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Positivity of Integrals for Higher Order $\nabla-$Convex and Completely Monotonic Functions

Faraz Mehmood[1*], Asif Raza Khan[2] and Muhammad Adnan[3]

Abstract. We extend the definitions of $\nabla-$convex and completely monotonic functions for two variables. Some general identities of Popoviciu type integrals $\int P(y)f(y)dy$ and $\int\int P(y,z)f(y,z)dydz$ are deduced. Using obtained identities, positivity of these expressions are characterized for higher order $\nabla-$convex and completely monotonic functions. Some applications in terms of generalized Cauchy means and exponential convexity are given.

1. Introduction

Over past few decades the notion of completely monotonic functions has gained much popularity among researchers in analysis and other related fields due to their interesting properties (see [10]) and higher applicability (see [3]). As it is evident from the following lines taken from a paper with the title "Completely monotone functions: a digest" [14] written by Milan Merkle. He writes "A brief search in MathSciNet reveals total of 286 items that mention this class of functions in the title from 1932 till the end of the year 2011; 98 of them have been published since the beginning of 2006". We would like to obtain general integral identities and inequalitiies for higher order differentiable function of one variable and two variables, respectively. These identities and inequalities would be generalization of several established results. We would also discuss the characterization of Popoviciu$-$type positivity of these general integrals involving $\nabla-$convex and completely monotonic functions. We would give new generalized, mean value theorems of Lagrange and Cauchy$-$type as well, and also discuss exponential convexity with the help of various examples.

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Let us recall, few useful definitions and significant results regarding the convex functions from [7, 20] (see also [8, 12, 13]). Throughout the paper I is an interval in $\mathbb{R}$. Also throughout the paper we would use the following notations for some subsets of $\mathbb{R}$, $\mathbb{R}_* = [0,\infty)$ and $\mathbb{R}_+ = (0,\infty)$.

Definition 1.1. The $m-$divided difference of a function $f : I \to \mathbb{R}$, at different points $y_i, y_{i+1}, \ldots, y_{i+m} \in I = [a,b] \subset \mathbb{R}$ where $i \in \mathbb{N}$, $m \geq 0$, is stated as:

$$[y_j; f] = f\left(y_j\right), \quad j \in \{i, i+1, \ldots, i+m\},$$

and

$$\begin{aligned}\Delta_{(m)} f(y_i) &= [y_i, \ldots, y_{i+m}; f] \\ &= \frac{[y_{i+1}, \ldots, y_{i+m}; f] - [y_i, \ldots, y_{i+m-1}; f]}{y_{i+m} - y_i}.\end{aligned}$$

Remark 1.2. The value $[y_i, \ldots, y_{i+m}; f]$ is independent of order of points $y_i, y_{i+1}, \ldots, y_{i+m}$. This definition can extended by including the cases for more than one point coincide by applying the respective limits.

Definition 1.3. A function $f : I \to \mathbb{R}$, is called an $m-$convex or mth order convex , if the inequality $\Delta_{(m)} f(y_i) \geq 0$ holds $\forall (m+1)$ different points $y_i, \ldots, y_{i+m} \in I$.

Further, if mth order derivative of the function exists, then function is convex of order m if and only if $f^{(m)} \geq 0$.

Definition 1.4. A function $f : I \to \mathbb{R}$, is said to be $m - \nabla-$convex or $\nabla-$convex of order m, if $\forall (m+1)$ different points $y_i, y_{i+1}, \ldots, y_{i+m} \in I$, we have $\nabla_{(m)} f(y_i) = (-1)^m \Delta_{(m)} f(y_i) \geq 0$.

Further, if mth order derivative of the function exists, then function is $\nabla-$convex of order m if and only if $(-1)^m f^{(m)} \geq 0$.

We extend all the aforementioned definitions up to order (m,n). For that let us denote $I \times J = [a,b] \times [c,d] \subset \mathbb{R}^2$.

Definition 1.5. Let $f : I \times J \to \mathbb{R}$, be a function, then $(m,n)-$divided difference or divided difference of order (m,n) of a function f at different points $y_i, \ldots, y_{i+m} \in I$, $z_j, \ldots, z_{j+n} \in J$ for some $i, j \in \mathbb{N}$, is stated as

$$\Delta_{(m,n)} f(y_i, z_j) = [y_i, \ldots, y_{i+m}; [z_j, \ldots, z_{j+n}; f]] .$$

Definition 1.6. A function $f : I \times J \to \mathbb{R}$, is said to be $(m,n)-$convex or convex of order (m,n), if $\forall$ different points $y_i, \ldots, y_{i+m} \in I$ and $z_j, \ldots, z_{j+n} \in J$ we have $\Delta_{(m,n)} f(y_i, z_j) \geq 0$.

Further, the f is $(m,n)-$convex if and only if $f_{(m,n)} \geq 0$, if the partial derivative $\frac{\partial^{m+n} f}{\partial y^m \partial z^n}$ denoted by $f_{(m,n)}$, exists.

Further, in this paper we would use the following notations: $I \times J = [a,b] \times [c,d] \subset \mathbb{R} \times \mathbb{R}$. For some real sequence (a_m), $m \in \mathbb{N}$ and $n \in \{2, 3, \ldots\}$:

$$\nabla^{(1)} a_m = \nabla a_m = a_m - a_{m+1}, \qquad \nabla^{(n)} a_m = \nabla(\nabla^{(n-1)} a_m).$$

Also for m distinct real numbers y_i, $i \in \{1, \dots, m\}$ and $n \geq 0$:

$$(y_k - y_i)^{\{n+1\}} = (y_k - y_i)(y_{k-1} - y_i) \dots (y_{k-n} - y_i), \qquad (y_k - y_i)^{\{0\}} = 1.$$

Definition 1.7. Any function $f : I \times J \to \mathbb{R}$ is called the $(m,n)-\nabla-$convex if inequality $\nabla_{(m,n)} f(y_i, z_j) = (-1)^{m+n} \Delta_{(m,n)} f(y_i, z_j) \geq 0$ holds $\forall$ different points $y_i, \dots, y_{i+m} \in I$, $z_j, \dots, z_{j+n} \in J$.

Let us recall a useful definition from [11] (also see [15]).

Definition 1.8. A function $f : I \to \mathbb{R}$ is called the completely monotonic (or totally monotonic) of order m or $m-$completely monotonic if all its derivatives $f^{(i)}$ exist and satisfy

$$(-1)^i f^{(i)}(y) \geq 0, \quad y \in (0, \infty); \quad i \in \{0, 1, \dots, m\}, \quad \text{where } m \text{ is finite.}$$

In the following, the above definition is given for two variables.

Definition 1.9. We would call the function $f : I \times J \to \mathbb{R}$ completely monotonic of order (m,n) or $(m,n)-$completely monotonic if all its $f_{(i,j)}$ partial derivatives exist and satisfy the condition below:

$$(-1)^{(j+i)} f_{(i,j)}(y, z) \geq 0, \quad y, z, \in (0, \infty); \qquad i \in \{0, 1, 2, \cdots, m-1, m\}, \; j \in \{0, 1, 2, \dots, n-1, n\}.$$

Remark 1.10. It is simple to observe that the notions of completely monotonic function of order m and (m,n) are generalized notions of $m-\nabla-$convex function and $(m,n)-\nabla-$convex function, respectively if there exists differentiability.

