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On New Integral Inequalities via Geometric-Arithmetic Convex Functions with Applications

Merve Avcı-Ardıç[1]*, Ahmet Ocak Akdemir[2] and Erhan Set[3]

Abstract. In this study, new Hermite-Hadamard type inequalities are generated for geometric-arithmetic functions with the help of an integral equation proved for differentiable functions. In proofs, some classical integral inequalities, such as Hölder's inequality, basic definitions and known mathematical analysis procedures are used. The third part of the study includes various applications confirming the accuracy of the generated results. A brief conclusion of the study has been given in the last part of the paper.

1. Introduction

Functions are classified according to their specific features and are used in various scientific fields, especially mathematical analysis. This classification has emerged as a result of the examination of the topological, algebraic and geometric properties of functions. Especially function classes with a geometric meaning have attracted more attention of mathematicians and numerous studies have been done on geometric function classes. We will give the definition of convex functions that has an important place in mathematical analysis, applied mathematics, numerical integration and statistics.

The function $f : I \subset \mathbb{R} \to \mathbb{R}$ is a convex function on I, if the following inequality

$$f\left(t\kappa_1 + (1-t)\kappa_2\right) \le tf\left(\kappa_1\right) + (1-t) f\left(\kappa_2\right),$$

holds for all $\kappa_1, \kappa_2 \in I$ and $t \in [0,1]$.

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Inequalities are frequently used in all branches of science, sometimes preferred due to their aesthetic structure. Also, application areas of inequalities are quite wide (see [15, 16]). Science is a phenomenon that is constantly expanding and renewing itself. This development, which is also valid for inequality theory, serves the effort to contribute to mathematics and related fields. In this direction, convexity, which can be considered as the meeting point of convex analysis with the theory of inequality, first gained meaning in the literature with the Hermite-Hadamard inequality. The Hermite-Hadamard inequality, which includes estimates of the Cauchy mean value of convex functions, is a famous inequality with the applications in different fields and has been studied by many mathematicians. Let's proceed by remembering this aesthetic inequality as following:

$$f\left(\frac{\kappa_1+\kappa_2}{2}\right) \leq \frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2} f(\kappa)\,d\kappa \leq \frac{f(\kappa_1)+f(\kappa_2)}{2}.$$

The concept of mean function, which is widely used in statistics and as distribution and trend measures in the analysis of data sets, is an important metric on the set of real numbers. Anderson *et al.* have mentioned mean function in [1] as following:

Definition 1.1. A function $M : (0,\infty)\times(0,\infty)\to(0,\infty)$ is called a mean function if

1. $M(\kappa_1,\kappa_2) = M(\kappa_2,\kappa_1)$,
2. $M(\kappa_1,\kappa_1) = \kappa_1$,
3. $\kappa_1 < M(\kappa_1,\kappa_2) < \kappa_2$, whenever $\kappa_1 < \kappa_2$,
4. $M(a\kappa_1, a\kappa_2) = aM(\kappa_1,\kappa_2)$ for all $a > 0$.

Based on the definition of mean function, let us recall special means of positive numbers (see [1]).

1. Arithmetic Mean: $A(\kappa_1,\kappa_2) = \frac{\kappa_1+\kappa_2}{2}$.
2. Geometric Mean: $G(\kappa_1,\kappa_2) = \sqrt{\kappa_1\kappa_2}$.
3. Harmonic Mean: $H(\kappa_1,\kappa_2) = 1/A\left(\frac{1}{\kappa_1},\frac{1}{\kappa_2}\right)$.
4. Logarithmic Mean: $L(\kappa_1,\kappa_2) = (\kappa_1-\kappa_2)/(\log\kappa_1-\log\kappa_2)$ for $\kappa_1\neq\kappa_2$ and $L(\kappa_1,\kappa_1)=\kappa_1$.
5. Identric Mean: $I(\kappa_1,\kappa_2) = (1/e)\left(\kappa_1^{\kappa_1}/\kappa_2^{\kappa_2}\right)^{1/(\kappa_1-\kappa_2)}$ for $\kappa_1\neq\kappa_2$ and $I(\kappa_1,\kappa_1)=\kappa_1$.
6. $p-$Logarithmic Mean: $L_p(\kappa_1,\kappa_2) = \left[\frac{\kappa_2^{p+1}-\kappa_1^{p+1}}{(p+1)(\kappa_2-\kappa_1)}\right]^{\frac{1}{p}}$ for $\kappa_1\neq\kappa_2$ and $L_p(\kappa_1,\kappa_1)=\kappa_1$ where $p\neq-1,0$.

In addition to these special kinds of means built on the concept of mean function, many new means have been handled by different researchers and given together with their usage areas such that Gini mean, Stolarsky mean and generalized means.

