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L_p-C^*-Semi-Inner Product Spaces

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Abstract. This article introduces the notion of L_p-C^*-semi-inner product space, a generalization of the concept of C^*-semi-inner product space introduced by Gamchi et al., where we consider Hölder's inequality instead of Cauchy Schwartz' inequality. We establish some basic results L_p-C^*-semi-inner product spaces, analogous to those valid for C^*-semi-inner product spaces and Hilbert C^*-modules.

1. Introduction

In 1961, Lumer introduced semi-inner product spaces (s.i.p. spaces) as those spaces on which there exists a form $[u, v]$ which is linear in the first variable (but non-necessarily in the second), strictly positive, and satisfies Schwarz inequality [15]. But it is clear that the axioms defining these spaces are weaker than those determining Hilbert spaces. After that, Giles in [10] explained more about the concept of Lumer and get more results. In 1970, Nath presented a paper in which he considered Hölder's inequality instead of Cauchy-Schwarz's inequality and he called his space generalized semi-inner product space [18]. Abo Hadi continued Nath's studies and called these spaces semi-inner product space of type (p), and obtained an analogue of the Riesz Representation theorem in [1]. Husain and Malviya in 1973, in [12], by using the concept of semi-inner product space, introduced the concept of semi-inner product algebra. Hilbert C^*-Modules are a generalization of Hilbert spaces in which a general C^*-algebra is employed instead of the complex scalars. This generalization of Hibert spaces was considered for the first time

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in the work of Kaplansky [13] in the case of commutative C*-algebras. Paschke [19] and Rieffel [21] continued working on the general theory of Hilbert C*-modules (i.e., for an arbitrary C*-algebra). Finsler modules, introduced by Phillips and Weaver in 1998, are another generalization of Hilbert spaces (see [20]). Gamchi et al. in [7] introduced a new generalization of Hilbert C*-modules that is called C*-semi-inner product space or semi-inner product $A-$module which is related to Finsler modules. The notion is a generalization of a semi-inner product space by replacing the complex field by a general C*-algebra. In this article, we generalize the results of Gamchi et al. by considering Hölder's inequality instead of Cauchy Schwartz's inequality in the definition introduced by them in [7].

In Section 2, we introduce L_p-C*-semi-inner product spaces and we establish some basic properties and theorems of these spaces. Section 3 is dedicated to the notion of orthogonality in L_p-C*-semi-inner product space. Recently, anti-derivations have been a significant topic. Ghahramani et al. have done a lot of research in this field (see, for example, [8], [9]). Fadaee and Ghahramani researched about anti-derivation at orthogonal elements in [6]. After that, Abulhamil et al. continued the research on this topic in [2]. In the last section, we shall try to define the natural operator algebra associated with an L_p-C*-semi-inner product space. We shall study derivations and anti-derivations on the operator algebra of an L_p-C*-semi-inner product space and those continuous linear maps which are derivable at the unit element.

2. L_p-C*- Semi-Inner Product Spaces

A generalized semi-inner product on a space V over the real or complex field $\mathbb{F}$, is a mapping $[.,.] : V \times V \longrightarrow \mathbb{F},\ (u,v) \mapsto [u,v]$ satisfying the following axioms:

(i) $[u,u] > 0$ for $u \neq 0$,
(ii) $[\lambda u, v] = \lambda\,[u,v]$,
(iii) $[u+v,w] = [u,w] + [v,w]$,
(iv) $|[u,v]| \leq [u,u]^{\frac{1}{p}}\,[v,v]^{\frac{p-1}{p}}$.

A vector space V, with a generalized semi-inner product defined on it, is called a generalized semi-inner product space. In this case, $\|u\| = [u,u]^{\frac{1}{p}}$ define a norm on V. By putting $p = 2$ in the previous definition, we find semi-inner product spaces. Inspired by this definition, we are going to introduce the following structure:

Definition 2.1. Let V be a left $A-$module over a C*-algebra A. A pre-L_p-C*-semi-inner product or L_p-C*-s.i.p. in brief, is a mapping

$$[.,.] : V \times V \longrightarrow A, \tag{2.1}$$

satisfying the following properties:

(i) $[u,u] \geq 0$ for all u and $[u,u] = 0$ implies $u = 0$,
(ii) $[\lambda u + \mu v, w] = \lambda [u,w] + \mu [v,w]$ for $u,v,w \in V$ and $\lambda, \mu \in \mathbb{C}$,
(iii) $[au, v] = a [u,v]$, for $u, v \in V$ and $a \in A$,
(iv) $|[u,v]| \leq \|[u,u]\|^{\frac{1}{p}} [v,v]^{\frac{p-1}{p}}$.

For an element a in a C*-algebra A, the module, $|a|$, is defined as $|a|^2 = a^*a$. Given $1 < p < \infty$ the element $|a|^p$ is obtained via functional calculus. The triple $(V, A, [.,.])$ is called a pre-L_p-C*-semi-inner product space (in brief, pre-L_p-C*-s.i.p. space) or we say that V is a pre-L_p-semi-inner product $A-$module (in brief, pre-L_p-C*-s.i.p.$A-$module). The property (iv) is called Hölder's inequality.