1.1. **Examples.** In present subsection, we would use variety of classes of completely monotonic function $F = \{f_v : v \in I \subset \mathbb{R}\}$ and construct examples of completely monotonic function.

Example 1.11. Take a family of functions $F_1 = \{\psi_v : \mathbb{R} \to \mathbb{R}_+ | v \in \mathbb{R}_+\}$ which is stated as

$$\psi_v(y) = \frac{e^{-vy}}{v^m}.$$

Since $(-1)^i \frac{d^i}{dy^i} \psi_v(y) > 0$, therefore ψ_v is $m-$completely monotonic on $\mathbb{R}$, for every $v \in \mathbb{R}_+$.

Example 1.12. Take a family of functions $F_2 = \{\phi_v : \mathbb{R}_+ \to \mathbb{R} | v \in \mathbb{R}_+\}$ which is stated as

$$\phi_v(y) = \begin{cases} \frac{v^{-y}}{(\ln v)^m}, & v \neq 1, \\ \frac{(-1)^i y^m}{m!}, & v = 1. \end{cases}$$

We have $(-1)^i \frac{d^i}{dy^i} \phi_v(y) \geq 0$, therefore ϕ_v is $m-$completely monotonic on $\mathbb{R}_+$ for each $v \in \mathbb{R}_+, y \geq 0$.

Remark 1.13. Other examples of completely monotonic functions include:

(i) $f(y) = c$ (a nonnegative real constant), $\quad \forall y \in \mathbb{R}$

(ii) $f(y) = \frac{\alpha}{y^{1-\alpha}}, \quad 0 \leq \alpha \leq 1,\ y > 0$
(iii) $f(y) = \frac{1}{(y+\alpha^2)^\beta}, \quad \alpha \geq 0,\ \beta \geq 0,\ y > 0$
(iv) $f(y) = -\ln y, \quad \forall y \in \mathbb{R}_*$
(v) $f(y) = -\ln(1 - 1/y), \quad \forall y \in \mathbb{R}_+$
(vi) $f(y) = e^{1/y}, \quad \forall y \in \mathbb{R}_+$.

Let us give brief explanation the formate of paper as follows, after introduction and some preliminaries, in second section, we consider one identity for the integral $\int P(y)f(y)dy$ which involves functions of higher order derivatives. This identity is basic tool for obtaining necessary and sufficient conditions for every $(m+1)-\nabla-$convex function in which $\int P(y)f(y)dy \geq 0$ holds and only necessary condition for $(m+1)-$completely monotonic function. The third section is devoted to the integral case for $(M+1, N+1)-\nabla-$convex functions and $(M+1, N+1)-$completely monotonic functions for two variables, then consider an identity of linear functional $\Lambda(f)$ in double integral. In fourth section we state some mean value theorems of Lagrange and Cauchy types. In fifth section, consider the nonnegative functional $\Lambda(f)$ and apply this on exponentially convex functions $\psi^{(q)}$ of certain type and give some properties. In sixth section, construct examples and applications of completely monotonic, exponentially convex functions by various classes of functions. In last section, give conclusion about the paper.

2. Integral Case for Function of One Variable

In the paper [16] the following result for a real sequence (a_M) was proved:

Proposition 2.1. *Let $p_i \in \mathbb{R}$ for $i \in \{1, \ldots, M\}$, then the following identity for any real sequence (a_M) holds:*

$$\sum_{i=1}^{M} p_i a_i = \sum_{k=0}^{m-1} \frac{1}{k!} \nabla^{(k)} a_{M-k} \sum_{i=1}^{M-k} (M-i)^{\{k\}} p_i + \frac{1}{(m-1)!} \sum_{k=1}^{M-m} \left(\sum_{i=1}^{k} (k-i+m-1)^{\{m-1\}} p_i \right) \nabla^{(m)} a_k. \tag{2.1}$$

Similar result was proved in [7] for the real function involving the operator ∇ and it is a generalization of (2.1) which may be stated as:

Proposition 2.2. *Let m, M be positive integers such that $m \leq M$ and let p_i be real number for $i \in \{1, 2, \cdots, M\}$. Let f be a function and y_i be non mutual element from interval I for $i \in \{1, 2, \ldots, M\}$, then the following identity holds:*

(2.2)
$$\sum_{i=1}^{M} p_i f(y_i) = \sum_{k=0}^{m-1} \left(\sum_{j=1}^{M-k} p_j (y_M - y_j)^{\{k\}} \right) \nabla_{(k)} f(y_{M-k})$$

$$+\sum_{k=1}^{M-m}\left(\sum_{j=1}^{k}p_j(y_{k+m-1}-y_j)^{\{m-1\}}\right)\nabla_{(m)}f(y_k)(y_{k+m}-y_k).$$

We can also prove an integral identity that is analogous to the above formula.

Theorem 2.3. *Let a function $f \in C^{(m+1)}$ and $P, f : I \to \mathbb{R}$, both be integrable functions, then*

(2.3)
$$\int_a^b f(y)P(y)dy = \sum_{i=0}^{m}\left(\int_a^b P(y)\frac{(b-y)^i}{i!}dy\right)(-1)^i f^{(i)}(b) + \int_a^b\left(\int_a^s P(y)\frac{(s-y)^m}{m!}dy\right)(-1)^{m+1}f^{(m+1)}(s)ds.$$

Proof. The function f by using Taylor expansion can be represented as

$$\begin{aligned} f(y) &= \sum_{i=0}^{m} f^{(i)}(b)\frac{(y-b)^i}{i!} + \int_b^y f^{(m+1)}(s)\frac{(y-s)^m}{m!}ds \\ &= \sum_{i=0}^{m}(-1)^i f^{(i)}(b)\frac{(b-y)^i}{i!} + \int_y^b (-1)^{m+1}f^{(m+1)}(s)\frac{(s-y)^m}{m!}ds. \end{aligned}$$

Multiplying the above equation by P and integrate it over $[a, b]$, then

$$\begin{aligned} \int_a^b f(y)P(y)dy &= \sum_{i=0}^{m}(-1)^i f^{(i)}(b)\int_a^b P(y)\frac{(b-y)^i}{i!}dy \\ &\quad + \int_a^b\left(\int_y^b (-1)^{m+1}f^{(m+1)}(s)ds\right)P(y)\frac{(s-y)^m}{m!}dy \\ &= \sum_{i=0}^{m}\left(\int_a^b P(y)\frac{(b-y)^i}{i!}dy\right)(-1)^i f^{(i)}(b) \\ &\quad + \int_a^b\left(\int_a^s P(y)\frac{(s-y)^m}{m!}dy\right)(-1)^{m+1}f^{(m+1)}(s)ds. \end{aligned}$$

We used the Fubini theorem in above last equation for the variables y and s. □

The following theorem is generalized form of result (see [18, pp. 121–122]) and (see also [7]).

Theorem 2.4. *Let supposition of the Theorem 2.3 be true, then following inequality holds*

$$\int_a^b f(y)P(y)dy \geq 0, \tag{2.4}$$

for all $(m+1)-\nabla-$*convex function* f, *if and only if*

$$\int_a^b P(y)\frac{(b-y)^i}{i!}dy = 0, \quad i \in \{1,\dots,m\} \tag{2.5}$$

$$\int_a^s P(y)\frac{(s-y)^m}{m!}dy \geq 0, \quad \forall s \in [a,b]. \tag{2.6}$$

Proof. If (2.5) holds, then the first sum is zero in (2.3) and required inequality (2.4) holds by applying (2.6).