When the definition of convex function is examined carefully, the basic relationship between convexity and mean function can be easily seen. This relationship is a natural consequence of linear components in the algebraic definition of convex functions and constitutes the strong link of convex analysis with statistics. In [1], Anderson *et al.* also have given the following definition on behalf of emphasize this relationship to use for moving to several different classes of convex functions:

Definition 1.2. Let $f : I \to (0, \infty)$ be continuous, where I is a subinterval of $(0, \infty)$. Let M and N be any two mean functions. We say f is MN-convex (concave) if

$$f\left(M\left(\kappa_1, \kappa_2\right)\right) \leq (\geq)\, N\left(f\left(\kappa_1\right), f\left(\kappa_2\right)\right),$$

for all $\kappa_1, \kappa_2 \in I$.

Among the function classes given on the basis of means, in [11] Niculescu has stated the most striking and attractive function classes as follows:

The $AG-$convex functions (usually known as $\log-$convex functions) are those functions $f : I \to (0, \infty)$ for which

$$\kappa_1, \kappa_2 \in I \text{ and } t \in [0,1] \quad \Rightarrow \quad f\left(t\kappa_1 + (1-t)\,\kappa_2\right) \leq \left[f\left(\kappa_1\right)\right]^t \left[f\left(\kappa_2\right)\right]^{1-t}, \tag{1.1}$$

i.e., for which $\log f$ is convex.

The $GG-$convex functions (called in what follows multiplicatively convex functions) are those functions $f : I \to J$ (acting on subintervals of $(0, \infty)$) such that

$$\kappa_1, \kappa_2 \in I \text{ and } t \in [0,1] \quad \Rightarrow \quad f\left(\kappa_1^t \kappa_2^{1-t}\right) \leq \left[f\left(\kappa_1\right)\right]^t \left[f\left(\kappa_2\right)\right]^{1-t}. \tag{1.2}$$

The class of all $GA-$convex functions is constituted by all functions $f : I \to \mathbb{R}$ (defined on subintervals of $(0, \infty)$) for which

$$\kappa_1, \kappa_2 \in I \text{ and } t \in [0,1] \quad \Rightarrow \quad f\left(\kappa_1^t \kappa_2^{1-t}\right) \leq t f\left(\kappa_1\right) + (1-t) f\left(\kappa_2\right). \tag{1.3}$$

Another feature of these function classes that stands out is that the criterions have been given for each classes. These criterions help us to classify the functions to different classes. For example, recall that the condition of $GA-$convexity is $\kappa^2 f''(\kappa) + \kappa f'(\kappa) \geq 0$ which implies all twice differentiable nondecreasing convex functions are also $GA-$convex (see [1]).

The Hermite-Hadamard inequality has many new generalizations, variants, iterations and improvements. Many new results on the left and right sides of this inequality are derived with new versions for each different class of convex functions. At this point, we should note the benefit of obtaining integral identity, which is one of the main ways for deriving inequalities. Any well-constructed integral equation gives rise to integral inequalities that give strong approaches.

In [1], Anderson et al. have defined different kinds of convex functions by using the mean function and have given several properties of these new classes of convexity. Also, they have proved some new findings for hypergeometric functions as applications. Based on this motivated paper, in [2], Akdemir et al. have established a new integral identity and proved several integral inequalities for $GA-$convex functions. In [3], the authors have given new approaches for $GG-$ and $GA-$convex functions via a new integral identity. In [4], Avcı-Ardıç et al. have provided some generalizations of Ostrowski type inequalities for geometrically convex functions. Another motivated papers have been performed to obtain some new estimations by Çoban et al. and Dragomir respectively (see [5, 6]). In [7], İşcan has used $GA-$convex functions to establish Jensen-Mercer type inequalities for the first time in the literature. With the increasing usage of fractional integral operators in inequality studies, in [8], Khurshid et al. have proved fractional integral inequalities via conformable integrals for geometrically convex function classes. Other important paper has been performed by Kunt ad İşcan (see [9]). In [10], Latif has presented some new inequalities and applications via $GA-$convexity. Niculescu has examined convexity concept according to mean functions (see [12]). In [13], Satnoianu has given several variants of previous results as improved versions. In [14], Wang and Shi have extended the discussion to $n-$time differentiable $GA-$convex functions and they presented some useful applications.

In this article, the main motivation point and the innovative direction are to establish a new integral identity and to prove some new Hermite-Hadamard type inequalities for $GA-$convex functions with the help of this identity. In addition, giving applications in the set of positive real numbers in order to reveal the effectiveness of the findings supported the results.

2. Main Results

This section includes our principal findings. Firstly, we will start with a new integral identity which we used to obtain our results.

Lemma 2.1. *Suppose that* $f : I \subseteq \mathbb{R}_+ = (0, \infty) \to \mathbb{R}$ *is a differentiable function on* I^o *and* $\kappa_1, \kappa_2 \in I^o$ *with* $\kappa_1 < \kappa_2$. *If* $f' \in L[\kappa_1, \kappa_2]$, *then the* *following identity holds:*

$$\begin{aligned}
&\kappa_2 f(\kappa_2) - \kappa_1 f(\kappa_1) - \int_{\kappa_1}^{\kappa_2} f(\kappa)\, d\kappa \\
&= (\log \kappa_3 - \log \kappa_1) \int_0^1 \kappa_3^{2t} \kappa_1^{2(1-t)} f'\left(\kappa_3^t \kappa_1^{1-t}\right) dt \\
&\quad + (\log \kappa_2 - \log \kappa_3) \int_0^1 \kappa_2^{2t} \kappa_3^{2(1-t)} f'\left(\kappa_2^t \kappa_3^{1-t}\right) dt,
\end{aligned}$$

for all $\kappa_3 \in [\kappa_1, \kappa_2]$.