Lemma 2.2. *Let V be a left $A-$module over a unital C*-algebra A. Let $(V, A, [.,.])$ be an pre-L_p-C*-s.i.p. space. If for any $u \in V$ we set $\|u\|_p = \|[u,u]\|^{\frac{1}{p}}$, then $\|u\|_p$ defines a norm on V.*

Proof. Clearly $\|u\|_p := \|[u,u]\|^{\frac{1}{p}} \geq 0$ and $\|u\|_p = 0$ results in that $[u,u]^{\frac{1}{p}} = 0$. So, $u = 0$. Now, let $\lambda \in \mathbb{F}$. If $\lambda = 0$, it's clear that $\|\lambda u\|_p = |\lambda| \, \|u\|_p$. So, let $\lambda \neq 0$. Since $[\lambda u, \lambda u] = \lambda [u, \lambda u]$, we have $|[\lambda u, \lambda u]| = |\lambda| \, |[u, \lambda u]|$. By property (i) of Definition 2.1, we can write $|[\lambda u, \lambda u]| = [\lambda u, \lambda u]$. Thus, by property (iv) of Definition 2.1, we see that for every $1 < p < \infty$,

$$|[u, \lambda u]| \leq \|[u,u]\|^{\frac{1}{p}} [\lambda u, \lambda u]^{\frac{p-1}{p}} .$$

Hence, we see

$$[\lambda u, \lambda u] \leq |\lambda| \, \|[u,u]\|^{\frac{1}{p}} [\lambda u, \lambda u]^{\frac{p-1}{p}} ,$$

and so,

$$|\lambda| \, \|[u,u]\|^{\frac{1}{p}} [\lambda u, \lambda u]^{\frac{p-1}{p}} 1 - [\lambda u, \lambda u] \geq 0.$$

On the other hand, since in the commutative C*-algebra generated by 1 and $[u,u]$ we have

$$\left(|\lambda| \, \|[u,u]\|^{\frac{1}{p}} 1 - [\lambda u, \lambda u]^{\frac{1}{p}} \right) \left([\lambda u, \lambda u]^{\frac{p-1}{p}} \right) \geq 0,$$

and $[\lambda u, \lambda u]^{\frac{p-1}{p}} \geq 0$, we deduce

$$|\lambda| \, \|[u,u]\|^{\frac{1}{p}} 1 - [\lambda u, \lambda u]^{\frac{1}{p}} \geq 0.$$

Therefore,

$$[\lambda u, \lambda u]^{\frac{1}{p}} \leq |\lambda| \, \|[u,u]\|^{\frac{1}{p}} . \tag{2.2}$$

So, we have $[\lambda u, \lambda u] \leq |\lambda|^p \|[u,u]\|$. By property (i) of Definition 2.1 and Theorem 2.2.5 in [17], we can coclude that $\|[\lambda u, \lambda u]\| \leq |\lambda|^p \|[u,u]\|$ and we get

$$\|[\lambda u, \lambda u]\|^{\frac{1}{p}} \leq |\lambda| \, \|[u,u]\|^{\frac{1}{p}}. \tag{2.3}$$

Furthermore, for every $\lambda \neq 0$,

$$[u,u]^{\frac{1}{p}} = \left[\frac{1}{\lambda}\lambda u, \frac{1}{\lambda}\lambda u\right]^{\frac{1}{p}}.$$

By (2.3),

$$\begin{aligned} \|[u,u]\|^{\frac{1}{p}} &= \left\| \left[\frac{1}{\lambda}(\lambda u), \frac{1}{\lambda}(\lambda u)\right] \right\|^{\frac{1}{p}} \\ &\leq \frac{1}{|\lambda|} \|[\lambda u, \lambda u]\|^{\frac{1}{p}}, \end{aligned}$$

and so

$$|\lambda| \, \|[u,u]\|^{\frac{1}{p}} \leq \|[\lambda u, \lambda u]\|^{\frac{1}{p}}. \tag{2.4}$$

By combining relationships (2.3) and (2.4), we see

$$\|[\lambda u, \lambda u]\|^{\frac{1}{p}} = |\lambda| \, \|[u,u]\|^{\frac{1}{p}}, \quad \text{i.e. } \|\lambda u\|_p = |\lambda| \, \|u\|_p.$$

To obtain the triangle inequality, we note that by property (i) of Definition 2.1,

$$[u+v, u+v] = [u, u+v] + [v, u+v].$$

So, we see

$$\begin{aligned} \|[u+v, u+v]\| &= \|[u, u+v] + [v, u+v]\| \\ &\leq \|[u, u+v]\| + \|[v, u+v]\| \end{aligned} \tag{2.5}$$

By property (iv) of Definition 2.1, for every $1 < p < \infty$,

$$|[u, u+v]| \leq \|[u,u]\|^{\frac{1}{p}} [u+v, u+v]^{\frac{p-1}{p}}. \tag{2.6}$$

Similarly, for every $1 < p < \infty$, we have

$$|[v, u+v]| \leq \|[v,v]\|^{\frac{1}{p}} [u+v, u+v]^{\frac{p-1}{p}}. \tag{2.7}$$

So, from (2.5), (2.6) and (2.7) is obtained,

$$\|[u+v, u+v]\| \leq \left[\|[u,u]\|^{\frac{1}{p}} + \|[v,v]\|^{\frac{1}{p}}\right] \|[u+v, u+v]\|^{\frac{p-1}{p}}.$$

Therefore,

$$\|[u+v, u+v]\|^{\frac{1}{p}} \leq \|[u,u]\|^{\frac{1}{p}} + \|[v,v]\|^{\frac{1}{p}}.$$

□

If V is complete with the norm $\|.\|_p$, we say that it is an L_p-C^*-semi-inner product space (in brief, L_p-C^*-s.i.p. space) or an L_p-semi-inner product $A-$module (in brief, L_p-C^*-s.i.p. $A-$module).