Conversely, if substitute the following functions in (2.4), then

$$f_1(y) = \frac{(b-y)^i}{i!}, \qquad f_2 = -f_1,$$

for $0 \leq i \leq m$ such that $(-1)^{m+1} f_l^{(m+1)}(y) \geq 0$, $l \in \{1,2\}$, then obtain required equality (2.5) i.e.

$$\int_a^b P(y)\frac{(b-y)^i}{i!}dy = 0, \quad 0 \leq i \leq m.$$

We get the last inequality (2.6) by assuming below function in (2.4), where $s \in [a,b]$

$$f_3(y) = \begin{cases} \frac{(s-y)^m}{m!}, & y < s, \\ 0, & y \geq s. \end{cases}$$

□

Theorem 2.5. *Let supposition of the Theorem* 2.3 *be true, then following inequality holds*

$$\int_a^b P(y)f(y)dy \geq 0, \tag{2.7}$$

for every completely monotonic function f *of order* $m+1$ *if*

$$\int_a^b P(y)\frac{(b-y)^i}{i!}dy = 0, \quad i \in \{1,\dots,m\}, \tag{2.8}$$

$$\int_a^s P(y)\frac{(s-y)^m}{m!}dy \geq 0, \quad \forall s \in [a,b]. \tag{2.9}$$

Proof. If (2.8) holds, then the first sum is zero in (2.3) and the required inequality (2.7) holds by using (2.9). □

Remark 2.6. Previous result also holds for every exponentially convex function [6]. Moreover, every completely monotonic function is log-convex so the stated result also holds for every log-convex function [14].

3. Integral Case for Functions of Two Variables

Under continuing heading, we can suppose function in y and z variables which is defined on the interval $I \times J = [a,b] \times [c,d]$. Moreover, $m, n, M, N \in$

$\mathbb{N}\cup\{0\}$ throughout the section, and useful notations are:

$$f_{(0,0)} = f, \qquad f_{(1,0)} = \frac{\partial f}{\partial y}, \qquad f_{(0,1)} = \frac{\partial f}{\partial z},$$

$$f_{(1,1)} = \frac{\partial^2 f}{\partial y \partial z} = \frac{\partial^2 f}{\partial z \partial y}, \qquad f_{(i,j)} = \frac{\partial^{i+j} f}{\partial y^i \partial z^j} = \frac{\partial^{i+j} f}{\partial z^j \partial y^i}.$$

Now we recall a result from [7] which would be helpful to prove our next main result:

Lemma 3.1. *Let f has continuous partial derivatives $f_{(i,j)}$ and $P, f : I \times J \to \mathbb{R}$ be both integrable functions, where $i \in \{0, 1, 2, \ldots, M, M+1\}$, $j \in \{0, 1, 2, \ldots, N, N+1\}$, then*

(3.1)

$$\begin{aligned}
&\int_a^b \int_c^d P(y,z) f(y,z)\,dz\,dy \\
&= \sum_{i=0}^{M} \sum_{j=0}^{N} \int_a^b \int_c^d P(s,t) \frac{(s-a)^i}{i!} \frac{(t-c)^j}{j!} f_{(i,j)}(a,c)\,dt\,ds \\
&\quad + \sum_{j=0}^{N} \int_a^b \int_y^b \int_c^d P(s,t) \frac{(s-y)^M}{M!} \frac{(t-c)^j}{j!} f_{(M+1,j)}(y,c)\,dt\,ds\,dy \\
&\quad + \sum_{i=0}^{M} \int_c^d \int_a^b \int_z^d P(s,t) \frac{(s-a)^i}{i!} \frac{(t-z)^N}{N!} f_{(i,N+1)}(a,z)\,dt\,ds\,dz \\
&\quad + \int_a^b \int_c^d \int_y^b \int_z^d P(s,t) \frac{(s-y)^M}{M!} \frac{(t-z)^N}{N!} f_{(M+1,N+1)}(y,z)\,dt\,ds\,dz\,dy.
\end{aligned}$$

Theorem 3.2. *Let f has continuous partial derivatives $f_{(i,j)}$ and $P, f : I \times J \to \mathbb{R}$ be both integrable functions, where $i \in \{0, 1, 2, \ldots, M, M+1\}$, $j \in \{0, 1, 2, \ldots, N, N+1\}$, then*

(3.2)

$$\begin{aligned}
&\int_a^b \int_c^d P(y,z) f(y,z)\,dz\,dy \\
&= \sum_{i=0}^{M} \sum_{j=0}^{N} \int_a^b \int_c^d P(y,z) \frac{(b-y)^i}{i!} \frac{(d-z)^j}{j!} (-1)^{i+j} f_{(i,j)}(b,d)\,dz\,dy \\
&\quad + \sum_{j=0}^{N} \int_a^b \int_a^s \int_c^d P(y,z) \frac{(s-y)^M}{M!} \frac{(d-z)^j}{j!} (-1)^{M+j+1} f_{(M+1,j)}(s,d)\,dz\,dy\,ds \\
&\quad + \sum_{i=0}^{M} \int_c^d \int_a^b \int_c^t P(y,z) \frac{(b-y)^i}{i!} \frac{(t-z)^N}{N!} (-1)^{i+N+1} f_{(i,N+1)}(b,t)\,dz\,dy\,dt \\
&\quad + \int_a^b \int_c^d \int_a^s \int_c^t P(y,z) \frac{(s-y)^M}{M!} \frac{(t-z)^N}{N!} (-1)^{M+N} f_{(M+1,N+1)}(s,t)\,dz\,dy\,dt\,ds.
\end{aligned}$$

Proof. We restate the identity given in Lemma 3.1 as follows:

(3.3)

$$\begin{aligned}
&\int_A^B \int_C^D P(y,z)f(y,z)dzdy \\
&\quad = \sum_{i=0}^{M}\sum_{j=0}^{N} \int_A^B \int_C^D P(s,t)\frac{(s-A)^i}{i!}\frac{(t-C)^j}{j!} f_{(i,j)}(A,C)dtds \\
&\qquad + \sum_{j=0}^{N} \int_A^B \int_y^B \int_C^D P(s,t)\frac{(s-y)^M}{M!}\frac{(t-C)^j}{j!} f_{(M+1,j)}(y,C)dtdsdy \\
&\qquad + \sum_{i=0}^{M} \int_C^D \int_A^B \int_z^D P(s,t)\frac{(s-A)^i}{i!}\frac{(t-z)^N}{N!} f_{(i,N+1)}(A,z)dtdsdz \\
&\qquad + \int_A^B \int_C^D \int_y^B \int_z^D P(s,t)\frac{(s-y)^M}{M!}\frac{(t-z)^N}{N!} f_{(M+1,N+1)}(y,z)dtdsdzdy.
\end{aligned}$$

Let us substitute $[A,B]=[b,a]$ and $[C,D]=[d,c]$. Then $\int_A^B = \int_b^a = -\int_a^b$ etc. and we change the variables names $y \leftrightarrow s$, $z \leftrightarrow t$, then the right hand side