Proof. Integrating by parts and by using the change of the variables, one can see that the equality holds. We omit the details. □

Theorem 2.2. *Suppose that* $f : I \subseteq \mathbb{R}_+ \to \mathbb{R}$ *is a differentiable function on* I^o, $\kappa_1, \kappa_2 \in I^o$ *with* $\kappa_1 < \kappa_2$ *and* $f' \in L[\kappa_1, \kappa_2]$. *If* $|f'|$ *is a* $GA-$*convex function on* $[\kappa_1, \kappa_2]$, *then we get the following inequality*

$$\begin{aligned}
&\left|\kappa_2 f(\kappa_2) - \kappa_1 f(\kappa_1) - \int_{\kappa_1}^{\kappa_2} f(\kappa)\, d\kappa\right| \\
&\le \left|f'(\kappa_3)\right| \left[\frac{L\left(\kappa_2^2, \kappa_3^2\right) - L\left(\kappa_3^2, \kappa_1^2\right)}{2}\right] \\
&\quad + \left|f'(\kappa_1)\right| \left[\frac{L\left(\kappa_3^2, \kappa_1^2\right) - \kappa_1^2}{2}\right] + \left|f'(\kappa_2)\right| \left[\frac{\kappa_2^2 - L\left(\kappa_2^2, \kappa_3^2\right)}{2}\right],
\end{aligned}$$

for all $\kappa_3 \in [\kappa_1, \kappa_2]$.

Proof. Via Lemma 2.1, the property of modulus and the $GA-$convexity of $|f'|$, we have

$$\begin{aligned}
&\left|\kappa_2 f(\kappa_2) - \kappa_1 f(\kappa_1) - \int_{\kappa_1}^{\kappa_2} f(\kappa)\, d\kappa\right| \\
&\le (\log \kappa_3 - \log \kappa_1) \int_0^1 \kappa_3^{2t} \kappa_1^{2(1-t)} \left|f'\left({\kappa_3}^t \kappa_1^{1-t}\right)\right| dt \\
&\quad + (\log \kappa_2 - \log \kappa_3) \int_0^1 \kappa_2^{2t} {\kappa_3}^{2(1-t)} \left|f'\left(\kappa_2^t {\kappa_3}^{1-t}\right)\right| dt \\
&\le (\log \kappa_3 - \log \kappa_1) \int_0^1 {\kappa_3}^{2t} \kappa_1^{2(1-t)} \left[t\left|f'(\kappa_3)\right| + (1-t)\left|f'(\kappa_1)\right|\right] dt \\
&\quad + (\log \kappa_2 - \log \kappa_3) \int_0^1 \kappa_2^{2t} {\kappa_3}^{2(1-t)} \left[t\left|f'(\kappa_2)\right| + (1-t)\left|f'(\kappa_3)\right|\right] dt.
\end{aligned}$$

By computing the above integrals, we obtain

$$\left|\kappa_2 f(\kappa_2) - \kappa_1 f(\kappa_1) - \int_{\kappa_1}^{\kappa_2} f(\kappa)\, d\kappa\right|$$

$$\leq (\log\kappa_3-\log\kappa_1)\left[\kappa_1^2\left|f'(\kappa_3)\right|\left(\frac{\left(\frac{\kappa_3}{\kappa_1}\right)^2\log\left(\frac{\kappa_3}{\kappa_1}\right)^2-\left(\frac{\kappa_3}{\kappa_1}\right)^2+1}{\left(\log\left(\frac{\kappa_3}{\kappa_1}\right)^2\right)^2}\right)\right.$$

$$\left.+\kappa_1^2\left|f'(\kappa_1)\right|\left(\frac{\left(\frac{\kappa_3}{\kappa_1}\right)^2-\log\left(\frac{\kappa_3}{\kappa_1}\right)^2-1}{\left(\log\left(\frac{\kappa_3}{\kappa_1}\right)^2\right)^2}\right)\right]$$

$$+(\log\kappa_2-\log\kappa_3)\left[\kappa_3^2\left|f'(\kappa_2)\right|\left(\frac{\left(\frac{\kappa_2}{\kappa_3}\right)^2\log\left(\frac{\kappa_2}{\kappa_3}\right)^2-\left(\frac{\kappa_2}{\kappa_3}\right)^2+1}{\left(\log\left(\frac{\kappa_2}{\kappa_3}\right)^2\right)^2}\right)\right.$$

$$\left.+\kappa_3^2\left|f'(\kappa_3)\right|\left(\frac{\left(\frac{\kappa_2}{\kappa_3}\right)^2-\log\left(\frac{\kappa_2}{\kappa_3}\right)^2-1}{\left(\log\left(\frac{\kappa_2}{\kappa_3}\right)^2\right)^2}\right)\right].$$