Example 2.3. Every Hilbert C^*-module is an example of L_p-C^*-s.i.p. space. But in general, the converse is not true. Example 2.4 illustrates this fact.

Example 2.4. Suppose I is a set and V_i is a generalized semi-inner product space with generalized semi-inner product $[.,.]_{V_i}, (i \in I)$ that $[.,.]_{V_i}$ is not linear or conjugate linear on the second variable (for some i) and norm

$$\|u\|_{V_i} = [u,u]_{V_i}^{\frac{1}{p}},$$

Let $B = \bigcup_i V_i$ and $\mathcal{A} = \ell_\infty(I)$, the set of all bounded complex-valued functions on I. Suppose V is the set of all maps $f : I \longrightarrow B$ such that $f(i) \in V_i$ for any $i \in I$, with $\sup_{i\in I} \|f(i)\|_{V_i} < \infty$. It is easy to see that V is an $\ell_\infty(I)-$module with pointwise module action. In addition, it is an example of an L_p-C^*-s.i.p. space with L_p-C^*-s.i.p. defined by

$$[f,h](i) = [f(i), h(i)]_{V_i}$$

for $i \in I$. V is not a Hilbert C^*-module because the L_p-C^*-s.i.p. is not linear or conjugate linear in both components.

Theorem 2.5. *Let $\psi : A \longrightarrow B$ be a $*-$isomorphism between two C^*-algebras A and B. If $(V, [.,.]_A, A)$ is an L_p-s.i.p. $A-$module, then V is a left $B-$module with the module action $\psi(a)u = au$ and it is an L_p-s.i.p. $B-$module with the L_p-C^*-s.i.p. defined by*

$$[.,.]_B = \psi([.,.]_A).$$

Proof. We shall only prove axiom (iv), of Definition 2.1, the proofs of axioms (i), (ii) and (iii) are straightforward. Let $v \in V$. ψ is a $*-$isomorphism from the C^*-algebra generated by 1 and $[v,v]_A$ to the C^*-algebra generated by 1 and $\psi([v,v]_A)$. Since $[v,v]_A$ is a (positive) normal element, by Theorem 2.1.14 in [17], we have

$$\psi\left([v,v]_A^{\frac{p-1}{p}}\right) = (\psi([v,v]_A))^{\frac{p-1}{p}}. \tag{2.8}$$

On the other hand, by $|[u,v]_A| \le \|[u,u]_A\|^{\frac{1}{p}} [u,v]_A^{\frac{p-1}{p}}$, we have

$$\psi(|[u,v]_A|) \le \|[u,u]_A\|^{\frac{1}{p}} \psi([v,v]_A)^{\frac{p-1}{p}}. \tag{2.9}$$

We also see that

$$\psi(|[u,v]_A|)^2 = \psi\left(|[u,v]_A|^2\right)$$

$$= \psi\left([u,v]_A^* [u,v]_A\right)$$
$$= \psi\left([u,v]_A\right)^* \psi\left([u,v]_A\right)$$
$$= \left|\psi\left([u,v]_A\right)\right|^2,$$

and so,

$$\psi\left(|[u,v]|\right) = \left|\psi\left([u,v]\right)\right|. \tag{2.10}$$

By (2.8), (2.9) and (2.10), we see

$$\left|\psi\left([u,v]_A\right)\right| \le \left\|\psi\left([u,u]_A\right)\right\|^{\frac{1}{p}} \psi\left([v,v]_A\right)^{\frac{p-1}{p}}.$$

Hence

$$\left|[u,v]_B\right| \le \left\|[u,u]_B\right\|^{\frac{1}{p}} [v,v]_B^{\frac{p-1}{p}}.$$

□

Theorem 2.6. *Suppose V is a left $A-$module over a C^*-algebra A, and let $[.,.]$ be an L_p-C^*-s.i.p. on V. Then, for any $a \in A$ and $u \in V$, we get*

$$\|au\|_p \le \|a\| \, \|u\|_p .$$

Proof. We see that

$$\|au\|_p^p = \|[au,au]\|$$
$$= \|a\,[u,au]\|$$
$$\le \|a\| \, \||[u,au]|\|$$
$$\le \|a\| \, \|[u,u]\|^{\frac{1}{p}} \, \|[au,au]\|^{\frac{p-1}{p}}.$$

Therefore,

$$\|au\|_p^p \le \|a\| \, \|u\|_p \, \|au\|^{p-1},$$

and so,

$$\|au\|_p \le \|a\| \, \|u\|_p .$$

□

Corollary 2.7. *In the L_p-C^*-s.i.p. space V, the module action $(a,u) \longrightarrow au$ is continuous.*

Definition 2.8. Suppose A is a C*-algebra. An L_p-C*-s.i.p. space V is called full if the linear span of $\{[u,v] : u,v \in V\}$, denoted by $[V,V]$, is dense in A.