(3.4)

$$\begin{aligned}
&\int_b^a \int_d^c P(y,z)f(y,z)dzdy \\
&\quad = \sum_{i=0}^{M}\sum_{j=0}^{N} \int_b^a \int_d^c P(y,z)\frac{(y-b)^i}{i!}\frac{(z-d)^j}{j!} f_{(i,j)}(b,d)dzdy \\
&\qquad + \sum_{j=0}^{N} \int_b^a \int_s^a \int_d^c P(y,z)\frac{(y-s)^M}{M!}\frac{(z-d)^j}{j!} f_{(M+1,j)}(s,d)dzdyds \\
&\qquad + \sum_{i=0}^{M} \int_d^c \int_b^a \int_t^c P(y,z)\frac{(y-b)^i}{i!}\frac{(z-t)^N}{N!} f_{(i,N+1)}(b,t)dzdydt \\
&\qquad + \int_b^a \int_d^c \int_s^a \int_t^c P(y,z)\frac{(y-s)^M}{M!}\frac{(z-t)^N}{N!} f_{(M+1,N+1)}(s,t)dzdydtds.
\end{aligned}$$

The left hand side of the (3.4) may be written as

$$\begin{aligned}
\int_b^a \int_d^c f(y,z)P(y,z)dzdy &= \int_a^b \int_c^d (-1)^2 f(y,z)P(y,z)dzdy \\
&= \int_a^b \int_c^d f(y,z)P(y,z)dzdy.
\end{aligned}$$

We can write the first summand on right hand side as

$$\sum_{i=0}^{M}\sum_{j=0}^{N}\int_b^a\int_d^c P(y,z)\frac{(y-b)^i}{i!}\frac{(z-d)^j}{j!}f_{(i,j)}(b,d)dzdy$$
$$=\sum_{i=0}^{M}\sum_{j=0}^{N}\int_a^b\int_c^d(-1)^2P(y,z)(-1)^i\frac{(b-y)^i}{i!}(-1)^j\frac{(d-z)^j}{j!}f_{(i,j)}(b,d)dzdy$$
$$=\sum_{i=0}^{M}\sum_{j=0}^{N}\int_a^b\int_c^d(-1)^{i+j}P(y,z)\frac{(b-y)^i}{i!}\frac{(d-z)^j}{j!}f_{(i,j)}(b,d)dzdy.$$

Also write the second summand on right hand side as

$$\sum_{j=0}^{N}\int_b^a\int_s^a\int_d^c P(y,z)\frac{(y-s)^M}{M!}\frac{(z-d)^j}{j!}f_{(M+1,j)}(s,d)dzdyds$$
$$=\sum_{j=0}^{N}\int_a^b\int_a^s\int_c^d(-1)^3P(y,z)(-1)^M\frac{(s-y)^M}{M!}(-1)^j\frac{(d-z)^j}{j!}f_{(M+1,j)}(s,d)dzdyds$$
$$=\sum_{j=0}^{N}\int_a^b\int_a^s\int_c^d(-1)^{M+1+j}P(y,z)\frac{(s-y)^M}{M!}\frac{(d-z)^j}{j!}f_{(M+1,j)}(s,d)dzdyds.$$

Similarly the third summand is rewritten as

$$\sum_{i=0}^{M}\int_d^c\int_b^a\int_t^c P(y,z)\frac{(y-b)^i}{i!}\frac{(z-t)^N}{N!}f_{(i,N+1)}(b,t)dzdydt$$
$$=\sum_{i=0}^{M}\int_c^d\int_a^b\int_c^t(-1)^3P(y,z)(-1)^i\frac{(b-y)^i}{i!}(-1)^N\frac{(t-z)^N}{N!}f_{(i,N+1)}(b,t)dzdydt$$
$$=\sum_{i=0}^{M}\int_c^d\int_a^b\int_c^t(-1)^{N+1+i}P(y,z)\frac{(b-y)^i}{i!}\frac{(t-z)^N}{N!}f_{(i,N+1)}(b,t)dzdydt.$$

Finally, last summand on right hand side rewritten as

$$\int_b^a\int_d^c\int_s^a\int_t^c P(y,z)\frac{(y-s)^M}{M!}\frac{(z-t)^N}{N!}f_{(M+1,N+1)}(s,t)dzdydtds$$
$$=\int_a^b\int_c^d\int_a^s\int_c^t(-1)^4P(y,z)(-1)^M\frac{(s-y)^M}{M!}(-1)^N\frac{(t-z)^N}{N!}f_{(M+1,N+1)}(s,t)dzdydtds$$
$$=\int_a^b\int_c^d\int_a^s\int_c^t(-1)^{M+N}P(y,z)\frac{(s-y)^M}{M!}\frac{(t-z)^N}{N!}f_{(M+1,N+1)}(s,t)dzdydtds.$$

By substituting all these expression in (3.4) we would arrive at our required result. □

Remark 3.3. (i) This result can also be obtained by using Taylor series expansion and by using Mathematical Induction.

(ii) If in Theorem 3.2, we replace $f(y,z)$ by $f(y)g(z)$, then we obtain the following statement.

Corollary 3.4. *Let $g \in C^{(N+1)}(J)$, $f \in C^{(M+1)}(I)$, be two different functions and $P : I \times J \to \mathbb{R}$, be an integrable function, then state following identity as:*

$$\begin{aligned}
&\int_a^b \int_c^d f(y,z)P(y,z)dzdy \\
&= \sum_{i=0}^{M}\sum_{j=0}^{N} \int_a^b \int_c^d P(y,z)(-1)^{i+j}\frac{(d-z)^j}{j!}g^{(j)}(d)\frac{(b-y)^i}{i!}f^{(i)}(b)dzdy \\
&\quad + \sum_{j=0}^{N} \int_a^b \int_a^s \int_c^d P(y,z)(-1)^{M+1+j}\frac{(d-z)^j}{j!}g^{(j)}(d)\frac{(s-y)^M}{M!}f^{(M+1)}(y)dzdyds \\
&\quad + \sum_{i=0}^{M} \int_c^d \int_a^b \int_c^t P(y,z)(-1)^{N+1+i}\frac{(t-z)^N}{N!}g^{(N+1)}(z)\frac{(b-y)^i}{i!}f^{(i)}(b)dzdydt \\
&\quad + \int_a^b \int_c^d \int_a^s \int_c^t P(y,z)(-1)^{M+N}\frac{(t-z)^N}{N!}g^{(N+1)}(z)\frac{(s-y)^M}{M!}f^{(M+1)}(y)dzdydtds.
\end{aligned}$$

We obtain necessary and sufficient conditions by using results of previous theorem that $\Lambda(f) \geq 0$ holds $\forall (M+1, N+1) - \nabla-$convex function and only necessary condition $\forall (M+1, N+1)-$completely monotonic function for two-variables function.

