]_\theta$.

Theorem 2.16. *Let I be an up-directed set and $\{h_\alpha : A_\alpha \to B_\alpha \mid \alpha \in I\}$ be a directed family of down closed embeddings. Then the directed colimit homomorphism induced by $h : \varinjlim A_\alpha \to \varinjlim B_\alpha$ is a down closed embedding.*

Proof. Let $(\varinjlim A_\alpha, f_\alpha), (\varinjlim B_\alpha, g_\alpha)$ be directed colimits of the directed systems $((A_\alpha)_{\alpha\in I}, (\psi_{\alpha\beta})_{\alpha\leq\beta})$ and $((B_\alpha)_{\alpha\in I}, (\varphi_{\alpha\beta})_{\alpha\leq\beta})$, respectively. Suppose $\{h_\alpha : A_\alpha \to B_\alpha \mid \alpha \in I\}$ is a directed family of down closed embeddings such that for every $\alpha \leq \beta$, $f_\beta\psi_{\alpha\beta} = f_\alpha$ and $g_\beta\varphi_{\alpha\beta} = g_\alpha$. Then $g_\beta h_\beta \psi_{\alpha\beta} = g_\beta\varphi_{\alpha\beta}h_\alpha = g_\alpha h_\alpha$. Thus $h = \varinjlim h_\alpha$ exists by the universal property of colimits. Consider $\varinjlim A_\alpha = \coprod_\alpha A_\alpha/\rho$ and $\varinjlim B_\alpha =$

$\coprod_\alpha B_\alpha/\rho'$. Recall from [12] that h is an embedding. So, it is enough to prove that h is a down closed map. Let $[b_j]_{\rho'} \leq h([a_i]_\rho) = [h_i(a_i)]_{\rho'}$. Therefore, $b_j \leq_{\rho'} h_i(a_i)$ and hence there exists $k \in I$ such that $k \geq i, j$ and $\varphi_{jk}(b_j) \leq \varphi_{ik} h_i(a_i) = h_k \psi_{ik}(a_i)$. Now, since h_k is a down closed embedding by hypothesis, we get $\varphi_{jk}(b_j) = h_k(a_k)$. Therefore, $h_k f_k(a_k) = h([a_k]_\rho) = g_k h_k(a_k) = g_k \varphi_{jk}(b_j) = g_j(b_j) = [b_j]_{\rho'}$ which shows that h is a down closed embedding. □

Corollary 2.17. **Pos**-S *has* $\mathcal{M}_{dc}$*-directed colimits.*

Proof. Assume that $(\varinjlim B_i, g_i)$ is the directed colimit of the directed system $((B_i)_{i\in I}, (\varphi_{ij})_{i\leq j})$, and $\{h_i : A \to B_i \mid i \in I\}$ is a directed family of down closed embeddings such that $g_j \varphi_{ij} = g_i$, for every $i \leq j$. Let $h : A \to \varinjlim B_i$ be the directed colimit of embeddings $h_i : A \to B_i$, $i \in I$. Recall that $h = \varinjlim h_i = g_i h_i = g_j h_j = g_k h_k = \ldots$. By [12], h is an embedding so we prove that h is down closed. Since each h_i is a down closed embedding and since each surjective map is down closed embedding, in view of Lemma 2.1 (i), h is a down closed embedding. □

Definition 2.18. Let $\mathcal{M}$ be a class of morphisms of a category $\mathcal{C}$. We say that $\mathcal{C}$ fulfills the $\mathcal{M}$-chain condition if for every directed system $((A_i)_{i\in I}, (\varphi_{ij})_{i\leq j\in I})$ whose index set I is a well-ordered chain with the least element 0, and $\varphi_{0i} \in \mathcal{M}$ for all i, there is a (so called "upper bound") family $(g_i : A_i \to A)_{i\in I}$ with $g_0 \in \mathcal{M}$ and $g_j \varphi_{ij} = g_i$.

Proposition 2.19. **Pos**-S *fulfills the* $\mathcal{M}_{dc}$*-chain condition.*

Proof. Take $A = \varinjlim A_i$ and let $g_i : A_i \to A$ be the colimit maps. Then, using Corollary 2.17, one gets the result . □

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Department of Mathematics, University of Maragheh, Maragheh, 55181-83111, Iran.

Email address: leilashahbaz@yahoo.com, l_shahbaz@maragheh.ac.ir