By a simple arrangement, we get the desired result. □

Theorem 2.3. *Suppose that $f : I \subseteq \mathbb{R}_+ \to \mathbb{R}$ is a differentiable function on I^o, $\kappa_1,\kappa_2 \in I^o$ with $\kappa_1<\kappa_2$ and $f' \in L[\kappa_1,\kappa_2]$. If $|f'|^q$ is a GA−convex function on $[\kappa_1,\kappa_2]$, then we get the following inequality*

$$\left|\kappa_2 f(\kappa_2)-\kappa_1 f(\kappa_1)-\int_{\kappa_1}^{\kappa_2} f(\kappa)\,d\kappa\right|$$

$$\leq (\log\kappa_3-\log\kappa_1)^{\frac{1}{p}}\left[L\left(\kappa_3^2,\kappa_1^2\right)\right]^{\frac{1}{p}}$$

$$\times\left[\left|f'(\kappa_3)\right|^q\left(\frac{\kappa_3^2-L\left(\kappa_3^2,\kappa_1^2\right)}{2}\right)+\left|f'(\kappa_1)\right|^q\left(\frac{L\left(\kappa_3^2,\kappa_1^2\right)-\kappa_1^2}{2}\right)\right]^{\frac{1}{q}}$$

$$+(\log\kappa_2-\log\kappa_3)^{\frac{1}{p}}\left[L\left(\kappa_2^2,\kappa_3^2\right)\right]^{\frac{1}{p}}$$

$$\times\left[\left|f'(\kappa_2)\right|^q\left(\frac{\kappa_2^2-L\left(\kappa_2^2,\kappa_3^2\right)}{2}\right)+\left|f'(\kappa_3)\right|^q\left(\frac{L\left(\kappa_2^2,\kappa_3^2\right)-\kappa_3^2}{2}\right)\right]^{\frac{1}{q}},$$

for all $\kappa_3 \in [\kappa_1,\kappa_2]$ and $q>1$, $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Since $|f'|^q$ is a $GA-$convex function on $[\kappa_1,\kappa_2]$, from Lemma 2.1 and by using Hölder integral inequality, we get

$$\left|\kappa_2 f(\kappa_2)-\kappa_1 f(\kappa_1)-\int_{\kappa_1}^{\kappa_2} f(\kappa)\,d\kappa\right|$$

$$\leq (\log\kappa_3 - \log\kappa_1)\left(\int_0^1 \kappa_3^{2t}{\kappa_1}^{2(1-t)}dt\right)^{\frac{1}{p}}$$

$$\times\left(\int_0^1 {\kappa_3}^{2t}{\kappa_1}^{2(1-t)}\left|f'\left({\kappa_3}^t{\kappa_1}^{1-t}\right)\right|^q dt\right)^{\frac{1}{q}}$$

$$+ (\log\kappa_2 - \log\kappa_3)\left(\int_0^1 \kappa_2^{2t}{\kappa_3}^{2(1-t)}dt\right)^{\frac{1}{p}}$$

$$\times\left(\int_0^1 {\kappa_2}^{2t}{\kappa_3}^{2(1-t)}\left|f'\left({\kappa_2}^t{\kappa_3}^{1-t}\right)\right|^q dt\right)^{\frac{1}{q}}$$

$$\leq \kappa_1^2(\log\kappa_3 - \log\kappa_1)\left(\int_0^1 \left(\frac{\kappa_3}{\kappa_1}\right)^{2t} dt\right)^{\frac{1}{p}}$$

$$\times\left(\int_0^1 \left(\frac{\kappa_3}{\kappa_1}\right)^{2t}\left[t\left|f'(\kappa_3)\right|^q + (1-t)\left|f'(\kappa_1)\right|^q\right]dt\right)^{\frac{1}{q}}$$

$$+ \kappa_3^2(\log\kappa_2 - \log\kappa_3)\left(\int_0^1 \left(\frac{\kappa_2}{\kappa_3}\right)^{2t} dt\right)^{\frac{1}{p}}$$

$$\times\left(\int_0^1 \left(\frac{\kappa_2}{\kappa_3}\right)^{2t}\left[t\left|f'(\kappa_2)\right|^q + (1-t)\left|f'(\kappa_3)\right|^q\right]dt\right)^{\frac{1}{q}}.$$

By a simple computation, we get the desired result. □

Theorem 2.4. *Under the assumptions of Theorem 2.3, the following inequality holds:*

$$\left|\kappa_2 f(\kappa_2) - \kappa_1 f(\kappa_1) - \int_{\kappa_1}^{\kappa_2} f(\kappa)\,d\kappa\right|$$

$$\leq (\log\kappa_3 - \log\kappa_1)\left[L\left(\kappa_3^{2p},\kappa_1^{2p}\right)\right]^{\frac{1}{p}}\left[A\left(\left|f'(\kappa_3)\right|^q,\left|f'(\kappa_1)\right|^q\right)\right]^{\frac{1}{q}}$$

$$+ (\log\kappa_2 - \log\kappa_3)\left[L\left(\kappa_2^{2p},\kappa_3^{2p}\right)\right]^{\frac{1}{p}}\left[A\left(\left|f'(\kappa_3)\right|^q,\left|f'(\kappa_2)\right|^q\right)\right]^{\frac{1}{q}}.$$

