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Fixed Point Results for F-Hardy-Rogers Contractions via Mann's Iteration Process in Complete Convex b-Metric Spaces

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Abstract. In this paper, we give a definition of the F-Hardy-Rogers contraction of Nadler type by eliminating the conditions ($F3$) and ($F4$). And, we obtain some fixed point theorems for such mappings using Mann's iteration process in complete convex b-metric spaces. We also give an example in order to support the main results, which generalize some results in [5, 6].

1. Introduction

There are many generalizations of the Banach Fixed Point Theory [9] in literature. Some of these generalizations are Kannan's fixed point theorem [23], Chatterjea's fixed point theorem [10], Reich's fixed point theorem [34], Ciric's fixed point theorem [13], Reich–Rus–Ciric's fixed point theorem [34–36, 38, 39] and Hardy and Rogers's fixed point theorem [18]. If (C, ρ) is a complete metric space and $S : C \to C$ is a mapping that satisfies any of the above contractive conditions, then the mapping S has a fixed point in C.

In recent decades, many authors have devoted themselves to extending the above theorem to all kinds of generalized metric spaces [2, 3, 12, 31–33]. One of them is b-metric space which was introduced by Bakhtin [8] and Czerwik [14]. Since then, many authors have studied the fixed points of different contraction mappings classes in such spaces (see [1, 4, 7, 19–22, 26, 28–30, 42]).

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In 1970, Takahashi [40] introduced the following concept of convexity in metric spaces. After then, many authors [17, 24, 25, 27, 43, 44] discussed convergence of different iterative processes and the existence of fixed points of various mappings in convex metric spaces. In 2020, Chen et al. [11] gave the definition of convex b-metric space and they proved some fixed point theorems in such spaces.

In 2012, Wardowski [41] introduced the idea of F-contraction which was later followed by many authors who delivered interesting results of F-contraction. One of them was presented by Cosentino et al. [15] who expanded F-contraction in F-contraction of Hardy Roger's type. Later, Asif et al. [5] defined F-Reich contraction while eliminating the condition $(F3)$ and $(F4)$ of F-contraction of Nadler type defined by Cosentino. They also obtained some interesting fixed point theorems using Mann's iteration process in the complete convex b-metric space.

Inspired by the above studies, we give the definition of F-Hardy-Rogers contraction of Nadler type by eliminating the conditions $(F3)$ and $(F4)$ in this paper. Moreover, we prove some fixed point theorems for such mappings using Mann's iteration process in complete convex b-metric spaces.

2. Preliminaries

Now, we will recall some main definitions related to our work as follows.

Definition 2.1 ([14])**.** Let C be a (nonempty) set and $\gamma \geq 1$ be a given real number. A function $\rho : C \times C \to [0, \infty)$ is a b-metric on C if for all $p, q, r \in C$, the following conditions hold:

- (b_1) $\rho(p, q) = 0$ if and only if $p = q$,
- (b_2) $\rho(p, q) = \rho(q, p)$,
- (b_3) $\rho(p, q) \leq \gamma[\rho(p, r) + \rho(r, q)]$.

Then, (C, ρ) is called a b-metric space.

It is clear that a b-metric space is a metric space for $\gamma = 1$.

Definition 2.2 ([14, 37])**.** Let (C, ρ) be a b-metric space and $\{p_n\}$ a sequence in C. Then:

1. The sequence $\{p_n\}$ is said to be convergent in (C, ρ) if there exists $p \in C$ such that $\lim\limits_{n\to\infty} \rho(p_n, p) = 0$.
2. The sequence $\{p_n\}$ is said to be a Cauchy sequence in (C, ρ), if for every $\epsilon > 0$ there exists a positive $n_0 \in \mathbb{N}$ such that $\rho(p_n, p_m) < \epsilon$ for all $n, m > n_0$.
3. (C, ρ) is called a complete b-metric space if every Cauchy sequence is convergent in C.

Each convergent sequence in a b-metric space has a unique limit and it is also a Cauchy sequence. In addition, in general, a b-metric is not necessarily continuous.

Definition 2.3 ([16])**.** Let $\gamma \geq 1$ be a real number. We denote by $\mathcal{F}$ the family of all functions. $F : \mathbb{R}^+ \to \mathbb{R}$ with the following properties:

- $(F1)$ F is strictly increasing;
- $(F2)$ for each sequence $\{p_n\} \subset \mathbb{R}^+$ of positive numbers $\lim_{n\to\infty} p_n = 0$ if and only if $\lim_{n\to\infty} F(p_n) = -\infty$;
- $(F3)$ for each sequence $\{p_n\} \subset \mathbb{R}^+$ of positive numbers with $\lim_{n\to\infty} p_n = 0$, there exists $m \in (0,1)$ such that $\lim_{n\to\infty} (p_n)^m F(p_n) = 0$;
- $(F4)$ for each sequence $\{p_n\} \subset \mathbb{R}^+$ of positive numbers such that $\tau + F(\gamma p_n) \leq F(\gamma p_{n-1})$ for all $n \in \mathbb{N}$ and some $\tau \in \mathbb{R}^+$, then $\tau + F(\gamma^n p_n) \leq F\left(\gamma^{n-1} p_{n-1}\right)$ for all $n \in \mathbb{N}$.

Example 2.4 ([16])**.** The following functions $F_i : \mathbb{R}^+ \to \mathbb{R}, i = 1,2,3$ belong to $\mathcal{F}$:

- (i) $F_1(p) = \ln\left(p^2 + p\right), \ p > 0$;
- (ii) $F_2(p) = -\frac{1}{\sqrt[2]{p}}$, $p > 0$;
- (iii) $F_3(p) = \ln p$, $p > 0$.

In our theorems, we will use the symbol $\digamma$ as the class of functions satisfying only $(F1)$ and $(F2)$.

Definition 2.5 ([16])**.** Let (C, ρ) be a b-metric space. A multivalued mapping $S : C \to CB(C)$ is called an F-contraction of Nadler type if there exist $F \in \digamma$ and $\tau \in \mathbb{R}^+$ such that

$$\tau + F(\gamma H(Sp, Sq)) \leq F(\rho(p,q)),$$

for all $p, q \in C$ with $Sp = Sq$. Here, $CB(C)$ is a collection of all nonempty closed and bounded subsets of C.

Definition 2.6 ([11])**.** A convex structure in a b-metric space (C, ρ) is a continuous mapping $\omega : C \times C \times [0,1] \to C$ satisfying, for all $p, q, r \in C$ and all $\mu \in [0,1]$,

$$\rho(r, \omega(p, q; \mu)) \leq \mu\rho(r, p) + (1-\mu)\rho(r, q).$$

Then the triplet (C, ρ, ω) is called a convex b-metric space.