Remark 2.9. Under the assumptions of Theorem 2.5, clearly V is a full L_p-s.i.p. A-module if and only if, it is a full L_p-s.i.p. B-module

Lemma 2.10. *Let A be a C^*-algebra and let V be a full L_p-C^*-s.i.p. space. Let $a \in A$ then $aV = \{av : v \in V\} = \{0\}$ if and only if $a = 0$.*

Proof. Let $a \in A$ be an arbitrary element. V is full, so there exists a sequence (x_n) of the form $x_n = \sum_{i=1}^{k_n} \lambda_{in} [u_{in}, v_{in}]$ for some $u_{in}, v_{in} \in V$ and $\lambda_{in} \in \mathbb{C}$ such that x_n tends to a^*. We have

$$\begin{aligned} aa^* &= \lim_n ax_n \\ &= \lim_n a \sum_{i=1}^{k_n} \lambda_{in} [u_{in}, v_{in}] \\ &= \lim_n \sum_{i=1}^{k_n} \lambda_{in} [au_{in}, v_{in}] \\ &= 0. \end{aligned}$$

So, $\|a\|^2 = \|a^*\|^2 = \|aa^*\| = 0$ and therefore $a = 0$. □

Theorem 2.11. *Let V be an L_p-s.i.p. $A-$module and a full L_p-s.i.p. $B-$module, and suppose $\psi : A \longrightarrow B$ is a $*-$map such that $au = \psi(a) u$ and $\psi([u, v]_A) = [u, v]_B$ where $u, v \in V$, $a \in A$. Then ψ is an $*-$isomorphism of C^*-algebras.*

Proof. At first, by $(\psi(a+b) - \psi(a) - \psi(b)) u = ((a+b) u - au - bu) = 0$, for each $u \in V$, we deduce $\psi(a+b) = \psi(a) + \psi(b)$. Similarly

$$\psi(\lambda a) = \lambda \psi(a),$$

and

$$\psi(ab) = \psi(a) \psi(b).$$

So, the $*-$mapping ψ is a $*-$homomorphism. We shall prove next that ψ is continuous. Let (a_n) be a sequence in A such that $(a_n) \to 0$ and $\psi(a_n) \to b$. Since the module actions of A and B on V are continuous, $a_n u \longrightarrow 0$, $\psi(a_n) u \longrightarrow bu$ and we deduce that $bu = 0$ for all $u \in V$. So, by Lemma 2.10 $b = 0$. Therefore, by closed graph theorem ψ is continuous. If $\psi(a) = 0$, then for every $u \in V$, we have $au = \psi(a) u = 0$. So, by Lemma 2.10, $a = 0$. Thus, ψ is one to one. Now, let $b \in B$ and $\epsilon > 0$. Since V is full L_p-s.i.p. $B-$module, there exist sequences $\{u_i\}_{1 \le i \le n}$ and $\{v_i\}_{1 \le i \le n}$ in V such that $\|b - \sum_{i=1}^{n} \lambda_i [u_i, v_i]_B\| < \epsilon$. So, $\|b - \psi(\sum_{i=1}^{n} \lambda_i [u_i, v_i]_A)\| < \epsilon$. Therefore, the range of ψ is dense. But ψ is a $*-$monomorphism between C^*-algebras and hence its range is closed. So, ψ is surjective. Hence, ψ is a $*-$isomorphism. □

Remark 2.12. Suppose A is a C^*-algebra and V is an L_p-C^*-s.i.p. space with L_p-C^*-s.i.p. denoted by $[u, v]$. If for any $a \in A$ and $u, v \in V$, we have $[u, av] = a^* [u, v]$, then one may see that $[\lambda u, v] = \overline{\lambda} [u, v]$, for all $u, v \in V$ and $\lambda \in \mathbb{C}$. Indeed, by hypothesis and Theorem 3.1.1 in [17] which affirms that every C*-algebra admits an approximate unit $\{u_\alpha\}$, we have

$$\begin{aligned}[u, \lambda v] &= \lim_\alpha u_\alpha [u, \lambda v] \\ &= \lim_\alpha [u, u_\alpha \lambda u] \\ &= \lim_\alpha (u_\alpha \lambda)^* [u, v] \\ &= \overline{\lambda} [u, v] .\end{aligned}$$

Corollary 2.13. *Let V be an L_p-s.i.p. $A-$module. If the equality $[u, av] = a^* [u, v]$ holds for any $u, v \in V$ and $a \in A$, then $p = 2$.*

Proof. By Lemma 2.2, $\|.\|_p$ is a norm on V and so, $\|\lambda u\|_p = |\lambda| \|u\|_p$, for all $u \in V$ and $\lambda \in \mathbb{F}$. On the other hand, by the previous remark,

$$\begin{aligned}\|\lambda u\|_p &= \|[\lambda u, \lambda u]\|^{\frac{1}{p}} \\ &= \left\|\lambda\overline{\lambda} [u, u]\right\|^{\frac{1}{p}} \\ &= |\lambda|^{\frac{2}{p}} \|u\|_p .\end{aligned}$$

Thus, $|\lambda| \|u\|_p = |\lambda|^{\frac{2}{p}} \|u\|_p$, for all $u \in V$ and $\lambda \in \mathbb{F}$ and so, $\dfrac{2}{p} = 1$. i.e. $p = 2$. □

Remark 2.14. By Corollary 2.13, if the equality $[u, av] = a^* [u, v]$ holds for any $u, v \in V$ and $a \in A$, then $p = 2$. So, every Hilbert C*-module is an L_2-C*-s.i.p. space.

3. Orthogonality in L_p-C*-Semi-Inner Product Spaces

Let $u, v \in V$, where V is a complex normed linear space . Then u is orthogonal to v in the Birkhoff-James sense, in short $u \perp_B v$, if

$$\|u + \lambda v\| \geq \|u\| , \quad \forall \lambda \in \mathbb{C}.$$

In this section, we want to express the relationship between orthogonality in L_p-C*-s.i.p. spaces and orthogonality in the Birkhoff-James sense.