Proof. Since $|f'|^q$ is a $GA-$convex function on $[\kappa_1,\kappa_2]$, from Lemma 2.1 and by using Hölder integral inequality, we get

$$\left|\kappa_2 f(\kappa_2) - \kappa_1 f(\kappa_1) - \int_{\kappa_1}^{\kappa_2} f(\kappa)\,d\kappa\right|$$

$$\leq \kappa_1^2(\log\kappa_3 - \log\kappa_1)\left(\int_0^1 \left(\frac{\kappa_3}{\kappa_1}\right)^{2tp} dt\right)^{\frac{1}{p}}\left(\int_0^1 \left|f'\left({\kappa_3}^t{\kappa_1}^{1-t}\right)\right|^q dt\right)^{\frac{1}{q}}$$

$$
\begin{aligned}
&+\kappa_3^2\left(\log\kappa_2-\log\kappa_3\right)\left(\int_0^1\left(\frac{\kappa_2}{\kappa_3}\right)^{2tp}dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f'\left(\kappa_2^t\kappa_3^{1-t}\right)\right|^q dt\right)^{\frac{1}{q}}\\
\leq\ &\kappa_1^2\left(\log\kappa_3-\log\kappa_1\right)\left(\int_0^1\left(\frac{\kappa_3}{\kappa_1}\right)^{2tp}dt\right)^{\frac{1}{p}}\\
&\times\left(\int_0^1\left[t\left|f'\left(\kappa_3\right)\right|^q+(1-t)\left|f'\left(\kappa_1\right)\right|^q\right]dt\right)^{\frac{1}{q}}\\
&+\kappa_3^2\left(\log\kappa_2-\log\kappa_3\right)\left(\int_0^1\left(\frac{\kappa_2}{\kappa_3}\right)^{2tp}dt\right)^{\frac{1}{p}}\\
&\times\left(\int_0^1\left[t\left|f'\left(\kappa_2\right)\right|^q+(1-t)\left|f'\left(\kappa_3\right)\right|^q\right]dt\right)^{\frac{1}{q}}.
\end{aligned}
$$

If we calculate the above integrals, we get the desired result. □

Theorem 2.5. *Under the assumptions of Theorem 2.3, the following inequality holds:*

$$
\begin{aligned}
&\left|\kappa_2 f(\kappa_2)-\kappa_1 f(\kappa_1)-\int_{\kappa_1}^{\kappa_2}f(\kappa)\,d\kappa\right|\\
&\leq\left(\log\kappa_3-\log\kappa_1\right)^{1-\frac{1}{q}}\left[\left|f'\left(\kappa_3\right)\right|^q\left(\frac{\kappa_3^{2q}-L\left(\kappa_3^{2q},\kappa_1^{2q}\right)}{2q}\right)\right.\\
&\left.+\left|f'\left(\kappa_1\right)\right|^q\left(\frac{L\left(\kappa_3^{2q},\kappa_1^{2q}\right)-\kappa_1^{2q}}{2q}\right)\right]^{\frac{1}{q}}\\
&+\left(\log\kappa_2-\log\kappa_3\right)^{1-\frac{1}{q}}\left[\left|f'\left(\kappa_3\right)\right|^q\left(\frac{L\left(\kappa_2^{2q},\kappa_3^{2q}\right)-\kappa_3^{2q}}{2q}\right)\right.\\
&\left.+\left|f'\left(\kappa_2\right)\right|^q\left(\frac{\kappa_2^{2q}-L\left(\kappa_2^{2q},\kappa_3^{2q}\right)}{2q}\right)\right]^{\frac{1}{q}}.
\end{aligned}
$$

Proof. By a similar argument to the proofs of Theorem 2.3 and Theorem 2.4, since $|f'|^q$ is a $GA-$convex function on $[\kappa_1,\kappa_2]$, from Lemma 2.1 and by using a version of Hölder integral inequality, we get

$$
\begin{aligned}
&\left|\kappa_2 f(\kappa_2)-\kappa_1 f(\kappa_1)-\int_{\kappa_1}^{\kappa_2}f(\kappa)\,d\kappa\right|\\
&\leq\kappa_1^2\left(\log\kappa_3-\log\kappa_1\right)
\end{aligned}
$$

$$\times\left(\int_0^1 dt\right)^{1-\frac{1}{q}}\left(\int_0^1\left(\frac{\kappa_3}{\kappa_1}\right)^{2qt}\left|f'\left(\kappa_3^t\kappa_1^{1-t}\right)\right|^q dt\right)^{\frac{1}{q}}$$

$$+\kappa_3^2\left(\log\kappa_2-\log\kappa_3\right)\left(\int_0^1 dt\right)^{1-\frac{1}{q}}$$

$$\times\left(\int_0^1\left(\frac{\kappa_2}{\kappa_3}\right)^{2qt}\left|f'\left(\kappa_2^t\kappa_3^{1-t}\right)\right|^q dt\right)^{\frac{1}{q}}$$