A nonempty subset K of C is said to be convex if $\omega(p, q; \mu) \in K$ for all $(p, q; \mu) \in K \times K \times [0,1]$.

Example 2.7. Let $C = \mathbb{R}$, and for any $p, q \in C$, Define metric ρ : $C \times C \to [0, \infty)$ by $\rho(p,q) = (p-q)^2$. Assume that the mapping w :

$C\times C\times\left\{\frac{1}{4}\right\}\to C$ is as follows:

$$w\left(p,q;\mu\right)=\frac{p+q}{4}.$$

Then, (C,ρ,w) is a convex b-metric space.

Let (C,ρ,w) be a convex b-metric space and $S:C\to C$ be a mapping. We can introduce Mann's iteration process in convex metric spaces as follows:

$$p_{n+1}=w\left(p_n,Sp_n;\mu_n\right),\quad n\in\mathbb{N},$$

where $p_n\in C$ and $\mu_n\in[0,1]$.

3. Main Results

In this section, we will show that the fixed points of F-Hardy-Rogers type contractions can be obtained via the Mann iteration process in complete convex b-metric spaces.

Definition 3.1. Suppose that $F\in\digamma$ and (C,ρ,ω) is a convex b-metric space with $\gamma>1$. Then $S:C\to C$ is called a F-Hardy-Rogers type contraction if for $f,g,h:C\times C\to\left[0,\frac{1}{2}\right)$ the following hold:

(3.1)
$$\tau+F\left(\gamma\rho\left(Sp,Sq\right)\right)\leq F\Big(f\left(p,q\right)\rho\left(p,q\right)+g\left(p,q\right)\left[\rho\left(p,Sp\right)+\rho\left(q,Sq\right)\right]$$
$$+h\left(p,q\right)\left[\rho\left(p,Sq\right)+\rho\left(q,Sp\right)\right]\Big)$$

for every $p,q\in C$.

Theorem 3.2. *Let (C,ρ,ω) be a complete convex b-metric space with a convex structure ω and $S:C\to C$ be a F-Hardy-Rogers type contraction. Suppose that the sequence $\{p_n\}$ is defined as follows*

$$p_n=\omega\left(p_{n-1},Sp_{n-1};\mu_{n-1}\right),\tag{3.2}$$

where $0<\mu_{n-1}\leq\frac{1}{4\gamma^2}$ for all $n\in\mathbb{N}$. If $f\left(p,q\right)+g\left(p,q\right)+2h\left(p,q\right)\leq\frac{1}{4\gamma^2}$, then S has a unique fixed point in C.

Proof. From the convex structure of the b-metric space and (3.2), we obtain

$$\begin{aligned}\rho(p_n,p_{n+1})&=\rho(p_n,w\left(p_n,Sp_n;\mu_n\right))\\&\leq\left(1-\mu_n\right)\rho(p_n,Sp_n),\end{aligned}\tag{3.3}$$

and

(3.4)
$$\begin{aligned}\rho(p_n,Sp_n)&=\rho\left(w\left(p_{n-1},Sp_{n-1};\mu_{n-1}\right),Sp_n\right)\\&\leq\mu_{n-1}\rho(p_{n-1},Sp_n)+\left(1-\mu_{n-1}\right)\rho(Sp_{n-1},Sp_n)\end{aligned}$$

$$
\begin{aligned}
&\leq \mu_{n-1}\gamma\rho(p_{n-1}, Sp_{n-1}) + \mu_{n-1}\gamma\rho(Sp_{n-1}, Sp_n) + \rho(Sp_{n-1}, Sp_n) \\
&= \mu_{n-1}\gamma\rho(p_{n-1}, Sp_{n-1}) + (1 + \mu_{n-1}\gamma)\,\rho(Sp_{n-1}, Sp_n) \\
&\leq \mu_{n-1}\gamma\rho(p_{n-1}, Sp_{n-1}) + \gamma\,(1 + \mu_{n-1})\,\rho(Sp_{n-1}, Sp_n).
\end{aligned}
$$

Using definition of F-Hardy-Rogers type contraction, we have

(3.5)

$$
\begin{aligned}
\tau + F\left(\gamma\rho(Sp_{n-1}, Sp_n)\right) \leq F\Big(& f(p_{n-1}, p_n)\rho(p_{n-1}, p_n) \\
&+ g(p_{n-1}, p_n)\left[\rho(p_{n-1}, Sp_{n-1}) + \rho(p_n, Sp_n)\right] \\
&+ h(p_{n-1}, p_n)\left[\rho(p_{n-1}, Sp_n) + (p_n, Sp_{n-1})\right]\Big).
\end{aligned}
$$

Then

$$
\begin{aligned}
F\left(\gamma\rho(Sp_{n-1}, Sp_n)\right) &\leq F\Big(f(p_{n-1}, p_n)\rho(p_{n-1}, p_n) \\
&\quad + g(p_{n-1}, p_n)\left[\rho(p_{n-1}, Sp_{n-1}) + \rho(p_n, Sp_n)\right] \\
&\quad + h(p_{n-1}, p_n)\left[\rho(p_{n-1}, Sp_n) + (p_n, Sp_{n-1})\right]\Big) - \tau \\
&\leq F\Big(f(p_{n-1}, p_n)\rho(p_{n-1}, p_n) \\
&\quad + g(p_{n-1}, p_n)\left[\rho(p_{n-1}, Sp_{n-1}) + \rho(p_n, Sp_n)\right] \\
&\quad + h(p_{n-1}, p_n)\left[\rho(p_{n-1}, Sp_n) + (p_n, Sp_{n-1})\right]\Big)
\end{aligned}
$$

which implies that

$$
\begin{aligned}
\gamma\rho(Sp_{n-1}, Sp_n) \leq\ & f(p_{n-1}, p_n)\rho(p_{n-1}, p_n) \\
&+ g(p_{n-1}, p_n)\left[\rho(p_{n-1}, Sp_{n-1}) + \rho(p_n, Sp_n)\right] \\
&+ h(p_{n-1}, p_n)\left[\rho(p_{n-1}, Sp_n) + (p_n, Sp_{n-1})\right].
\end{aligned}
$$

From (3.2), we also know that

$$
\rho(p_n, Sp_n) \leq \mu_{n-1}\gamma\rho(p_{n-1}, Sp_{n-1}) + \gamma\,(1 + \mu_{n-1})\,\rho(Sp_{n-1}, Sp_n).
$$