Definition 3.1. Suppose V is an L_p-C*-s.i.p. and $u, v \in V$. We say u is orthogonal to v, if $[u, v] = 0$ and it is denoted by $u \perp_p v$.

Theorem 3.2. *Suppose V is a left $A-$module and $[.,.]$ is an L_p-C*-s.i.p. on V. If $u, v \in V$ and $v \perp_p u$ then, $u \perp_B v$.*

Proof. Let $[v, u] = 0$. If $u = 0$, we clearly have $u \perp_B v$. So, we suppose $u \neq 0$. Then, for all $\lambda \in \mathbb{K}$,

$$\|u\|_p^p - |\lambda| \|[v, u]\| \leq \|[u + \lambda v, u]\|$$

$$= \||[u+\lambda v, u]|\|$$
$$\leq \|[u+\lambda v, u+\lambda v]\|^{\frac{1}{p}} \left\|[u,u]^{\frac{p-1}{p}}\right\|$$
$$\leq \|u+\lambda v\|_p \|u\|_p^{p-1}.$$

Therefore,

$$- |\lambda| \|[v,u]\| \leq \|u\|_p^{p-1} \left(\|u+\lambda v\|_p - \|u\|_p\right). \tag{3.1}$$

But $u \neq 0$ and $[v,u]=0$, so by (3.1), we see

$$\|u+\lambda v\|_p \geq \|u\|_p,$$

and this shows that $u \perp_B v$. □

Definition 3.3. An L_p-C*-s.i.p. $[.,.]$ on a left $A-$module V is continuous, if for every $u,v \in V$ the equality

$$\lim_{\lambda\to 0} Re\,[u, v+\lambda u] = Re\,[u,v], \quad \lambda \in \mathbb{R},$$

is satisfied.

Theorem 3.4. *Suppose A is a unital C*-algebra and V is a left $A-$module. Let $[.,.]$ be a continuous L_p-C*-s.i.p. on V such that $[u,v] \in A_{sa}$, for every $u,v \in V$. Suppose that for $u,v \in V$ and $\lambda \in \mathbb{R}$, the inequality*

$$[u+\lambda v, u+\lambda v]^{\frac{1}{p}} \geq \|u\|_p,$$

holds. Then $[v,u]=0$ for all $u,v \in V$.

Proof. By assumption, for $u,v \in V$ and any $\lambda \in \mathbb{R}$,

$$[u+\lambda v, u+\lambda v]^{\frac{1}{p}} \geq \|u\|_p, \quad \text{i.e.}\ [u+\lambda v, u+\lambda v]^{\frac{1}{p}} \geq \|u\|_p 1. \tag{3.2}$$

So, Theorem 2.2.5(2) from [17] implies that by multiplying both sides of (3.2) by $[u+\lambda v, u+\lambda v]^{\frac{p-1}{2p}}$, we have

$$[u+\lambda v, u+\lambda v] \geq \|u\|_p [u+\lambda v, u+\lambda v]^{1-\frac{1}{p}}$$
$$\geq |[u, u+\lambda v]|$$
$$\geq [u, u+\lambda v].$$

Therefore, we deduce that

$$[u, u+\lambda v] + \lambda [v, u+\lambda v] \geq [u, u+\lambda v].$$

So, for any $\lambda \in \mathbb{R}$, we see $\lambda [v, u+\lambda v] \geq 0$. Therefore, for $\lambda \geq 0$, we have $[v, u+\lambda v] \geq 0$ and for $\lambda \leq 0$, we see $[v, u+\lambda v] \leq 0$. Now, since by assumption $[.,.]$ is a continuous L_p-C*-s.i.p. and A_+ (the set of all positive element of A) is closed subset of A, we get

$$0 \geq \lim_{\lambda\to 0^-} [v, u+\lambda v]$$

$$\begin{aligned} &= [v, u] \\ &= \lim_{\lambda \to 0^+} [v, u + \lambda v] \\ &\geq 0. \end{aligned}$$

So, $[v, u] = 0$. □

4. Derivation and Anti-Derivation on the Operator Algebras of an L_p-C*-s.i.p. Space

In this section, we introduce the operator algebra associated with an L_p-C*-s.i.p. space and we study derivations and anti-derivations on this algebra.

4.1. **Operator Algebra of L_p-C*-Semi-Inner Product Space.** Let V and W be L_p-C*-s.i.p. spaces. A bounded $\mathbb{C}-$linear $A-$homomorphism [i.e., $T(av) = aT(v)$ for each $a \in A$ and $v \in V$] from V to W is called an operator from V to W. By $Hom_{AL_p}(V, W)$ we shall denote the set of all operators from V to W. If $V = W$, then $Hom_{AL_p}(V, V)$ is denoted by $End_{AL_p}(V)$ and it is a Banach algebra.

Let V be an L_p-C*-s.i.p. space over a C*-algebra A. For any fixed vector $v \in V$, let define a map $f_v : V \longrightarrow A$ by $f_v(u) = [u, v]$ and denote the set $\{f_v; v \in V\}$ by V'.

In the following theorem, we prove f_v is an $A-$linear bounded operator and we try to find it's operator norm.

