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Sahand Communications in Mathematical Analysis

Print ISSN: 2322-5807
Online ISSN: 2423-3900
Volume: 19
Number: 2
Pages: 33-47

Sahand Commun. Math. Anal.
DOI: 10.22130/scma.2022.521137.888

Sahand Communications in Mathematical Analysis (SCMA) Vol. 19 No. 2 (2022), 33-47
http://scma.maragheh.ac.ir
DOI: 10.22130/scma.2022.521137.888

Essential Norm of the Generalized Integration Operator from Zygmund Space into Weighted Dirichlet Type Space

Fariba Alighadr[1], Hamid Vaezi[2*] and Mostafa Hassanlou[3]

Abstract. Let $H(\mathbb{D})$ be the space of all analytic functions on the open unit disc $\mathbb{D}$ in the complex plane $\mathbb{C}$. In this paper, we investigate the boundedness and compactness of the generalized integration operator

$$I_{g,\varphi}^{(n)}(f)(z) = \int_0^z f^{(n)}(\varphi(\xi))g(\xi)\; d\xi, \quad z \in \mathbb{D},$$

from Zygmund space into weighted Dirichlet type space, where φ is an analytic self-map of $\mathbb{D}$, $n \in \mathbb{N}$ and $g \in H(\mathbb{D})$. Also we give an estimate for the essential norm of the above operator.

1. Introduction and Preliminaries

Let $\mathbb{D}$ be the open unit disc in the complex plane $\mathbb{C}$ and $H(\mathbb{D})$ be the space of all analytic functions on $\mathbb{D}$. Let $p \in (0, \infty)$, $\beta > -1$ and $\mathcal{A}_\beta^p$ denotes the space of all $f \in H(\mathbb{D})$ for which

$$\|f\|_{\mathcal{A}_\beta^p}^p = \int_{\mathbb{D}} |f(z)|^p \left(\log \frac{1}{|z|}\right)^\beta dA(z) < \infty,$$

where dA denotes the normalized Lebesgue area measure on $\mathbb{D}$. The space $\mathcal{A}_\beta^p$ is called the weighted Bergman space. The weighted Bergman space $\mathcal{A}_\beta^p$, for $p \geq 1$ is a Banach space, and for $p = 2$ is a Hilbert space. It is well-known that $f \in \mathcal{A}_\beta^p$ if and only if

$$\int_{\mathbb{D}} |f(z)|^p \left(1 - |z|^2\right)^\beta dA(z) < \infty.$$

2020 *Mathematics Subject Classification.* 47B38, 30H99, 47G10.
Key words and phrases. Generalized integration operator, Dirichlet type space, Zygmund space, Essential norm.
Received: 14 December 2020, Accepted: 20 February 2022.
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Let p and β be as above. The weighted Dirichlet type space $\mathcal{D}^p_\beta$ is the space of all functions $f \in H(\mathbb{D})$ for which

$$\|f\|^p_{\mathcal{D}^p_\beta} = \int_{\mathbb{D}} \left|f'(z)\right|^p \left(\log \frac{1}{|z|}\right)^\beta dA(z) < \infty.$$

Thus, $f \in \mathcal{D}^p_\beta$ if and only if $f' \in \mathcal{A}^p_\beta$.

We denote the space of all continuous functions on $\overline{\mathbb{D}}$ by $C(\overline{\mathbb{D}})$. Denote by $\mathcal{Z}$ the class of all $f \in H(\mathbb{D}) \cap C(\overline{\mathbb{D}})$ such that

$$\|f\| = \sup_{e^{i\theta} \in \partial\mathbb{D},\ h>0} \frac{\left|f\left(e^{i(\theta+h)}\right) + f\left(e^{i(\theta-h)}\right) - 2f\left(e^{i\theta}\right)\right|}{h} < \infty.$$

The class $\mathcal{Z}$ becomes a Banach space with the following norm

$$|f\|_{\mathcal{Z}} = |f(0)| + \left|f'(0)\right| + \sup_{z\in\mathbb{D}} \left(1-|z|^2\right) \left|f''(z)\right|.$$

The Banach space $\mathcal{Z}$ with the above norm is called the Zygmund space. Note that by Theorem 5.3 of [8], an analytic function f on $\mathbb{D}$ belongs to $\mathcal{Z}$ if and only if

$$\sup_{z\in\mathbb{D}} \left(1-|z|^2\right) \left|f''(z)\right| < \infty.$$

Let u be an analytic function on $\mathbb{D}$ and φ a nonconstant analytic self-map of $\mathbb{D}$. The weighted composition operator uC_φ induced by u and φ is defined on $H(\mathbb{D})$ as follows:

$$uC_\varphi(f) = ufo\varphi.$$

Putting $u \equiv 1, uC_\varphi$ reduces to the composition operator C_φ. For general background on composition operators, we refer to [7, 17] and for weighted composition operators acting between some spaces of analytic functions we refer for example to [1, 6, 10, 16].