Combine the last two inequalities in the above, we get

$$
\begin{aligned}
\rho(p_n, Sp_n) \leq\ & \mu_{n-1}\gamma\rho(p_{n-1}, Sp_{n-1}) + (1 + \mu_{n-1})\Big\{ f(p_{n-1}, p_n)\rho(p_{n-1}, p_n) \\
&+ g(p_{n-1}, p_n)\left[\rho(p_{n-1}, Sp_{n-1}) + \rho(p_n, Sp_n)\right] \\
&+ h(p_{n-1}, p_n)\left[\rho(p_{n-1}, Sp_n) + (p_n, Sp_{n-1})\right]\Big\}.
\end{aligned}
$$

From (3.3), we obtain

$$
\begin{aligned}
\rho(p_n, Sp_n) \leq\ & \mu_{n-1}\gamma\rho(p_{n-1}, Sp_{n-1}) \\
&+ (1 + \mu_{n-1})\,(1 - \mu_{n-1})\Big\{ f(p_{n-1}, p_n)\rho(p_{n-1}, Sp_{n-1}) \\
&+ g(p_{n-1}, p_n)\left[\rho(p_{n-1}, Sp_{n-1}) + \rho(p_n, Sp_n)\right]
\end{aligned}
$$

$$+ h(p_{n-1},p_n)\left[\rho(p_{n-1},Sp_n)+(p_n,Sp_{n-1})\right]\Big\}.$$

Then

(3.6)

$$\begin{aligned}
\rho(p_n,Sp_n) &\leq \mu_{n-1}\gamma\rho(p_{n-1},Sp_{n-1}) + \left(1-\mu_{n-1}^2\right) f(p_{n-1},p_n)\rho(p_{n-1},Sp_{n-1}) \\
&\quad + (1+\mu_{n-1})\, g(p_{n-1},p_n)\left[\rho(p_{n-1},Sp_{n-1})+\rho(p_n,Sp_n)\right] \\
&\quad + (1+\mu_{n-1})\, h(p_{n-1},p_n)\left[\rho(p_{n-1},Sp_n)+(p_n,Sp_{n-1})\right] \\
&\leq \mu_{n-1}\gamma\rho(p_{n-1},Sp_{n-1}) + \left(1-\mu_{n-1}^2\right) f(p_{n-1},p_n)\rho(p_{n-1},Sp_{n-1}) \\
&\quad + (1+\mu_{n-1})\, g(p_{n-1},p_n)\left[\rho(p_{n-1},Sp_{n-1})+\rho(p_n,Sp_n)\right] \\
&\quad + (1+\mu_{n-1})\, h(p_{n-1},p_n)\gamma\Big[\rho(p_{n-1},p_n)+\rho(p_n,Sp_n) \\
&\quad + \rho(p_n,p_{n-1})+\rho(p_{n-1},Sp_{n-1})\Big] \\
&\leq \mu_{n-1}\gamma\rho(p_{n-1},Sp_{n-1}) + \left(1-\mu_{n-1}^2\right) f(p_{n-1},p_n)\rho(p_{n-1},Sp_{n-1}) \\
&\quad + (1+\mu_{n-1})\, g(p_{n-1},p_n)\left[\rho(p_{n-1},Sp_{n-1})+\rho(p_n,Sp_n)\right] \\
&\quad + (1+\mu_{n-1})\, h(p_{n-1},p_n)\gamma\Big[\,(1-\mu_{n-1})\,\rho(p_{n-1},Sp_{n-1}) \\
&\quad + \rho(p_n,Sp_n)\,(1-\mu_n)\,\rho(p_n,Sp_n)+\rho(p_{n-1},Sp_{n-1})\Big] \\
&= \Big[\mu_{n-1}\gamma + \left(1-\mu_{n-1}^2\right) f(p_{n-1},p_n) + (1+\mu_{n-1})\, g(p_{n-1},p_n) \\
&\quad + (1+\mu_{n-1})\, h(p_{n-1},p_n)\gamma\,(2-\mu_{n-1})\,\Big]\rho(p_{n-1},Sp_{n-1}) \\
&\quad + \Big[\,(1+\mu_{n-1})\, g(p_{n-1},p_n) \\
&\quad + (1+\mu_{n-1})\, h(p_{n-1},p_n)\gamma\,(2-\mu_{n-1})\,\Big]\rho(p_n,Sp_n)
\end{aligned}$$

which implies that

$$\begin{aligned}
&\left[1-\left((1+\mu_{n-1})\, g(p_{n-1},p_n)+(1+\mu_{n-1})\, h(p_{n-1},p_n)\gamma\,(2-\mu_{n-1})\right)\right]\rho(p_n,Sp_n) \\
&\qquad \leq \Big[\mu_{n-1}\gamma + \left(1-\mu_{n-1}^2\right) f(p_{n-1},p_n) + (1+\mu_{n-1})\, g(p_{n-1},p_n) \\
&\qquad\quad + (1+\mu_{n-1})\, h(p_{n-1},p_n)\gamma\,(2-\mu_{n-1})\,\Big]\rho(p_{n-1},Sp_{n-1}).
\end{aligned}$$

By hypothesis, we know that $g(p,q)+2h(p,q) \leq \frac{1}{4\gamma^2}$ and $0 < \mu_{n-1} \leq \frac{1}{4\gamma^2}$. Using these conditions, we obtain that

(3.7)
$$\begin{aligned}
&(1+\mu_{n-1})\, g(p_{n-1},p_n) + (1+\mu_{n-1})\, h(p_{n-1},p_n)\gamma\,(2-\mu_{n-1}) \\
&\quad \leq (1+\mu_{n-1})\left[g(p_{n-1},p_n)+2h(p_{n-1},p_n)\gamma\right] \\
&\quad \leq (1+\mu_{n-1})\left[g(p_{n-1},p_n)+2h(p_{n-1},p_n)\right]\gamma
\end{aligned}$$