$$\leq\kappa_1^2\left(\log\kappa_3-\log\kappa_1\right)$$

$$\times\left(\int_0^1\left(\frac{\kappa_3}{\kappa_1}\right)^{2qt}\left[t\left|f'\left(\kappa_3\right)\right|^q+(1-t)\left|f'\left(\kappa_1\right)\right|^q\right]dt\right)^{\frac{1}{q}}$$

$$+\kappa_3^2\left(\log\kappa_2-\log\kappa_3\right)$$

$$\times\left(\int_0^1\left(\frac{\kappa_2}{\kappa_3}\right)^{2qt}\left[t\left|f'\left(\kappa_2\right)\right|^q+(1-t)\left|f'\left(\kappa_3\right)\right|^q\right]dt\right)^{\frac{1}{q}}.$$

By computing the above integrals, we deduce

$$\left|\kappa_2 f(\kappa_2)-\kappa_1 f(\kappa_1)-\int_{\kappa_1}^{\kappa_2} f(\kappa)\,d\kappa\right|$$

$$\leq\left(\log\kappa_3-\log\kappa_1\right)^{1-\frac{1}{q}}\left[\left|f'\left(\kappa_3\right)\right|^q\left(\frac{\kappa_3^{2q}}{2q}-\frac{\kappa_3^{2q}-\kappa_1^{2q}}{2q\left(\log\kappa_3^{2q}-\log\kappa_1^{2q}\right)}\right)\right.$$

$$\left.+\left|f'\left(\kappa_1\right)\right|^q\left(\frac{\kappa_3^{2q}-\kappa_1^{2q}}{2q\left(\log\kappa_3^{2q}-\log\kappa_1^{2q}\right)}-\frac{\kappa_1^{2q}}{2q}\right)\right]^{\frac{1}{q}}$$

$$+\left(\log\kappa_2-\log\kappa_3\right)^{1-\frac{1}{q}}\left[\left|f'\left(\kappa_2\right)\right|^q\left(\frac{\kappa_2^{2q}}{2q}-\frac{\kappa_2^{2q}-\kappa_3^{2q}}{2q\left(\log\kappa_2^{2q}-\log\kappa_3^{2q}\right)}\right)\right.$$

$$\left.+\left|f'\left(\kappa_3\right)\right|^q\left(\frac{\kappa_2^{2q}-\kappa_3^{2q}}{2q\left(\log\kappa_2^{2q}-\log\kappa_3^{2q}\right)}-\frac{\kappa_3^{2q}}{2q}\right)\right]^{\frac{1}{q}},$$

which completes the proof. □

Theorem 2.6. *Under the assumptions of Theorem 2.3, the following inequality holds:*

$$\left|\kappa_2 f(\kappa_2)-\kappa_1 f(\kappa_1)-\int_{\kappa_1}^{\kappa_2} f(\kappa)\,d\kappa\right|$$

$$\leq (\log\kappa_3-\log\kappa_1)^{1-\frac{1}{q}}\frac{1}{(2p)^{\frac{1}{q}}}\left(L\left(\kappa_3^{2\left(\frac{q-p}{q-1}\right)},\kappa_1^{2\left(\frac{q-p}{q-1}\right)}\right)\right)^{\frac{1}{p}}$$
$$\times\left(\left|f'(\kappa_3)\right|^q\left(\kappa_3^{2p}-L\left(\kappa_3^{2p},\kappa_1^{2p}\right)\right)+\left|f'(\kappa_1)\right|^q\left(L\left(\kappa_3^{2p},\kappa_1^{2p}\right)-\kappa_1^{2p}\right)\right)^{\frac{1}{q}}$$
$$+(\log\kappa_2-\log\kappa_3)^{1-\frac{1}{q}}\frac{1}{(2p)^{\frac{1}{q}}}\left(L\left(\kappa_2^{2\left(\frac{q-p}{q-1}\right)},\kappa_3^{2\left(\frac{q-p}{q-1}\right)}\right)\right)^{\frac{1}{p}}$$
$$\times\left(\left|f'(\kappa_2)\right|^q\left(\kappa_2^{2p}-L\left(\kappa_2^{2p},\kappa_3^{2p}\right)\right)+\left|f'(\kappa_3)\right|^q\left(L\left(\kappa_2^{2p},\kappa_3^{2p}\right)-\kappa_3^{2p}\right)\right)^{\frac{1}{q}}.$$

Proof. We can remind the Hölder integral inequality as

$$\int_{\kappa_1}^{\kappa_2}|S(\kappa)|\left|f'(\kappa)\right|d\kappa$$
$$\leq\left[\int_{\kappa_1}^{\kappa_2}|S(\kappa)|^{\frac{q-p}{q-1}}d\kappa\right]^{\frac{q-1}{q}}\left[\int_{\kappa_1}^{\kappa_2}|S(\kappa)|^p\left|f'(\kappa)\right|^q d\kappa\right]^{\frac{1}{q}}.$$