Theorem 4.1. *Suppose A is a C*-algebra and V is an L_p-C*-s.i.p. space. Then, for every $v \in V$, the mapping f_v is an $A-$linear continuous operator endowed with the norm $\|.\|_p$. Moreover, $\|f_v\| = \|v\|_p^{p-1}$.*

Proof. Clearly, by properties (ii) and (iii) of Definition 2.1, f_v is a linear and an $A-$linear operator. Now, by using property (iv) of Definition 2.1, we get

$$\begin{aligned} \|f_v(u)\| &= \|[u, v]\| \\ &\leq \|[u, u]\|^{\frac{1}{p}} \|[v, v]\|^{\frac{p-1}{p}} \\ &\leq \|u\|_p \|v\|_p^{p-1}. \end{aligned}$$

Thus,

$$\|f_v\| \leq \|v\|_p^{p-1},$$

and so f_v is bounded. On the other hand, we see

$$\|f_v\| \geq \left\| f_v\left(\frac{v}{\|v\|_p}\right) \right\|$$

$$
\begin{aligned}
&= \frac{\|f_v(v)\|}{\|v\|_p} \\
&= \frac{\|[v,v]\|}{\|v\|_p} \\
&= \frac{\|v\|_p^p}{\|v\|_p} \\
&= \|v\|_p^{p-1}.
\end{aligned}
$$

Hence, $\|f_v\| = \|v\|_p^{p-1}$. □

For any $w \in W$ and any $\mathbb{C}$-linear, bounded $A-$homomorphism $f : V \longrightarrow A$, we define the mapping $\theta_{w,f} : V \longrightarrow W$, by $\theta_{w,f}(v) = f(v)\, w$. Let V and W be L_p-C*-s.i.p. spaces, $v, v_0 \in V$ and $w \in W$. Similar to what we had in Hilbert C*-modules, we define the operator $\theta_{w,f_{v_0}}$ on V by,

$$
\begin{aligned}
\theta_{w,f_{v_0}}(v) &= f_{v_0}(v)\, w \\
&= [v, v_0]\, w.
\end{aligned} \tag{4.1}
$$

Clearly, $\theta_{w,f_{v_0}} : V \longrightarrow W$ is in $Hom_{AL_p}(V, W)$.

Lemma 4.2. *Let V and W be L_p-C*-s.i.p. spaces over a C*-algebra A. Let $f : V \to A$, $g : V \to A$ and $h : V \to A$ be $A-$homomorphism. Then, for any $v \in V$, $w \in W$ and $T \in Hom_{AL_p}(W, V)$, we have:*

- (i) $\theta_{w,f} \circ T = \theta_{w,f\circ T}$,
- (ii) $T\theta_{w,f} = \theta_{Tw,f}$,
- (iii) *If, in addition, A is commutative and $a \in A$, then* $\theta_{v,g}\theta_{w,h} = g(w)\,\theta_{v,h}$ *and* $\theta_{aw,g} = a\theta_{w,g}$.

Proof. (i) For each $w' \in W$, $\theta_{w,f}(T)(w') = \theta_{w,f}(T(w')) = f(T(w'))\, w = \theta_{w,foT}(w')$. So, we have

$$\theta_{w,f}(T) = \theta_{w,foT}.$$

(ii) For each $u \in V$, $T\theta_{w,f}(u) = T(\theta_{w,f}(u)) = T(f(u)\, w) = f(u)\, Tw = \theta_{Tw,f}(u)$. Therefore $T\theta_{w,f} = \theta_{Tw,f}$.

(iii) For each $u \in V$, $\theta_{v,g}\theta_{w,h}(u) = \theta_{v,g}(h(u)\, w) = h(u)\,\theta_{v,g}(w) = h(u)\, g(w)\, v = g(w)\, h(u)\, v = g(w)\,\theta_{v,h}(u)$. Since $u \in V$ is arbitrary, we have

$$\theta_{v,g}\theta_{w,h} = g(w)\,\theta_{v,h},$$

and

$$
\begin{aligned}
\theta_{aw,g}(u) &= g(u)\, aw \\
&= ag(u)\, w
\end{aligned}
$$

$$= a\theta_{w,g}(u).$$

So, $\theta_{aw,g} = a\theta_{w,g}$.

□

4.2. **Derivations and Anti-Derivations.** Let V be a Banach $A-$bimodule over an algebra A. A linear mapping δ from A to V is called a "derivation", if for any $a, b \in A$ the following identity holds:

$$\delta(ab) = \delta(a)b + a\delta(b). \tag{4.2}$$

If there exists $v \in V$ such that

$$\delta(a) = av - va, \quad \forall a \in A,$$

δ is called an "inner derivation". A linear mapping δ is called an "anti-derivation", if

$$\delta(ab) = b\delta(a) + \delta(b)a, \quad (a, b \in A). \tag{4.3}$$

When a mapping is said that is derivable or anti-derivation at a point like w, that's mean, the equations above, (4.2) and (4.3) hold only for $a, b \in A$ such that $w = ab$. In 1997, Semrl [22] introduced the notion of $2-$local derivations on the algebra $B(H)$ of all bounded linear operators on the infinite-dimensional separable Hilbert space H. A similar description for the finite-dimensional case appeared later in [14] and [16]. A non-necessarily linear map $\delta : A \longrightarrow V$ is called a $2-$local derivation, if for any $a, b \in A$, there exists a derivation $\delta_{a,b} : A \longrightarrow V$ such that, $\delta(a) = \delta_{a,b}(a)$ and $\delta(b) = \delta_{a,b}(b)$. The concept of $2-$local derivation is actually an important and interesting property for an algebra. For a given algebra A, the main problem concerning these notions is to prove that they automatically become a derivation. In Theorem 4.8, we will investigate about checking this fact on $End_{AL_p}(V)$. (see [3] and [4] for further reading). A Jordan algebra is an (associative) algebra A over a field with commutative multiplication that we denote by $\circ$ and satisfies the equation,

$$(a \circ b) \circ a^2 = a \circ \left(b \circ a^2\right), \quad (a, b \in A). \tag{4.4}$$

The identity (4.4) is called the "Jordan identity".