Theorem 3.5. *Let suppositions of Theorem* 3.2 *be true, then following inequality holds;*

$$\begin{aligned}
\Lambda(f) &= \int_a^b \int_c^d P(y,z)f(y,z)dzdy \\
&\geq 0,
\end{aligned} \tag{3.5}$$

for all $(M+1, N+1) - \nabla-$convex function f on $I \times J$, iff

$$\int_a^b \int_c^d P(y,z)\frac{(b-y)^i}{i!}\frac{(d-z)^j}{j!}dzdy = 0, \quad i \in \{0,\ldots,M\}; j \in \{0,\ldots,N\} \tag{3.6}$$

$$\int_a^s \int_c^d P(y,z)\frac{(s-y)^M}{M!}\frac{(d-z)^j}{j!}dzdy = 0, \quad j \in \{0,\ldots,N\}; \forall s \in [a,b] \tag{3.7}$$

$$\int_a^b \int_c^t P(y,z)\frac{(b-y)^i}{i!}\frac{(t-z)^N}{N!}dzdy = 0, \quad i \in \{0,\ldots,M\}; \forall t \in [c,d] \tag{3.8}$$

$$\int_a^s \int_c^t P(y,z)\frac{(s-y)^M}{M!}\frac{(t-z)^N}{N!}dzdy \geq 0, \quad \forall s \in [a,b]; \forall t \in [c,d]. \tag{3.9}$$

Proof. If (3.6), (3.7) and (3.8) hold, then first, second and third sums are zero in (3.2), then by using (3.9) we obtain the required inequality (3.5).

Conversely, if substitute the following functions in (3.5), then

$$f^1(y,z)=\frac{(b-y)^m}{m!}\frac{(d-z)^n}{n!},\qquad f^2=-f^1,$$

for $0\le m\le M$ and $0\le n\le N$ such that $(-1)^{M+N}f^l_{(M+1,N+1)}\ge 0$, $l\in\{1,2\}$, then obtain the desired equation (3.6) i.e.

$$\int_a^b\int_c^d P(y,z)\frac{(b-y)^m}{m!}\frac{(d-z)^n}{n!}dzdy=0,\quad \le m\le M,\, 0\le n\le N.$$

In the similar manner, if take the following functions in (3.5) $\forall s\in[a,b]$ and $0\le n\le N$

$$f^3(y,z)=\begin{cases}\frac{(s-y)^M}{M!}\frac{(d-z)^n}{n!}, & y<s\\ 0, & y\ge s\end{cases}\qquad f^4=-f^3,$$

such that $(-1)^{M+N}f^l_{(M+1,N+1)}\ge 0$, $l\in\{3,4\}$, we obtain desired equation (3.7) i.e.

$$\int_a^s\int_c^d P(y,z)\frac{(s-y)^M}{M!}\frac{(d-z)^n}{n!}dzdy=0,\quad 0\le n\le N;\,\forall s\in[a,b].$$

Similarly, if take the following functions in (3.5) $\forall t\in[c,d]$ and $0\le m\le M$

$$f^5(y,z)=\begin{cases}\frac{(b-y)^m}{m!}\frac{(t-z)^N}{N!}, & z<t\\ 0, & z\ge t\end{cases}\qquad f^6=-f^5,$$

such that $(-1)^{M+N}f^l_{(M+1,N+1)}\ge 0$, $l\in\{5,6\}$, we can obtain above equation (3.8) i.e.

$$\int_a^b\int_c^t P(y,z)\frac{(b-y)^m}{m!}\frac{(t-z)^N}{N!}dzdy=0,\quad 0\le m\le M;\,\forall t\in[c,d].$$

By considering the below function in (3.5), obtain the last inequality (3.9) for $s\in[a,b],\ t\in[c,d]$

$$f^7(y,z)=\begin{cases}\frac{(s-y)^M}{M!}\frac{(t-z)^N}{N!}, & y<s, \quad z<t\\ 0, & y\ge s \text{ or } z\ge t.\end{cases}$$ □

Theorem 3.6. *Let suppositions of Theorem* 3.2 *be true, then the following inequality holds;*

$$\Lambda(f)=\int_a^b\int_c^d P(y,z)f(y,z)dzdy \ge 0, \tag{3.10}$$

for all completely monotonic functions f of order $(M+1,N+1)$ on $I\times J$ if

(3.11)
$$\int_a^b\int_c^d P(y,z)\frac{(b-y)^i}{i!}\frac{(d-z)^j}{j!}dzdy=0,\quad i\in\{0,\dots,M\};j\in\{0,\dots,N\}$$

(3.12)
$$\int_a^s \int_c^d P(y,z)\frac{(s-y)^M}{M!}\frac{(d-z)^j}{j!}dzdy = 0, \quad j \in \{0,\dots,N\}; \forall s \in [a,b]$$
(3.13)
$$\int_a^b \int_c^t P(y,z)\frac{(b-y)^i}{i!}\frac{(t-z)^N}{N!}dzdy = 0, \quad i \in \{0,\dots,M\}; \forall t \in [c,d]$$
(3.14)
$$\int_a^s \int_c^t P(y,z)\frac{(s-y)^M}{M!}\frac{(t-z)^N}{N!}dzdy \geq 0, \quad \forall s \in [a,b]; \forall t \in [c,d].$$

Proof. If (3.11), (3.12) and (3.13) hold, then first, second and third sums are zero in (3.2), then by using (3.14) obtain the desired inequality (3.10). ☐

Remark 3.7. If we simply put $f(y,z) = f(y)g(z)$ in previous two results, then we obtain the similar results for two functions f and g.

For the study of discrete case for functions and sequences of two dimension see our paper [9].

4. Mean Value Theorems

It is known fact that the mean value theorem is valueable tool for obtaining interesting and important results of classical real analysis. In the field of differential calculus, the most demanding theorems are Lagrange and Cauchy mean value theorems. For more materials on this topic see [22]. Here, we would give some generalized mean value theorems of Lagrange and Cauchy-type.

Theorem 4.1. *Let $P : I \times J \to \mathbb{R}$, be an integrable function and $f \in C^{(M+1,N+1)}(I \times J)$, be a $(M+1, N+1)-\nabla-$convex function on the interval $I \times J$. Let Λ be a linear functional as stated in* (3.5) *and the conditions* (3.6)*,* (3.7)*,* (3.8) *and* (3.9) *be true for function P in Theorem* 3.5*, then there exists $(\eta, \zeta) \in I \times J$, such that*

$$\Lambda(f) = \Lambda(G_0) f_{(M+1,N+1)}(\eta, \zeta), \tag{4.1}$$

where
$$G_0(y,z) = (-1)^{M+N}\frac{y^{M+1}}{(M+1)!}\frac{z^{N+1}}{(N+1)!}.$$

Proof. Let
$$U = \max_{(y,z)\in I\times J} (-1)^{M+N} f_{(M+1,N+1)}(y,z),$$
$$L = \min_{(y,z)\in I\times J} (-1)^{M+N} f_{(M+1,N+1)}(y,z).$$

Then the function
$$G(y,z) = U(-1)^{M+N}\frac{y^{M+1}}{(M+1)!}\frac{z^{N+1}}{(N+1)!} - f(y,z)$$

$$= UG_0(y,z) - f(y,z),$$

gives us

$$(-1)^{M+N} G_{(M+1,N+1)}(y,z) = U - (-1)^{M+N} f_{(M+1,N+1)}(y,z) \geq 0$$

i.e. G is $\nabla-$convex function of order $(M+1, N+1)$ on $I \times J$. Hence $\Lambda(G) \geq 0$ using Theorem 3.5 and we would summerize that $\Lambda(f) \leq U\Lambda(G_0)$. Similarly, $L\Lambda(G_0) \leq \Lambda(f)$. Now, we can write the above two inequalities as: $L\Lambda(G_0) \leq \Lambda(f) \leq U\Lambda(G_0)$, which gives the required result (4.1). □

Theorem 4.2. *Let $f, g \in C^{(M+1,N+1)}(I \times J)$ be two $\nabla-$convex functions of order $(M+1, N+1)$ on the interval and $P : I \times J \to R$ be an integrable function. Let Λ be a linear functional as stated in* (3.5) *and the conditions* (3.6)*,* (3.7)*,* (3.8) *and* (3.9) *be true for function P in Theorem* 3.5*, then there exists $(\eta, \zeta) \in I \times J$, such that*

$$\frac{\Lambda(f)}{\Lambda(g)} = \frac{f_{(M+1,N+1)}(\eta,\zeta)}{g_{(M+1,N+1)}(\eta,\zeta)},$$

considering with non-zero denominators.