By using the Hölder's inequality as above and the $GA-$convexity of $|f'|^q$, we get

$$\left|\kappa_2 f(\kappa_2)-\kappa_1 f(\kappa_1)-\int_{\kappa_1}^{\kappa_2}f(\kappa)\,d\kappa\right|$$
$$\leq\kappa_1^2(\log\kappa_3-\log\kappa_1)\left[\int_0^1\left(\frac{\kappa_3}{\kappa_1}\right)^{2t\frac{q-p}{q-1}}dt\right]^{\frac{1}{p}}$$
$$\times\left[\int_0^1\left(\frac{\kappa_3}{\kappa_1}\right)^{2tp}\left[t\left|f'(\kappa_3)\right|^q+(1-t)\left|f'(\kappa_1)\right|^q\right]dt\right]^{\frac{1}{q}}$$
$$+\kappa_3^2(\log\kappa_2-\log\kappa_3)\left[\int_0^1\left(\frac{\kappa_2}{\kappa_3}\right)^{2t\frac{q-p}{q-1}}dt\right]^{\frac{1}{p}}$$
$$\times\left[\int_0^1\left(\frac{\kappa_2}{\kappa_3}\right)^{2tp}\left[t\left|f'(\kappa_2)\right|^q+(1-t)\left|f'(\kappa_3)\right|^q\right]dt\right]^{\frac{1}{q}}.$$

By a simple computation, we get the result. □

3. Applications to Special Means

The following propositions hold:

Proposition 3.1. *Let $\kappa_1,\kappa_2\in\mathbb{R}_+$ and $s>0$, $s\neq1$. Then, we have*

$$(\kappa_3-\kappa_1)L_{s-1}^{s-1}(\kappa_3,\kappa_1)L\left(\kappa_3^2,\kappa_1^2\right)+(\kappa_2-\kappa_3)L_{s-1}^{s-1}(\kappa_2,\kappa_3)L\left(\kappa_2^2,\kappa_3^2\right)$$
$$\leq(\kappa_2-\kappa_1)L_{s+1}^{s+1}(\kappa_2,\kappa_1),$$

for all $\kappa_3\in[\kappa_1,\kappa_2]$.

Proof. The assertion follows immediately by applying Theorem 2.2 with $f(\kappa)=\frac{\kappa^{s+1}}{s+1}$, $\kappa\in\mathbb{R}_+$, $s>0$. □

Proposition 3.2. *Let* $\kappa_1,\kappa_2\in\mathbb{R}_+$ *and* $s>0$, $q>1$, $\frac{1}{p}+\frac{1}{q}=1$ *and* $sq\neq 1$. *Then, we have*

$$
\begin{aligned}
&(\kappa_2-\kappa_1)\,L_{s+1}^{s+1}L(\kappa_2,\kappa_1)\\
&\quad\le(\log\kappa_3-\log\kappa_1)^{\frac{1}{p}}\left(\frac{\kappa_3-\kappa_1}{2}\right)^{\frac{1}{q}}\left[L\left(\kappa_3^2,\kappa_1^2\right)\right]^{\frac{1}{p}}\\
&\qquad\times\left[(sq+2)\,L_{sq+1}^{sq+1}(\kappa_3,\kappa_1)-sqL\left(\kappa_3^2,\kappa_1^2\right)L_{sq-1}^{sq-1}(\kappa_3,\kappa_1)\right]^{\frac{1}{q}}\\
&\qquad+(\log\kappa_2-\log\kappa_3)^{\frac{1}{p}}\left(\frac{\kappa_2-\kappa_3}{2}\right)^{\frac{1}{q}}\left[L\left(\kappa_2^2,\kappa_3^2\right)\right]^{\frac{1}{p}}\\
&\qquad\times\left[(sq+2)\,L_{sq+1}^{sq+1}(\kappa_2,\kappa_3)-sqL\left(\kappa_2^2,\kappa_3^2\right)L_{sq-1}^{sq-1}(\kappa_2,\kappa_3)\right]^{\frac{1}{q}},
\end{aligned}
$$

for all $\kappa_3\in[\kappa_1,\kappa_2]$.

Proof. The assertion follows immediately by applying Theorem 2.3 with $f(\kappa)=\frac{\kappa^{s+1}}{s+1}$, $\kappa\in\mathbb{R}_+$, $s>0$. □

Proposition 3.3. *Let* $\kappa_1,\kappa_2\in\mathbb{R}_+$ *and* $s>0$, $q>1$, $\frac{1}{p}+\frac{1}{q}=1$. *Then, we have*

$$
\begin{aligned}
L_{s+1}^{s+1}(\kappa_2,\kappa_1)L(\kappa_2,\kappa_1)\le{}&\frac{\log\kappa_3-\log\kappa_1}{\log\kappa_2-\log\kappa_1}\left[L\left(\kappa_3^{2p},\kappa_1^{2p}\right)\right]^{\frac{1}{p}}\left[A\left(\kappa_3^{sq},\kappa_1^{sq}\right)\right]^{\frac{1}{q}}\\
&+\frac{\log\kappa_2-\log\kappa_3}{\log\kappa_2-\log\kappa_1}\left[L\left(\kappa_2^{2p},\kappa_3^{2p}\right)\right]^{\frac{1}{p}}\left[A\left(\kappa_3^{sq},\kappa_2^{sq}\right)\right]^{\frac{1}{q}},
\end{aligned}
$$

for all $\kappa_3\in[\kappa_1,\kappa_2]$.