$$\leq \left(1+\frac{1}{4\gamma^2}\right)\frac{1}{4\gamma^2}\gamma < 1,$$

and

$$\begin{aligned}
&\left(1-\mu_{n-1}^2\right) f(p_{n-1},p_n) + (1+\mu_{n-1})\, g(p_{n-1},p_n) \\
&\quad + (1+\mu_{n-1})\, h(p_{n-1},p_n)\gamma\,(2-\mu_{n-1}) \\
&\leq (1+\mu_{n-1})\, f(p_{n-1},p_n) + (1+\mu_{n-1})\, g(p_{n-1},p_n) \\
&\quad + 2\,(1+\mu_{n-1})\, h(p_{n-1},p_n)\gamma \\
&= (1+\mu_{n-1})\left[f(p_{n-1},p_n) + g(p_{n-1},p_n) + 2h(p_{n-1},p_n)\gamma\right] \\
&\leq (1+\mu_{n-1})\left[f(p_{n-1},p_n) + g(p_{n-1},p_n) + 2h(p_{n-1},p_n)\right]\gamma \\
&\leq \left(1+\frac{1}{4\gamma^2}\right)\frac{1}{4\gamma^2}\gamma \\
&< 1,
\end{aligned}$$

hence

$$\rho(p_n, Sp_n) \leq \left[\frac{\begin{array}{c}\mu_{n-1}\gamma + (1+\mu_{n-1})\, f(p_{n-1},p_n) \\ + (1+\mu_{n-1})\, g(p_{n-1},p_n) \\ + (1+\mu_{n-1})\, h(p_{n-1},p_n)\gamma\,(2-\mu_{n-1})\end{array}}{1-\left(\begin{array}{c}(1+\mu_{n-1})\, g(p_{n-1},p_n) \\ +2\,(1+\mu_{n-1})\, h(p_{n-1},p_n)\gamma\end{array}\right)}\right]\rho(p_{n-1}, Sp_{n-1}). \tag{3.8}$$

Denote

$$\theta_{n-1} = \frac{\begin{array}{c}\mu_{n-1}\gamma + (1+\mu_{n-1})\, f(p_{n-1},p_n) + (1+\mu_{n-1})\, g(p_{n-1},p_n) \\ + (1+\mu_{n-1})\, h(p_{n-1},p_n)\gamma\,(2-\mu_{n-1})\end{array}}{1-\left((1+\mu_{n-1})\, g(p_{n-1},p_n) + 2\,(1+\mu_{n-1})\, h(p_{n-1},p_n)\gamma\right)}.$$

Then

$$\begin{aligned}
\theta_{n-1} &= \frac{\begin{array}{c}\mu_{n-1}\gamma + (1+\mu_{n-1})\, f(p_{n-1},p_n) + (1+\mu_{n-1})\, g(p_{n-1},p_n) \\ + (1+\mu_{n-1})\, h(p_{n-1},p_n)\gamma\,(2-\mu_{n-1})\end{array}}{1-\left((1+\mu_{n-1})\, g(p_{n-1},p_n) + 2\,(1+\mu_{n-1})\, h(p_{n-1},p_n)\gamma\right)} \\
&\leq \frac{\begin{array}{c}\mu_{n-1}\gamma + (1+\mu_{n-1})\, f(p_{n-1},p_n) + (1+\mu_{n-1})\, g(p_{n-1},p_n) \\ +2\,(1+\mu_{n-1})\, h(p_{n-1},p_n)\gamma\end{array}}{1-\left((1+\mu_{n-1})\, g(p_{n-1},p_n) + 2\,(1+\mu_{n-1})\, h(p_{n-1},p_n)\gamma\right)}
\end{aligned} \tag{3.9}$$

$$
\begin{aligned}
&< \frac{\begin{array}{c}\mu_{n-1}\gamma + (1+\mu_{n-1})\, f(p_{n-1},p_n) + (1+\mu_{n-1})\, g(p_{n-1},p_n) \\ +2\,(1+\mu_{n-1})\, h(p_{n-1},p_n)\gamma\end{array}}{1-\left(\begin{array}{c}(1+\mu_{n-1})\, f(p_{n-1},p_n) + (1+\mu_{n-1})\, g(p_{n-1},p_n) \\ +2\,(1+\mu_{n-1})\, h(p_{n-1},p_n)\gamma\end{array}\right)} \\
&< \frac{1+\frac{1}{4\gamma}}{1-\left(\begin{array}{c}(1+\mu_{n-1})\, f(p_{n-1},p_n) + (1+\mu_{n-1})\, g(p_{n-1},p_n) \\ +2\,(1+\mu_{n-1})\, h(p_{n-1},p_n)\gamma\end{array}\right)} - 1 \\
&\le \frac{8\gamma^2+1}{16\gamma^3-4\gamma^2-1} \\
&< 1.
\end{aligned}
$$

Therefore, from (3.8) we have

$$
\begin{aligned}
\rho(p_n, Sp_n) &\le \theta_{n-1}\rho(p_{n-1}, Sp_{n-1}) \\
&< \frac{8\gamma^2+1}{16\gamma^3-4\gamma^2-1}\rho(p_{n-1}, Sp_{n-1}).
\end{aligned} \tag{3.10}
$$

Applying Condition $(F1)$, we get that

$$
\begin{aligned}
F\left(\rho(p_n, Sp_n)\right) &\le F\left(\frac{8\gamma^2+1}{16\gamma^3-4\gamma^2-1}\rho(p_{n-1}, Sp_{n-1})\right) - \tau \\
&< F\left(\rho(p_{n-1}, Sp_{n-1})\right) - \tau.
\end{aligned}
$$

Using similar way, we have

$$
F\left(\rho(p_{n-1}, Sp_{n-1})\right) < F\left(\rho(p_{n-2}, Sp_{n-2})\right) - \tau.
$$

Thus, we obtain that

$$
\begin{aligned}
F\left(\rho(p_n, Sp_n)\right) &< F\left(\rho(p_{n-1}, Sp_{n-1})\right) - \tau \\
&< F\left(\rho(p_{n-2}, Sp_{n-2})\right) - 2\tau \\
&\ \vdots \\
&< F\left(\rho(p_0, Sp_0)\right) - n\tau.
\end{aligned}
$$

Taking limit on both sides as $n \to \infty$ in the above inequality, we have that

$$
\lim_{n\to\infty} F\left(\rho(p_n, Sp_n)\right) = -\infty.
$$

It follows from Condition $(F2)$ that

$$
\lim_{n\to\infty} \rho(p_n, Sp_n) = 0. \tag{3.11}
$$

From (3.3) and (3.11), we have

$$
\lim_{n\to\infty} \rho(p_n, p_{n+1}) = 0.
$$

Now, we will show that $\{p_n\}$ is a Cauchy sequence. For this, we will use similar method at the proof in [5]. Assume that $\{p_n\}$ is not a Cauchy sequence. Then, there exist $\overline{\epsilon}$ and the subsequences $\left\{p_{r(k)}\right\}$ and $\left\{p_{t(k)}\right\}$ of $\{p_n\}$, such that $r(k)$ is the smallest natural index with $r(k) > t(k) > k$,

$$\rho\left(p_{r(k)}, p_{t(k)}\right) \geq \overline{\epsilon}$$

and

$$\rho\left(p_{r(k)-1}, p_{t(k)}\right) < \overline{\epsilon}.$$

Then, we have

$$\begin{aligned} \overline{\epsilon} &\leq \rho\left(p_{r(k)}, p_{t(k)}\right) \\ &\leq \gamma\rho\left(p_{r(k)}, p_{t(k)+1}\right) + \gamma\rho\left(p_{t(k)+1}, p_{t(k)}\right). \end{aligned} \tag{3.12}$$