Proof. Let $u \in C^{(M+1,N+1)}$ be a $\nabla-$convex function of order $(M+1, N+1)$ on the interval $I \times J$, be stated as: $u = \Lambda(g)f - \Lambda(f)g$. Applying Theorem 4.1, there exists (η, ζ), such that

$$\begin{aligned} 0 &= \Lambda(u) \\ &= u_{(M+1,N+1)}(\eta,\zeta)\Lambda(G_0), \end{aligned}$$

or $\left[\Lambda(g) f_{(M+1,N+1)}(\eta,\zeta) - \Lambda(f) g_{(M+1,N+1)}(\eta,\zeta)\right] \Lambda(G_0) = 0$, which gives desired result. □

Corollary 4.3. *Let Λ be a linear functional as stated in* (3.5) *and the conditions* (3.6)*,* (3.7)*,* (3.8) *and* (3.9) *be true for function P with $N = M$ in Theorem* 3.5*, then there exists $(\eta, \zeta) \in I \times J$, such that*

$$(\eta\zeta)^{p-q} = \frac{[(q+1)q(q-1)\cdots(q-M+2)(q-M+1)]^2 \Lambda((yz)^{p+1})}{[(p+1)p(p-1)\cdots(p-M+2)(p-M+1)]^2 \Lambda((yz)^{q+1})},$$

where $p, q \notin \{-1, 0, 1, \ldots, M-1\}$, but lie on $-\infty < p \neq q < +\infty$.

Proof. Let $f(y,z) = (yz)^{p+1}$ and $g(y,z) = (yz)^{q+1}$. Then

$$f_{(M+1,M+1)} = [(p+1)p(p-1)\ldots(p-M+2)(p-M+1)]^2 (yz)^{p-M}$$

and

$$g_{(M+1,M+1)} = [(q+1)q(q-1)\ldots(q-M+2)(q-M+1)]^2 (yz)^{q-M}$$

$$\frac{f_{(M+1,M+1)}(\eta,\zeta)}{g_{(M+1,M+1)}(\eta,\zeta)} = \frac{[(p+1)p(p-1)\ldots(p-M+2)(p-M+1)]^2 (\eta\zeta)^{p-M}}{[(q+1)q(q-1)\ldots(q-M+2)(q-M+1)]^2 (\eta\zeta)^{q-M}}$$

$$= \frac{[(p+1)p(p-1)\dots(p-M+2)(p-M+1)]^2}{[(q+1)q(q-1)\dots(q-M+2)(q-M+1)]^2}(\eta\zeta)^{p-q}.$$

On the other hand, $\Lambda(f) = \Lambda((yz)^{p+1})$. So if we put all these in the equality of Theorem 4.2, we get

$$\frac{\Lambda(f)}{\Lambda(g)} = \frac{f_{(M+1,M+1)}(\eta,\zeta)}{g_{(M+1,M+1)}(\eta,\zeta)}$$

$$\frac{\Lambda((yz)^{p+1})}{\Lambda((yz)^{q+1})} = \frac{[(p+1)p(p-1)\dots(p-M+2)(p-M+1)]^2}{[(q+1)q(q-1)\dots(q-M+2)(q-M+1)]^2}(\eta\zeta)^{p-q}$$

$$(\eta\zeta)^{p-q} = \frac{[(q+1)q(q-1)\dots(q-M+2)(q-M+1)]^2\Lambda((yz)^{p+1})}{[(p+1)p(p-1)\dots(p-M+2)(p-M+1)]^2\Lambda((yz)^{q+1})}.$$

□

Remark 4.4. Here we observe that for the case $M = N$ the $(M+1, N+1)-\nabla-$convex function becomes $(M+1, M+1)-$convex function and hence we retrieve the results from [7].

5. Exponential Convexity

Here, In this section $J = (a, b) \subseteq \mathbb{R}$.

Definition 5.1 ([2])**.** A function $\omega : J \to \mathbb{R}$ is exponentially convex on open interval J, if ω is continuous and

$$\sum_{i,j=1}^{m} \rho_i\rho_j\omega(y_i + y_j) \geq 0,$$

$\forall m \in \mathbb{N}$ and $\forall\ \rho_i, \rho_j \in \mathbb{R}$; such that $y_i + y_j \in J$ and $i, j \in \{1, \dots, m\}$.

Example 5.2. A function $y \mapsto ce^{ky}$ is an example of exponentially convex for $k \in \mathbb{R}$ and $c \geq 0$.

Proposition 5.3 ([1])**.** *Take $\omega : J \to \mathbb{R}$, then given statements are similar:*
(i) *ω is exponentially convex on open interval J.*
(ii) *ω is continuous and*

$$\sum_{i,j=1}^{m} \rho_i\rho_j\omega\left(\frac{y_i + y_j}{2}\right) \geq 0,$$

$\forall \rho_i, \rho_j \in \mathbb{R}$ and every $y_i, y_j \in J$; $i, j \in \{1, 2, 3, \dots, m-1, m\}$.

Corollary 5.4. *If ω is exponentially convex function on open interval J, then the following matrix is a positive semi-definite matrix.*

$$\left[\omega\left(\frac{y_i + y_j}{2}\right)\right]_{i,j=1}^{m}.$$

Particularly, i.e. det $\left[\omega\left(\frac{y_i+y_j}{2}\right)\right]_{i,j=1}^{m} \geq 0$, $\forall m \in \mathbb{N}, y_i, y_j \in J$; $i, j \in \{1, 2, 3, \cdots, m-1, m\}$.

Corollary 5.5. *If $\omega : J \to \mathbb{R}_+$ is an exponentially convex function, then ω is a* log*-convex function, where $\forall y, z \in J$ and $\forall \lambda \in [0,1]$, we have*

$$\omega(\lambda y + (1-\lambda)z) \leq \omega^{\lambda}(y)\omega^{1-\lambda}(z).$$

Let $D = \{\psi^{(q)} : (0,\infty) \times (0,\infty) \to \mathbb{R} | q \in \mathbb{R}\}$ be a family of functions stated as:

$$\psi^{(q)}(y,z) = \begin{cases} \dfrac{(y+k_1)^q(z-k_2)^q}{[q(q-1)\cdots(q-M)]^2}, & q \notin \{0,1,\ldots,M\} \\ \dfrac{(y+k_1)^q(z-k_2)^q[\log(y+k_1)(z-k_2)]^2}{2[q!(M-q)!]^2}, & q \in \{0,1,\ldots,M\}. \end{cases}$$