Every associative algebra equipped with the Jordan product denoted by $a \circ b = \frac{1}{2}(ab + ba)$ is a Jordan algebra.

A linear map $\delta : A \longrightarrow V$, from an (associative) algebra A into a Banach $A-$module V, satisfying $\delta(a \circ b) = \delta(a) \circ b + a \circ \delta(b)$, is called "Jordan derivation". Clearly, each derivation and each anti-derivation is a Jordan derivation. We recall that that an algebra A is called semi-prime algebra, if $\{0\}$ be the only its two-sided ideal I for which $I^2 = \{0\}$. One can easily check that A is semi-prime if and only if for any $a \in A$, the equality $aAa = \{0\}$ implies $a = 0$.

Lemma 4.3. *Let V be an L_p-C^*-s.i.p. space on C^*-algebra A. Then, $End_{AL_p}(V)$ is a semi-prime Banach algebra.*

Proof. Suppose $T \in End_{AL_p}(V)$ and for any $S \in End_{AL_p}(V)$, $TST = 0$. Especially, for each $v \in V$ and $f \in V'$, we get

$$\begin{aligned} T\theta_{v,f}Tv &= \theta_{Tv,f\circ T}v \\ &= f(Tv)\,Tv \\ &= 0. \end{aligned}$$

Let $w = Tv$ and $f = f'_w$. By (4.1), We see $[w,w]\,w = 0$. So

$$\begin{aligned} [[w,w]\,w,w] &= [w,w]^2 \\ &= 0. \end{aligned}$$

The element $[w,w]$ is self-adjoint, therefore $[w,w] = 0$, and so $w = 0$. Hence $Tv = 0$, for all $v \in V$ and so $T = 0$. Thus, $End_{AL_p}(V)$ is semi-prime. □

Remark 4.4. Let δ be an anti-derivation from $End_{AL_p}(V)$ into itself. It is clear that $\delta(S \circ T) = \delta(S) \circ T + S \circ \delta(T)$, $\left(S, T \in End_{AL_p}(V)\right)$, and hence, δ is Jordan derivation. On the other hand, by Lemma 4.3, $End_{AL_p}(V)$ is a semi-prime Banach algebra. By the results in [23], each Jordan derivation defined on a semi-prime Banach algebra is a derivation, so we deduce that δ is a derivation. By researching in L_p-C*-s.i.p. space, we find that many theorems on derivations on operator algebras of L_p-C*-s.i.p. spaces are similar to the corresponding theorems on derivations on operator algebra of Hilbert C*-modules. Here, we state some of these theorems, we shall see how their proofs are very similar to those given for the operator algebra of a Hilbert C*-module.

Theorem 4.5. *Suppose V is an L_p-C^*-s.i.p. space on a unital commutative C^*-algebra A. Then, every $A-$module homomorphism derivation on $End_{AL_p}(V)$ is continuous.*

Proof. The arguments in the proof of Lemma 2.2. in [11] work here. □

Theorem 4.6. *Let V be an L_p-C^*-s.i.p. space on a unital commutative C^*-algebra A. Then every $A-$module homomorphism derivation on $End_{AL_p}(V)$ is an inner derivation.*

Proof. The arguments in the proof of Theorem 2.3. in [11] work here. □

Lemma 4.7. *Let V be an L_p-C^*-s.i.p. space on a unital commutative C^*-algebra A. Let $v_i \in V$ and $f_i \in V'$. Then, $\sum_{i=1}^{n} \theta_{v_i,f_i} = 0$ implies $\sum_{i=1}^{n} f_i(v_i) = 0$.*

Proof. The arguments in the proof of Lemma 3.1. in [11] work here. □

Theorem 4.8. *Let V be L_p-C^*-s.i.p. space on a unital commutative C^*-algebra A. Then every $2-$local derivation that is $A-$module homomorphism on $End_{AL_p}(V)$ is a derivation.*

Proof. By Lemma 4.7, the arguments in the proof of Theorem 3.2. in [11] work here. □

Theorem 4.9. *Let V be an L_p-C^*-s.i.p. space on a unital commutative C^*-algebra A. Then every anti-derivation on $End_{AL_p}(V)$ that is an $A-$module homomorphism, is continuous.*

Proof. Let δ be an $A-$module anti-derivation on $End_{AL_p}(V)$ where V is an L$_p$-C*-s.i.p. space on commutative unital C*-algebra. Let (T_n) be a sequence in $End_{AL_p}(V)$ which converges to 0 and $\delta(T_n) \longrightarrow T$. We want to show that $T=0$. Then the desired result will follow from the closed graph theorem. By hypothesis, for every $f \in V'$ and $u, v \in V$, by Lemma 4.2, we have

$$\begin{aligned} \delta\left(\theta_{u,f} T_n \theta_{v,g}\right) &= \delta\left(\theta_{f(T_n v)u,g}\right) \\ &= \delta\left(f\left(T_n(v)\right)\theta_{u,g}\right) \\ &= f\left(T_n(v)\right)\delta\left(\theta_{u,g}\right) \longrightarrow 0, \quad as \quad n \to +\infty. \end{aligned} \tag{4.5}$$