Using (3.3) and $\lim_{n\to\infty} \rho(p_n, p_{n+1}) = 0$, we obtain

$$\frac{\overline{\epsilon}}{\gamma} \leq \limsup_{k\to\infty} \rho\left(p_{r(k)}, p_{t(k)+1}\right).$$

Also, we have

$$\begin{aligned} \rho\left(p_{r(k)}, p_{t(k)+1}\right) &= \rho\left(w\left(p_{r(k)-1}, Sp_{r(k)-1}; \mu_{r(k)-1}\right), p_{t(k)+1}\right) \\ &\leq \mu_{r(k)-1}\rho\left(p_{r(k)-1}, p_{t(k)+1}\right) \\ &\quad + \left(1-\mu_{r(k)-1}\right)\rho\left(Sp_{r(k)-1}, p_{t(k)+1}\right) \\ &\leq \mu_{r(k)-1}\rho\left(p_{r(k)-1}, p_{t(k)+1}\right) \\ &\quad + \left(1-\mu_{r(k)-1}\right)\gamma\rho\left(Sp_{r(k)-1}, Sp_{t(k)+1}\right) \\ &\quad + \left(1-\mu_{r(k)-1}\right)\gamma\rho\left(Sp_{t(k)+1}, p_{t(k)+1}\right). \end{aligned} \tag{3.13}$$

Using (3.1), we have

$$\begin{aligned} &F\left(\gamma\rho\left(Sp_{r(k)-1}, Sp_{t(k)+1}\right)\right) \\ &\quad \leq F\Big(f\left(p_{r(k)-1}, p_{t(k)+1}\right)\rho\left(p_{r(k)-1}, p_{t(k)+1}\right) \\ &\qquad + g\left(p_{r(k)-1}, p_{t(k)+1}\right)\left[\rho\left(p_{r(k)-1}, Sp_{r(k)-1}\right) + \rho\left(p_{t(k)+1}, Sp_{t(k)+1}\right)\right] \\ &\qquad + h\left(p_{r(k)-1}, p_{t(k)+1}\right)\left[\rho\left(p_{r(k)-1}, Sp_{t(k)+1}\right) + \rho\left(p_{t(k)+1}, Sp_{r(k)-1}\right)\right]\Big) \\ &\qquad - \tau. \end{aligned}$$

Thus

$$\begin{aligned} &F\Big[\mu_{r(k)-1}\rho\left(p_{r(k)-1}, p_{t(k)+1}\right) + \left(1-\mu_{r(k)-1}\right)\gamma\rho\left(Sp_{r(k)-1}, Sp_{t(k)+1}\right) \\ &\qquad + \left(1-\mu_{r(k)-1}\right)\gamma\rho\left(Sp_{t(k)+1}, p_{t(k)+1}\right)\Big] \\ &\quad \leq F\Bigg[\mu_{r(k)-1}\rho\left(p_{r(k)-1}, p_{t(k)+1}\right) \end{aligned}$$

$$
\begin{aligned}
&+\left(1-\mu_{r(k)-1}\right)\Big[f\left(p_{r(k)-1},p_{t(k)+1}\right)\rho\left(p_{r(k)-1},p_{t(k)+1}\right)\\
&+g\left(p_{r(k)-1},p_{t(k)+1}\right)\left[\rho\left(p_{r(k)-1},Sp_{r(k)-1}\right)+\rho\left(p_{t(k)+1},Sp_{t(k)+1}\right)\right]\\
&+h\left(p_{r(k)-1},p_{t(k)+1}\right)\left[\rho\left(p_{r(k)-1},Sp_{t(k)+1}\right)+\rho\left(p_{t(k)+1},Sp_{r(k)-1}\right)\right]\Big]\\
&+\left(1-\mu_{r(k)-1}\right)\gamma\rho\left(Sp_{t(k)+1},p_{t(k)+1}\right)\Bigg]-\tau.
\end{aligned}
$$