Proof. The assertion follows immediately by applying Theorem 2.4 with $f(\kappa)=\frac{\kappa^{s+1}}{s+1}$, $\kappa\in\mathbb{R}_+$, $s>0$. □

Proposition 3.4. *Let* $\kappa_1,\kappa_2\in\mathbb{R}_+$ *and* $s>0$, $q>1$, $\frac{1}{p}+\frac{1}{q}=1$ *and* $sq\neq 1$. *Then, we have*

$$
\begin{aligned}
&L_{s+1}^{s+1}(\kappa_2,\kappa_1)L(\kappa_2,\kappa_1)\\
&\quad\le\frac{\log\kappa_3-\log\kappa_1}{\log\kappa_2-\log\kappa_1}\left[\frac{L(\kappa_3,\kappa_1)}{2}\right]^{\frac{1}{q}}\\
&\qquad\times\left[(s+2)L_{q(s+2)-1}^{q(s+2)-1}(\kappa_3,\kappa_1)-sL(\kappa_3^{2q},\kappa_1^{2q})L_{sq-1}^{sq-1}(\kappa_3,\kappa_1)\right]^{\frac{1}{q}}\\
&\qquad+\frac{\log\kappa_2-\log\kappa_3}{\log\kappa_2-\log\kappa_1}\left[\frac{L(\kappa_2,\kappa_3)}{2}\right]^{\frac{1}{q}}
\end{aligned}
$$

$$\times\left[(s+2)L_{q(s+2)-1}^{q(s+2)-1}(\kappa_2,\kappa_3)-sL(\kappa_2^{2q},\kappa_3^{2q})L_{sq-1}^{sq-1}(\kappa_2,\kappa_3)\right]^{\frac{1}{q}},$$

for all $\kappa_3\in[\kappa_1,\kappa_2]$.

Proof. The assertion follows immediately by applying Theorem 2.5 with $f(\kappa)=\frac{\kappa^{s+1}}{s+1}$, $\kappa\in\mathbb{R}_+$, $s>0$. □

Proposition 3.5. *Let* $\kappa_1,\kappa_2\in\mathbb{R}_+$ *and* $s>0$, $q>1$, $\frac{1}{p}+\frac{1}{q}=1$ *and* $sq\neq1$. *Then, we have*

$$\begin{aligned}
&L_{s+1}^{s+1}(\kappa_2,\kappa_1)L(\kappa_2,\kappa_1)\\
&\quad\leq\frac{\log\kappa_3-\log\kappa_1}{\log\kappa_2-\log\kappa_1}\left[\frac{L(\kappa_3,\kappa_1)}{2p}\right]^{\frac{1}{q}}\left(L\left(\kappa_3^{2\left(\frac{q-p}{q-1}\right)},\kappa_1^{2\left(\frac{q-p}{q-1}\right)}\right)\right)^{\frac{1}{p}}\\
&\quad\times\left[(sq+2p)L_{sq+2p-1}^{sq+2p-1}(\kappa_3,\kappa_1)-sqL(\kappa_3^{2p},\kappa_1^{2p})L_{sq-1}^{sq-1}(\kappa_3,\kappa_1)\right]^{\frac{1}{q}}\\
&\quad+\frac{\log\kappa_2-\log\kappa_3}{\log\kappa_2-\log\kappa_1}\left[\frac{L(\kappa_3,\kappa_2)}{2p}\right]^{\frac{1}{q}}\left(L\left(\kappa_2^{2\left(\frac{q-p}{q-1}\right)},\kappa_3^{2\left(\frac{q-p}{q-1}\right)}\right)\right)^{\frac{1}{p}}\\
&\quad\times\left[(sq+2p)L_{sq+2p-1}^{sq+2p-1}(\kappa_2,\kappa_3)-sqL(\kappa_2^{2p},\kappa_3^{2p})L_{sq-1}^{sq-1}(\kappa_2,\kappa_3)\right]^{\frac{1}{q}},
\end{aligned}$$

for all $\kappa_3\in[\kappa_1,\kappa_2]$.

Proof. The assertion follows immediately by applying Theorem 2.6 with $f(\kappa)=\frac{\kappa^{s+1}}{s+1}$, $\kappa\in\mathbb{R}_+$, $s>0$. □

4. Conclusions

Studies on inequalities are based on discovering new inequalities and strengthening classical approaches by using new concepts and methods. The modern theory of inequality continues as a deep-rooted field of mathematics without losing its importance for centuries. The issues of inequalities continue to be a branch that is constantly studied, still plays an active role in research and is fascinating. Researchers interested in the subject can obtain different versions or generalizations of the integral identity given in the main findings section. Several new variants and improvements can be established by this way. Based on the lemma used in the study, new integral identities can be established in coordinates and also new integral inequalities can be obtained for geometrically convex functions by using fractional integral operators.

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