Clearly

$$\begin{aligned} \psi^{(q)}_{(M+1,M+1)}(y,z) &= [(y+k_1)(z-k_2)]^{q-M-1} \\ &= e^{(q-M-1)\log[(y+k_1)(z-k_2)]} \end{aligned}$$

for $(y+k_1, z-k_2) \in (0,\infty)\times(0,\infty)$ and $\psi^{(q)}$ is completely monotonic function of order $(M+1, M+1)$ since $(-1)^{2i}\psi^{(q)}_{(i+1,i+1)}(y,z) \geq 0$ and $q \mapsto \psi^{(q)}_{(M+1,M+1)}$ is an exponentially convex function f on real numbers. We can say that every positive exponentially convex is log−convex function, by using the above mention Corollary 5.5. Now, at this stage we can give next theorem which is stated as:

Theorem 5.6. *Let Λ be a linear functional as stated in* (3.5) *and the conditions* (3.6), (3.7), (3.8) *and* (3.9) *be true for function P in Theorem* 3.5 *and $\psi^{(q)}$ be a completely monotonic function defined in previous, then the following points hold:*

(i) *$q \mapsto \Lambda(\psi^{(q)})$ is continuous on $\mathbb{R}$.*

(ii) *$q \mapsto \Lambda(\psi^{(q)})$ is exponentially convex function on $\mathbb{R}$.*

(iii) *If $q \mapsto \Lambda(\psi^{(q)})$ is positive function on $\mathbb{R}$, then the $q \mapsto \Lambda(\psi^{(q)})$ is log-convex on $\mathbb{R}$. Moreover, the following inequality holds for $r < s < t$; $r, s, t \in J$*

$$[\Lambda_k(f_s)]^{t-r} \leq [\Lambda_k(f_r)]^{t-s}[\Lambda_k(f_t)]^{s-r}. \tag{5.1}$$

(iv) *For every $m \in \mathbb{N}$ and $q_1, \ldots, q_m \in \mathbb{R}$, the following matrix is positive semi-definite.*

$$\left[\Lambda(\psi^{(\frac{q_i+q_j}{2})})\right]_{i,j=1}^{m}.$$

Particularly,

$$\det\left[\Lambda(\psi^{(\frac{q_i+q_j}{2})})\right]_{i,j=1}^{m} \geq 0.$$

(v) *If $q \mapsto \Lambda(\psi^{(q)})$ is differentiable on $\mathbb{R}$, and $\forall s, t, u, v \in \mathbb{R}$, such that $t \leq v$ and $s \leq u$, then we have*

$$\mathfrak{M}_{s,t}(y,z) \leq \mathfrak{M}_{u,v}(y,z), \tag{5.2}$$

where

$$\mathfrak{M}_{s,t}(y,z) = \begin{cases} \left(\frac{\Lambda(\psi^{(s)})}{\Lambda(\psi^{\{t\}})}\right)^{\frac{1}{s-t}}, & s \neq t, \\ \exp\left(\frac{\frac{d}{ds}\Lambda(\psi^{(s)})}{\Lambda(\psi^{(s)})}\right), & s = t, \end{cases}$$

for $\psi^{(s)}, \psi^{(t)} \in D$.

Proof. (i) For fixed $M \in \mathbb{N} \cup \{0\}$, using $L^{'}$Hôpital rule two times and then apply limit, we obtain

$$\begin{aligned}\lim_{q\to 0} \Lambda(\psi^{(q)}) &= \lim_{q\to 0} \frac{\int_a^b \int_a^b P(y,z)(y+k_1)^q (z-k_2)^q dzdy}{[q(q-1)\cdots(q-M)]^2} \\ &= \frac{\int_a^b \int_a^b P(y,z)[\log(y+k_1)(z-k_2)]^2 dzdy}{2[M!]^2} \\ &= \Lambda(\psi^{(0)}).\end{aligned}$$

Similarly, we can show

$$\lim_{q\to k} \Lambda(\psi^{(q)}) = \Lambda(\psi^{(k)}), \quad k \in \{1,2,\ldots,M\}.$$

(ii) Let us define the function

$$\eta(y,z) = \sum_{i,j=1}^{k} \alpha_i \alpha_j \psi^{\left(\frac{q_i+q_j}{2}\right)}(y,z),$$

$q_i \in \mathbb{R}, \alpha_i \in \mathbb{R}$ and $i \in \{1,2,\ldots,k\}$.
Since the function $q \mapsto \psi^{(q)}_{(M+1,M+1)}$ is exponentially convex function, we write

$$\begin{aligned}\eta_{(M+1,M+1)} &= \sum_{i,j=1}^{k} \alpha_i \alpha_j \psi^{\left(\frac{q_i+q_j}{2}\right)}_{(M+1,M+1)} \\ &\geq 0,\end{aligned}$$

which implies that η is $\nabla-$convex function of order $(M+1, M+1)$ on $\mathbb{R}_+ \times \mathbb{R}_+$ and we have $\Lambda(\eta) \geq 0$. Hence

$$\sum_{i,j=1}^{k} \alpha_i \alpha_j \Lambda\left(\psi^{\left(\frac{q_i+q_j}{2}\right)}\right) \geq 0.$$

On the behalf of above working we can summarize that function $q \to \Lambda\left(\psi^{(q)}\right)$ is exponentially convex on real numbers.

(iii) It follows from (ii) and Corollary 5.5. As the function $t \mapsto \Lambda(f_t)$ is log-convex i.e. $\ln \Lambda(f_t)$ is convex. Now using the definition of convex function from [20, p. 2], we have

$$(y_3 - y_2)\, f(y_1) + (y_1 - y_3)\, f(y_2) + (y_2 - y_1)\, f(y_3) \geq 0,$$

holds for each $y_1, y_2, y_3 \in I$ such that $y_1 < y_2 < y_3$, which gives (5.1), i.e.,

$$\ln[\Lambda(f_s)]^{t-r} \leq \ln[\Lambda(f_r)]^{t-s} + \ln[\Lambda(f_t)]^{s-r}.$$

(iv) It is consequence of Corollary 5.4.

(v) We recall another definition of convex function φ from [20, p.2]

$$\frac{\varphi(s) - \varphi(t)}{s-t} \leq \frac{\varphi(u) - \varphi(v)}{u-v}, \tag{5.3}$$

$\forall s, t, u, v \in J, \ni s \leq u, t \leq v, s \neq t, u \neq v.$

Since by (iii), $\Lambda(\psi^{(q)})$ is log-convex, so by setting $\varphi(y) = \log \Lambda(\psi^{(y)})$ in (5.3) we have

$$\frac{\log \Lambda(\psi^{(s)}) - \log \Lambda(\psi^{(t)})}{s-t} \leq \frac{\log \Lambda(\psi^{(u)}) - \log \Lambda(\psi^{(v)})}{u-v}, \tag{5.4}$$

for $s \leq u, t \leq v, s \neq t, u \neq v$, which is similar to (5.2). The cases for $s = t$ and/or $u = v$ are simply getting from (5.4) by using the respective limits. □

6. Examples with Applications

In this section of our paper, we would construct different examples of completely monotonic, exponentially convex functions and applications by using different classes of functions $F = \{f^q : q \in I \subset \mathbb{R}\}$. Let us consider the following examples.