Using $(F1)$ and (3.13), we write

$$
\begin{aligned}
&\rho\left(p_{r(k)},p_{t(k)+1}\right)\\
&\quad\le\mu_{r(k)-1}\rho\left(p_{r(k)-1},p_{t(k)+1}\right)\\
&\qquad+\left(1-\mu_{r(k)-1}\right)\Big[f\left(p_{r(k)-1},p_{t(k)+1}\right)\rho\left(p_{r(k)-1},p_{t(k)+1}\right)\\
&\qquad+g\left(p_{r(k)-1},p_{t(k)+1}\right)\left[\rho\left(p_{r(k)-1},Sp_{r(k)-1}\right)+\rho\left(p_{t(k)+1},Sp_{t(k)+1}\right)\right]\\
&\qquad+h\left(p_{r(k)-1},p_{t(k)+1}\right)\left[\rho\left(p_{r(k)-1},Sp_{t(k)+1}\right)+\rho\left(p_{t(k)+1},Sp_{r(k)-1}\right)\right]\\
&\qquad+\gamma\rho\left(Sp_{t(k)+1},p_{t(k)+1}\right)\Big]\\
&\le\mu_{r(k)-1}\gamma\left[\rho\left(p_{r(k)-1},p_{t(k)}\right)+\rho\left(p_{t(k)},p_{t(k)+1}\right)\right]+\left(1-\mu_{r(k)-1}\right)\\
&\qquad\times\Bigg[f\left(p_{r(k)-1},p_{t(k)+1}\right)\gamma\left[\rho\left(p_{r(k)-1},p_{t(k)}\right)+\rho\left(p_{t(k)},p_{t(k)+1}\right)\right]\\
&\qquad+g\left(p_{r(k)-1},p_{t(k)+1}\right)\left[\rho\left(p_{r(k)-1},Sp_{r(k)-1}\right)+\rho\left(p_{t(k)+1},Sp_{t(k)+1}\right)\right]\\
&\qquad+h\left(p_{r(k)-1},p_{t(k)+1}\right)\Big[\gamma\rho\left(p_{r(k)-1},p_{t(k)}\right)+\gamma\rho\left(p_{t(k)},Sp_{t(k)+1}\right)\\
&\qquad+\gamma\rho\left(p_{t(k)+1},p_{r(k)}\right)+\gamma\rho\left(p_{r(k)},Sp_{r(k)-1}\right)\Big]\\
&\qquad+\gamma\rho\left(Sp_{t(k)+1},p_{t(k)+1}\right)\Bigg]\\
&\quad\le\mu_{r(k)-1}\gamma\left[\rho\left(p_{r(k)-1},p_{t(k)}\right)+\rho\left(p_{t(k)},p_{t(k)+1}\right)\right]+\left(1-\mu_{r(k)-1}\right)\\
&\qquad\times\Bigg[f\left(p_{r(k)-1},p_{t(k)+1}\right)\gamma\left[\rho\left(p_{r(k)-1},p_{t(k)}\right)+\rho\left(p_{t(k)},p_{t(k)+1}\right)\right]\\
&\qquad+g\left(p_{r(k)-1},p_{t(k)+1}\right)\left[\rho\left(p_{r(k)-1},Sp_{r(k)-1}\right)+\rho\left(p_{t(k)+1},Sp_{t(k)+1}\right)\right]\\
&\qquad+h\left(p_{r(k)-1},p_{t(k)+1}\right)\Big[\gamma\rho\left(p_{r(k)-1},p_{t(k)}\right)+\gamma^2\rho\left(p_{t(k)},p_{t(k)+1}\right)\\
&\qquad+\gamma^2\rho\left(p_{t(k)+1},Sp_{t(k)+1}\right)+\gamma\rho\left(p_{t(k)+1},p_{r(k)}\right)\\
&\qquad+\gamma^2\rho\left(p_{r(k)},p_{r(k)-1}\right)+\gamma^2\rho\left(p_{r(k)-1},Sp_{r(k)-1}\right)\Big]
\end{aligned}
$$

$$+ \gamma\rho\left(Sp_{t(k)+1}, p_{t(k)+1}\right)\Bigg].$$

Taking lim sup as $k \to \infty$ in the above inequality, we obtain

$$\begin{aligned}
&\limsup_{k\to\infty} \rho\left(p_{r(k)}, p_{t(k)+1}\right) \\
&\quad \le \limsup_{k\to\infty} \Big\{ \mu_{r(k)-1}\gamma\rho\left(p_{r(k)-1}, p_{t(k)}\right) \\
&\qquad + \left(1-\mu_{r(k)-1}\right) f\left(p_{r(k)-1}, p_{t(k)+1}\right)\gamma\rho\left(p_{r(k)-1}, p_{t(k)}\right) \\
&\qquad + h\left(p_{r(k)-1}, p_{t(k)+1}\right)\gamma^2 2\rho\left(p_{r(k)-1}, p_{t(k)}\right)\Big\},
\end{aligned}$$

which implies that

$$\frac{\overline{\epsilon}}{\gamma} \le \limsup_{k\to\infty} \rho\left(p_{r(k)}, p_{t(k)+1}\right) \le \gamma\overline{\epsilon}\left[\frac{1}{4\gamma^2} + \frac{1}{4\gamma^2}\right].$$

Then, we have

$$\frac{\overline{\epsilon}}{\gamma} \le \limsup_{k\to\infty} \rho\left(p_{r(k)}, p_{t(k)+1}\right) \le \frac{\overline{\epsilon}}{2\gamma},$$

which is a contradiction. Thus, $\{p_n\}$ is a Cauchy sequence in C. Since (C, ρ, ω) is a complete convex b-metric space, there exists $p^* \in C$ such that $p_n \to p^* \in C$ as $n \to \infty$. Now, we will show that p^* is a fixed point of S. Note that

$$\begin{aligned}
\rho\left(p^*, Sp^*\right) &\le \gamma\left[\rho\left(p^*, p_n\right) + \rho\left(p_n, Sp^*\right)\right] \\
&\le \gamma\rho\left(p^*, p_n\right) + \gamma^2\rho\left(p_n, Sp_n\right) + \gamma^2\rho\left(Sp_n, Sp^*\right).
\end{aligned} \tag{3.14}$$

Noticing that

$$\begin{aligned}
F\left(\gamma\rho\left(Sp_n, Sp^*\right)\right) \le F\Big(& f\left(p_n, p^*\right)\rho\left(p_n, p^*\right) \\
&+ g\left(p_n, p^*\right)\left[\rho\left(p_n, Sp_n\right) + \rho\left(p^*, Sp^*\right)\right] \\
&+ h\left(p_n, p^*\right)\left[\rho\left(p_n, Sp^*\right) + \rho\left(p^*, Sp_n\right)\right]\Big) - \tau.
\end{aligned} \tag{3.15}$$

From (3.14) and (3.15), we get

$$\begin{aligned}
F\left(\rho\left(p^*, Sp^*\right)\right) &\le F\left(\gamma\rho\left(p^*, p_n\right) + \gamma^2\rho\left(p_n, Sp_n\right) + \gamma^2\rho\left(Sp_n, Sp^*\right)\right) \\
&\le F\Big(\gamma\rho\left(p^*, p_n\right) + \gamma^2\rho\left(p_n, Sp_n\right) + \gamma f\left(p_n, p^*\right)\rho\left(p_n, p^*\right) \\
&\quad + \gamma g\left(p_n, p^*\right)\left[\rho\left(p_n, Sp_n\right) + \rho\left(p^*, Sp^*\right)\right] \\
&\quad + \gamma h\left(p_n, p^*\right)\left[\rho\left(p_n, Sp^*\right) + \rho\left(p^*, Sp_n\right)\right]\Big) - \tau \\
&\le F\Big(\gamma\rho\left(p^*, p_n\right) + \gamma^2\rho\left(p_n, Sp_n\right) + \gamma f\left(p_n, p^*\right)\rho\left(p_n, p^*\right) \\
&\quad + \gamma g\left(p_n, p^*\right)\left[\rho\left(p_n, Sp_n\right) + \rho\left(p^*, Sp^*\right)\right]
\end{aligned}$$

$$+\gamma h(p_n,p^*)\Big[\gamma\rho(p_n,p^*)+\gamma\rho(p^*,Sp^*)$$
$$+\gamma\rho(p^*,p_n)+\gamma\rho(p_n,Sp_n)\Big]\Big)-\tau.$$