Also,

$$\begin{aligned} \delta\left(\theta_{u,f} T_n \theta_{v,g}\right) &= \delta\left(\theta_{u,f}\theta_{T_n v,g}\right) \\ &= \theta_{T_n v,g}\delta\left(\theta_{u,f}\right) + \delta\left(\theta_{T_n v,g}\right)\theta_{u,f} \\ &= T_n\theta_{v,g}\delta\left(\theta_{u,f}\right) + \delta\left(T_n\theta_{v,g}\right)\theta_{u,f} \\ &= T_n\theta_{v,g}\delta\left(\theta_{u,f}\right) + \theta_{v,g}\delta\left(T_n\right)\theta_{u,f} + \delta\left(\theta_{v,g}\right)T_n\theta_{u,f}. \end{aligned}$$

The convergence of (T_n) to zero and (4.5) imply that

$$\begin{aligned} \lim_{n\to\infty} \delta\left(\theta_{u,f} T_n \theta_{v,g}\right) &= \lim_{n\to\infty} \theta_{v,g}\delta\left(T_n\right)\theta_{u,f} \\ &= \theta_{v,g} T \theta_{u,f} \\ &= 0. \end{aligned}$$

So, $g(Tu)\theta_{v,f} = 0$. Now, let $a = g(Tu)$, we have $a\theta_{v,f} = \theta_{av,f} = 0$. So, for every $w \in V$, we see

$$\begin{aligned} \theta_{av,f} w &= f(w)\, av \\ &= 0. \end{aligned} \tag{4.6}$$

Let, $f = f'_{av}$ and $w = av$ in (4.6), then, by (4.1), $av = 0$. So $g(Tu)v = 0$. Take $g = g'_{Tu}$ and $v = Tu$. Then, by (4.1) $Tu = 0$ and so $T = 0$. □

Theorem 4.10. *Let V be an L_p-C^*-s.i.p. space over a C^*-algebra A. Let δ be a bounded linear map on $End_{AL_p}(V)$ that for any $S, T \in End_{AL_p}(V)$ with $ST = I$, satisfying*

$$\delta(ST) = \delta(T)S + T\delta(S).$$

Then δ is a Jordan derivation ("I" is the identity mapping on $End_{AL_p}(V)$.).

Proof. At first, $\delta(I) = \delta(II) = 2\delta(I)$. So $\delta(I) = 0$. Let $S \in End_{AL_p}(V)$. Consider the operator $I - \lambda S$, for any scalars λ that $|\lambda| < \frac{1}{\|S\|}$. It is known that

$$(I - \lambda S)^{-1} = \sum_{n=0}^{\infty} \lambda^n S^n.$$

δ is bounded and linear, so

$$\begin{aligned}
0 &= \delta(I) \\
&= \delta\left((I - \lambda S)(I - \lambda S)^{-1}\right) \\
&= \delta(I - \lambda S)^{-1}(I - \lambda S) + (I - \lambda S)^{-1}\delta(I - \lambda S) \\
&= \delta\left(\sum_{n=0}^{\infty} \lambda^n S^n\right)(I - \lambda S) + \left(\sum_{n=0}^{\infty} \lambda^n S^n\right)(-\lambda\delta(S)) \\
&= \sum_{n=0}^{\infty} \lambda^n \delta(S^n)(I - \lambda S) - \sum_{n=0}^{\infty} \lambda^{n+1} S^n \delta(S) \\
&= \sum_{n=0}^{\infty} \lambda^n \delta(S^n) - \sum_{n=0}^{\infty} \lambda^{n+1}\delta(S^n)S - \sum_{n=0}^{\infty} \lambda^{n+1} S^n \delta(S) \\
&= \delta(I) + \sum_{n=1}^{\infty} \lambda^n \delta(S^n) - \sum_{n=1}^{\infty} \lambda^n \delta\left(S^{n-1}\right)S - \sum_{n=1}^{\infty} \lambda^n S^{n-1}\delta(S) \\
&= \sum_{n=1}^{\infty} \lambda^n \left(\delta(S^n) - \delta\left(S^{n-1}\right)S - S^{n-1}\delta(S)\right).
\end{aligned}$$

So,

$$\sum_{n=1}^{\infty} \lambda^n \left(\delta(S^n) - \delta\left(S^{n-1}\right)S - S^{n-1}\delta(S)\right) = 0. \tag{4.7}$$

The equation (4.7) is true for all λ that $|\lambda| < \frac{1}{\|S\|}$. Therefore,

$$\delta(S^n) - \delta\left(S^{n-1}\right)S - S^{n-1}\delta(S) = 0. \quad \forall n \in \mathbb{N} \tag{4.8}$$

In the case n = 2 we have $\delta\left(S^2\right) - \delta(S)S - S\delta(S) = 0$, for every $S \in End_{AL_p}(V)$. Hence, δ is a Jordan derivation. ∎

Corollary 4.11. *Suppose V is L_p-C^*-s.i.p. space. Suppose δ is a bounded linear map from $End_{AL_p}(V)$ into itself that for any $S, T \in End_{AL_p}(V)$ that $ST = I$, satisfying*

$$\delta(ST) = \delta(T)S + T\delta(S). \tag{4.9}$$

Then δ is a derivation.

Proof. By Theorem 4.10, δ is Jordan derivation . By [5], every Jordan derivation on a semi-prime Banach algebra will be derivation. So, the result is obtained. □

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