Example 6.1. Let $F_1 = \{\zeta^q : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_* | q \in \mathbb{R}\}$ be a family of functions which is stated as

$$\zeta^q(y,z) = \begin{cases} \frac{e^{q(y+z)}}{q^{2M+2}}, & q \neq 0, \\ \frac{(y+z)^{2M+2}}{(2M+2)!}, & q = 0. \end{cases}$$

Clearly $\zeta^q_{(M+1,M+1)}(y,z) = e^{q(y+z)} > 0$ and the function $\zeta^q(y,z)$ is a $(M+1, M+1)-$completely monotonic on $\mathbb{R}_+ \times \mathbb{R}_+$ since $(-1)^{2i} \zeta^q_{(i+1,i+1)}(y,z) \geq 0$, $\forall\ q \in \mathbb{R}$ and $q \to \zeta^q_{(M+1,M+1)}(y,z)$ is exponentially convex by definition.

Example 6.2. Let $F_2 = \{\phi^q : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+ | q \in \mathbb{R}_+\}$ be a family of functions which is stated as

$$\phi^q(y,z) = \begin{cases} \frac{(yz)^q}{[q(q-1)\cdots(q-M)]^2}, & q \notin \{0,1,2,\ldots,M-1,M\}, \\ \frac{(yz)^q \ln(yz)^2}{2[q!(M-q)!]^2}, & q \in \{0,1,2,\ldots,M-1,M\}. \end{cases}$$

Clearly $\phi^q_{(M+1,M+1)}(y,z) = e^{(q-M-1)\ln(yz)} > 0$ and the function $\phi^q(y,z)$ is a $(M+1, M+1)-$completely monotonic on $\mathbb{R}_+ \times \mathbb{R}_+$ since $(-1)^{2i} \phi^q_{(i+1,i+1)}(y,z) \geq 0$, $\forall\ q \in \mathbb{R}$ and $q \to \phi^q(y,z)_{(M+1,M+1)}$ is exponentially convex by definition.

In all the above examples we can use similar arguments as in the proof of Theorem 5.6.

7. Conclusion

At the end of paper, we give conclusion that we have used variety of classes of completely monotonic function and on the basis of this we successfully constructed examples and applications of completely monotonic function. We obtained identity for the integral $\int_a^b P(y)f(y)dy$ which involves function of higher order derivatives. We have obtained necessary and sufficient conditions by result of Theorem 2.3 for every $(m+1)-\nabla-$convex function in which $\int_a^b P(y)f(y)dy \geq 0$ holds and also found positivity of result of Theorem 2.3 for $(m+1)-$completely monotonic function. We got general integral identity $\int_a^b \int_c^d P(y,z)f(y,z)dzdy$ in the interval $I \times J$ for higher order differentiable function of two independent variables of Popoviciu type and also obtained similar consequence as Theorem 3.2 for both functions f and g. We have also obtained necessary and sufficient conditions by result of Theorem 3.2 using linear functional $\Lambda(f)$ in double integral for all $(M+1, N+1)-\nabla-$convex function for two-variables and also found positivity of result of Theorem 3.2 using linear functional $\Lambda(f)$ for all $(M+1, N+1)-$completely monotonic function with two variables. Further, we have obtained some generalized mean value theorems of Lagrange and Cauchy-type involving $\Lambda(f)$ for $(M+1, N+1)-\nabla-$convex functions for two-variables. We applied nonnegative functional $\Lambda(f)$ on exponentially convex functions $\psi^{(q)}$ of certain type and gave some properties and in last constructed examples and applications of completely monotonic, exponentially convex functions by applying various classes of functions. Moreover, we conclude that every $\nabla-$convex function is a subclass of completely monotonic function if there exists differentiability.

References

1. M. Anwar, J. Jakšetić, J. Pečarić and A. Rehman, *Exponential convexity, positive semi-definite matrices and fundamental inequalities*, J. Math. Inequal., 4(2) (2010), pp. 171-189.
2. S.N. Bernstien, *Sur les fonctions absolument monotones*, Acta Math., 52 (1929), pp. 1-66.
3. G. Fasshauer, *Meshfree approximation methods with MATLAB*, World Scientific Publishing, Hackensack, 2007.
4. F. Hausdorff, *Summationsmethoden und Momentfolgen I*, Math. Z., 9 (1921), pp. 74-109.
5. F. Hausdorff, *Momentprobleme fur̈ ein endliches Intervall*, Math. Z., 16 (1923), pp. 220-248.
6. J. Jakšetić and J.E. Pečarić, *Exponential convexity method*, J. Convex Anal., 20(1) (2013), pp. 181-197.
7. A.R. Khan, J.E. Pečarić and S. Varošanec, *Popoviciu type characterization of positivity of sums and integrals for convex functions of higher order*, J. Math. Ineq., 7(2) (2013), pp. 195-212.
8. A.R. Khan and F. Mehmood, *Some Remarks on Functions with Non-decreasing Increments*, J. Math. Anal., 11(1) (2020), pp. 1-16.
9. A.R. Khan and F. Mehmood, *Positivity of Sums for Higher Order* $\nabla-$*Convex Sequences and Functions*, Glob. J. Pure Appl. Math., 16(1) (2020), pp. 93-105.

10. C.H. Kimberling, *A probabilistic interpretation of complete monotonicity*, Aequationes Math., 10 (1974), pp. 152-164.
11. A. Mahajan and D.K. Ross, *A note on completely and absolutely monotone functions*, Canad. Math. Bull., 25(2) (1982), pp. 143-148.
12. F. Mehmood, *On Function with Nondecreasing Increments*, (Unpublished doctoral dissertation), Department of Mathemtics, University of Karachi, Karachi, Pakistan, 2019.
13. F. Mehmood, A.R. Khan and M.A.U. Siddique, *Some Results Related to Convexifiable Functions*, J. Mech. Cont. & Math. Sci., 15 (12) (2020), pp. 36-45.
14. M. Merkle, *Completetly monotone functions: A Digest*, Analytic Number Theory, Approximation Theory, and Special Functions, (2014) (2014), pp. 347-364.
15. K.S. Miller, S.G. Samko, *Completely monotonic functions*, Integral Transforms Spec. Funct., 12(4) (2001), pp. 389-402.
16. I.Ž. Milovanović and J.E. Pečarić, *On some inequalities for* $\nabla-$*convex sequences of higher order*, Period. Math. Hung., 17 (1986), pp. 21-24.
17. J.E. Pečarić, *An inequality for* m*-convex sequences*, Mat. Vesnik, 5(18)(33) (1981), pp. 201-203.
18. J.E. Pečarić, *Convex functions: Inequalities*, Naučna knjiga, Beograd, 1987.
19. J.E. Pečarić, B.A. Mesihović, I.Ž. Milovanović and N. Stojanović, *On some inequalities for convex and* $\nabla-$*convex sequences of higher order II*, Period. Math. Hung., 17(4) (1986), pp. 313-320.
20. J. Pečarić, F. Proschan and Y.L. Tong, *Convex functions, Partial Orderings and Statistical Applications*, Academic Press, New York, 1992.
21. T. Popoviciu, *Introduction à la théorie des dif and only iférences divisées*, Bull. Math. Soc. Roumaine des Sciences, 42(1) (1940), pp. 65-78.
22. P.K. Sahoo and T. Riedel, *Mean value theorems and functional equations*, World Scientific Publishing Co., Inc., River Edge, NJ, 1998.
23. D.V. Widder, *Necessary and sufficient conditions for the representation of a function as a Laplace integral*, Trans. Amer. Math. Soc., 33 (1931), pp. 851-892.
24. D.V. Widder, *The inversion of the Laplace integral and the related moment problem*, Trans. Amer. Math. Soc., 36 (1934), pp. 107-200.

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