From above inequality, $(F1)$ and (3.10), we have

$$\begin{aligned}
&F\left[\left(1-\gamma g(p_n,p^*)-\gamma^2 h(p_n,p^*)\right)\rho(p^*,Sp^*)\right]\\
&\quad< F\Big(\gamma\rho(p^*,p_n)+\gamma^2\rho(p_n,Sp_n)+\gamma f(p_n,p^*)\rho(p_n,p^*)\\
&\qquad+\gamma g(p_n,p^*)\rho(p_n,Sp_n)\\
&\qquad+\gamma h(p_n,p^*)\left[\gamma\rho(p_n,p^*)+\gamma\rho(p^*,p_n)+\gamma\rho(p_n,Sp_n)\right]\Big)-\tau\\
&\quad< F\left(\gamma\rho(p^*,p_n)+\gamma^2\left(\frac{8\gamma^2+1}{16\gamma^3-4\gamma^2-1}\right)^n\rho(p_0,Sp_0)\right.\\
&\qquad+\gamma f(p_n,p^*)\rho(p_n,p^*)+\gamma g(p_n,p^*)\left(\frac{8\gamma^2+1}{16\gamma^3-4\gamma^2-1}\right)^n\rho(p_0,Sp_0)\\
&\qquad\left.+\gamma h(p_n,p^*)\left[\gamma\rho(p_n,p^*)+\gamma\rho(p^*,p_n)+\left(\frac{8\gamma^2+1}{16\gamma^3-4\gamma^2-1}\right)^n\rho(p_0,Sp_0)\right]\right)\\
&\qquad-\tau.
\end{aligned}$$

From (3.9), we know that $\frac{8\gamma^2+1}{16\gamma^3-4\gamma^2-1}<1$. This implies that

$$\begin{aligned}
&\lim_{n\to\infty}\left\{\gamma\rho(p^*,p_n)+\gamma^2\left(\frac{8\gamma^2+1}{16\gamma^3-4\gamma^2-1}\right)^n\rho(p_0,Sp_0)\right.\\
&\qquad+\gamma f(p_n,p^*)\rho(p_n,p^*)+\gamma g(p_n,p^*)\left(\frac{8\gamma^2+1}{16\gamma^3-4\gamma^2-1}\right)^n\rho(p_0,Sp_0)\\
&\qquad\left.+\gamma h(p_n,p^*)\left[\gamma\rho(p_n,p^*)+\gamma\rho(p^*,p_n)+\left(\frac{8\gamma^2+1}{16\gamma^3-4\gamma^2-1}\right)^n\rho(p_0,Sp_0)\right]\right\}\\
&\quad=0.
\end{aligned}$$

Thus

$$\begin{aligned}
&\lim_{n\to\infty}F\left[\left(1-\gamma g(p_n,p^*)-\gamma^2 h(p_n,p^*)\right)\rho(p^*,Sp^*)\right]\\
&\quad=\lim_{n\to\infty}F\left(\gamma\rho(p^*,p_n)+\gamma^2\left(\frac{8\gamma^2+1}{16\gamma^3-4\gamma^2-1}\right)^n\rho(p_0,Sp_0)\right.\\
&\qquad+\gamma f(p_n,p^*)\rho(p_n,p^*)+\gamma g(p_n,p^*)\left(\frac{8\gamma^2+1}{16\gamma^3-4\gamma^2-1}\right)^n\rho(p_0,Sp_0)\\
&\qquad\left.+\gamma h(p_n,p^*)\left[\gamma\rho(p_n,p^*)+\gamma\rho(p^*,p_n)+\left(\frac{8\gamma^2+1}{16\gamma^3-4\gamma^2-1}\right)^n\rho(p_0,Sp_0)\right]\right)\\
&\quad=-\infty,
\end{aligned}$$

consequently,

$$\lim_{n\to\infty}\left(1-\gamma g\left(p_n,p^*\right)-\gamma^2 h\left(p_n,p^*\right)\right)\rho\left(p^*,Sp^*\right)=0. \tag{3.16}$$

By assumptions, we know that $\mu_{n-1}\le\frac{1}{4\gamma^2}$ for all $n\in\mathbb{N}$ and $f(p,q)+g(p,q)+2h(p,q)\le\frac{1}{4\gamma^2}$ for every $p,q\in C$. This implies that

$$\gamma g\left(p_n,p^*\right)+\gamma^2 h\left(p_n,p^*\right)<1,$$

for all $n\in\mathbb{N}$. Using (3.16), we obtain $\rho\left(p^*,Sp^*\right)=0$. That is, p^* is a fixed point of the mapping S. Finally, we will show the uniqueness of the fixed point of the mapping S. Suppose, on the contrary, that q^* is another fixed point of S. Then

$$\begin{aligned}
F\left(\gamma\rho\left(p^*,q^*\right)\right)&=F\left(\gamma\rho\left(Sp^*,Sq^*\right)\right)\\
&\le F\Big(f\left(p^*,q^*\right)\rho\left(p^*,q^*\right)+g\left(p^*,q^*\right)\left[\rho\left(p^*,Sp^*\right)+\rho\left(q^*,Sq^*\right)\right]\\
&\quad+h\left(p^*,q^*\right)\left[\rho\left(p^*,Sq^*\right)+\rho\left(q^*,Sp^*\right)\right]\Big)-\tau\\
&=F\left(f\left(p^*,q^*\right)\rho\left(p^*,q^*\right)+h\left(p^*,q^*\right)\left[\rho\left(p^*,Sq^*\right)+\rho\left(q^*,Sp^*\right)\right]\right)-\tau\\
&=F\left(f\left(p^*,q^*\right)\rho\left(p^*,q^*\right)+2h\left(p^*,q^*\right)\rho\left(p^*,q^*\right)\right)-\tau\\
&=F\left[\rho\left(p^*,q^*\right)\left(f\left(p^*,q^*\right)+2h\left(p^*,q^*\right)\right)\right]-\tau,
\end{aligned}$$

which is a contradiction. Therefore, $\rho\left(p^*,q^*\right)=0$. This shows the uniqueness of the fixed point of the mapping S. That is, $p^*=q^*$. □

Choosing $f=g=0$ in the above theorem, the following result for F-Chatterjea type contraction in a complete convex b-metric space is obvious.

Theorem 3.3. *Let (C,ρ,ω) be a complete convex b-metric space with a convex structure ω and $S:C\to C$ be F-Chatterjea type contraction is as follows. Suppose there exists $h:C\times C\to\left[0,\frac{1}{4\gamma^2}\right]$ the following hold:*

$$\tau+F\left(\gamma\rho\left(Sp,Sq\right)\right)\le F\left(h\left(p,q\right)\left[\rho\left(p,Sq\right)+\rho\left(q,Sp\right)\right]\right)$$

for every $p,q\in C$. Suppose that the sequence $\{p_n\}$ is defined as follows

$$p_n=\omega\left(p_{n-1},Sp_{n-1};\mu_{n-1}\right),$$

where $0<\mu_{n-1}\le\frac{1}{4\gamma^2}$ for all $n\in\mathbb{N}$. If $2h(p,q)\le\frac{1}{4\gamma^2}$, then S has a unique fixed point in C.

Choosing $f=h=0$ in Theorem 3.2, we obtain that Theorem 2.2 in [5]. Choosing also $h=0$ in Theorem 3.2, we have also the following result.

Theorem 3.4. *Let (C,ρ,ω) be a complete convex b-metric space with a convex structure ω and $S: C \to C$ be F-Reich type contraction as follows. Suppose there exist $f,g : C\times C \to \left[0,\frac{1}{4\gamma^2}\right]$ the following holds:*

$$\tau + F\left(\gamma\rho\left(Sp,Sq\right)\right) \leq F\left(f\left(p,q\right)\rho\left(p,q\right) + g\left(p,q\right)\left[\rho\left(p,Sp\right)+\rho\left(q,Sq\right)\right]\right)$$

for every $p,q\in C$. Suppose that the sequence $\{p_n\}$ is defined as follows

$$p_n = \omega\left(p_{n-1}, Sp_{n-1};\mu_{n-1}\right),$$

where $0<\mu_{n-1}\leq \frac{1}{4\gamma^2}$ for all $n\in\mathbb{N}$. If $f\left(p,q\right)+g\left(p,q\right)\leq \frac{1}{4\gamma^2}$, then S has a unique fixed point in C.

Remark 3.5. Theorem 3.4 expands the Theorem 3.2 in [5], relaxing the contraction condition from $f\left(p,q\right)+2g\left(p,q\right)\leq\frac{1}{4\gamma^2}$ to $f\left(p,q\right)+g\left(p,q\right)\leq \frac{1}{4\gamma^2}$.

Finally, we will give the following example that satisfies the conditions of the main Theorem 3.2.

Example 3.6. Let $C=[0,\infty)$ and $Sp=\frac{p}{9}$ for all $p\in C$. For any $p,q\in C$, we define mapping $\rho : C\times C\to[0,\infty)$ by the formula $\rho\left(p,q\right) = \left(p-q\right)^2$, while the mapping $\omega : C\times C\times[0,1]\to C$ is defined as

$$\omega\left(p,q;\mu\right)\leq \mu p+\left(1-\mu\right)q, \qquad \text{for all } p,q\in C \text{ and all } \mu\in[0,1].$$

Set $p_n=\omega\left(p_{n-1},Sp_{n-1};\mu_{n-1}\right)$ and $\mu_{n-1}=\frac{1}{6\gamma^2}$. Then (C,ρ,ω) is a complete convex b-metric space with $\gamma=2$. Now, we define $f,g,h : C\times C\to\left[0,\frac{1}{2}\right)$ as

$$f\left(p,q\right)=\begin{cases}\frac{1}{8\gamma^2}, & \text{if } p<q,\\ \frac{1}{8\gamma^2+10}, & \text{otherwise,}\end{cases}$$

$g\left(p,q\right)=\frac{1}{8\gamma^2}$ and $h\left(p,q\right)=0$ for all $p,q\in C$. It is easy to see that $f\left(p,q\right)+g\left(p,q\right)+2h\left(p,q\right)\leq\frac{1}{4\gamma^2}$. Then the mapping S holds all conditions of Theorem 3.2. That is, S has a unique fixed point in C. Indeed,

$$\begin{aligned}\ln\left[\gamma\rho\left(Sp,Sq\right)\right] &= \ln\left[\gamma\left(\frac{p}{9}-\frac{q}{9}\right)^2\right]\\ &= \ln\left[\frac{2}{81}\left(p-q\right)^2\right]\\ &\leq \ln\left[f\left(p,q\right)\rho\left(p,q\right)\right]\\ &\leq \ln\Big[f\left(p,q\right)\rho\left(p,q\right)+g\left(p,q\right)\left[\rho\left(p,Sp\right)+\rho\left(q,Sq\right)\right]\\ &\quad + h\left(p,q\right)\left[\rho\left(p,Sq\right)+\rho\left(q,Sp\right)\right]\Big],\end{aligned}$$

that is,

$$F\left[\gamma\rho\left(Sp,Sq\right)\right]\leq F\Big[f\left(p,q\right)\rho\left(p,q\right)+g\left(p,q\right)\left[\rho\left(p,Sp\right)+\rho\left(q,Sq\right)\right] + h\left(p,q\right)\left[\rho\left(p,Sq\right)+\rho\left(q,Sp\right)\right]\Big].$$

Also, $F(x)=\ln x$ satisfies the conditions $(F1)$ and $(F2)$. Then, the inequality (3.1) is satisfied for $\tau\leq\ln\left(\frac{81}{66}\right)$. We choose $p_0\in C\setminus\{0\}$. Combining with $p_n=\omega\left(p_{n-1},Sp_{n-1};\mu_{n-1}\right)$, $\mu_{n-1}=\frac{1}{6\gamma^2}=\frac{1}{24}$ and $Sp=\frac{p}{9}$, we obtain

$$\begin{aligned}p_n&=\omega\left(p_{n-1},Sp_{n-1};\mu_{n-1}\right)\\&=\mu_{n-1}p_{n-1}+\left(1-\mu_{n-1}\right)Sp_{n-1}\\&=\frac{1}{24}p_{n-1}+\left(1-\frac{1}{24}\right)\frac{p_{n-1}}{9}\\&=\frac{4}{27}p_{n-1}.\end{aligned}$$

Similarly, we have

$$p_{n-1}=\frac{4}{27}p_{n-2},\qquad p_{n-2}=\frac{4}{27}p_{n-3},\qquad \ldots\qquad p_1=\frac{4}{27}p_0.$$

Therefore,

$$p_n=\left(\frac{4}{27}\right)^n p_0,\qquad Sp_n=\frac{1}{9}\left(\frac{4}{27}\right)^n p_0.$$

If we take limits of the above sequences as $n\to\infty$, we get $p_n\to 0$ and $Sp_n\to 0$. That is 0 is a fixed point of S.

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