In this paper, we consider an integration operator $I^{(n)}_{g,\varphi}$ which is defined on $H(\mathbb{D})$ by

$$I^{(n)}_{g,\varphi}(f)(z) = \int_0^z f^{(n)}(\varphi(\xi))g(\xi)\ d\xi, \quad z \in \mathbb{D},$$

where φ is an analytic self-map of $\mathbb{D}$, $n \in \mathbb{N}$ and $g \in H(\mathbb{D})$. This operator is called the generalized integration operator, which was introduced in [18]. This operator is a generalization of the Riemann-Stieltjes operator I_g induced by g, defined by

$$I_g f(z) = \int_0^z f(\zeta)g'(\zeta)d\zeta, \quad z \in \mathbb{D}.$$

Yu and Liu in [22] characterized the boundedness and compactness of the Riemann-Stieltjes operator I_g from weighted Bloch spaces into Bergman-type spaces. The essential norm of the integral operator I_g on some spaces of analytic functions was studied by Liu, Lou and Xiong in [14].

In fact, the operator $I_{g,\varphi}^{(n)}$ can induce some known operators. For example, when $n = 1$, $I_{g,\varphi}^{(n)}$ reduces to an integration operator recently studied by Li and Stevic in [11–13]. Taking $n = 1$ and $g(z) = \varphi'(z)$, we obtain the composition operator C_φ defined by $C_\varphi f = f(\varphi) - f(\varphi(0)), f \in H(D)$.

Recently years, Sharma and Sharma in [18] have characterized the boundedness and compactness of generalized integration operator $I_{g,\varphi}^{(n)}$ from Bloch-type spaces to weighted BMOA.

Boundedness and compactness of Riemann-Stieltjes operator from mixed norm spaces to Zygmund-type spaces on the unit ball was studied by Liu and Yu in [15]. Zhu in [23] investigated the boundedness and compactness of generalized integration operators from H^∞ to Zygmund-type spaces. He and Cao in [9] investigated the boundedness and compactness of generalized integration operators between Bloch-type spaces and $F(p, q, s)$ spaces. The equivalent conditions for the boundedness and compactness of the generalized integration operator between the Bloch-type space and weighted Dirichlet-type spaces are given in [2] by Alighadr-Ardebili, Vaezi and Hassanlou. For related integral-type operators on unit disc and also in $\mathbb{C}^n$, see for example [4, 5, 19, 21].

Motivated by the above results, in this article, first we give an equivalent condition for the boundedness and compactness of the generalized integration operator $I_{g,\varphi}^{(n)}$ from Zygmund space into weighted Dirichlet type space. Then, we give an estimate for the essential norm of this operator.

Throughout this paper, the notation $a \preceq b$ means that there exists a positive constant C such that $a \leq Cb$. If both $a \preceq b$ and $b \preceq a$ occur, then $a \sim b$.

2. The Boundedness and Compactness of $I_{g,\varphi}^{(n)} : \mathcal{Z} \longrightarrow \mathcal{D}_\beta^p$

In this section, we characterize the boundedness and compactness of the generalized integration operator $I_{g,\varphi}^{(n)}$ from the Zygmund space into the weighted Dirichlet-type space.

Lemma 2.1 ([3, Lemma 3])**.** *Let $k \in \mathbb{N}$ and $k \geq 2$. Then there exists a positive constant C_k independent of $f \in \mathcal{Z}$ and $z \in \mathbb{D}$ such that*

$$|f^{(k)}(z)| \leq \frac{C_k \|f\|_{\mathcal{Z}}}{(1 - |z|^2)^{k-1}}.$$

Lemma 2.2 ([3, Lemma 2]). *For each $f \in \mathcal{Z}$ and $z \in \mathbb{D}$, it follows that*

$$|f'(z)| \leq \log \frac{e}{1-|z|^2} \|f\|_{\mathcal{Z}}.$$

Theorem 2.3. *Let $g \in H(\mathbb{D})$, $n \in \mathbb{N}$, φ be an analytic self-map of $\mathbb{D}$, $0 < p < \infty$ and $\beta > -1$. Then the following statements are equivalent:*

(i) $I_{g,\varphi}^{(n)} : \mathcal{Z} \longrightarrow \mathcal{D}_\beta^p$ *is bounded.*

(ii)

$$M = \int_{\mathbb{D}} \frac{|g(z)|^p}{(1-|\varphi(z)|^2)^{p(n-1)}} \left(\log \frac{1}{|z|}\right)^\beta dA(z) < \infty.$$

Proof. $(i) \Rightarrow (ii)$. Suppose that $I_{g,\varphi}^{(n)} : \mathcal{Z} \longrightarrow \mathcal{D}_\beta^p$ is bounded. According to [11] define

$$h(z) = (z-1)\left[\left(1 + \log \frac{1}{1-z}\right)^2 + 1\right], \tag{2.1}$$

and

$$f_a(z) = \frac{h(\bar{a}z)}{\bar{a}} \left(\log \frac{1}{1-|a|^2}\right)^{-1}, \tag{2.2}$$

for every $a \in \mathbb{D}$ with $1/\sqrt{2} < |a| < 1$. Then

$$f_a'(z) = \left(\log \frac{1}{1-\bar{a}z}\right)^2 \left(\log \frac{1}{1-|a|^2}\right)^{-1},$$

$$f_a''(z) = \frac{2\bar{a}}{1-\bar{a}z} \left(\log \frac{1}{1-\bar{a}z}\right) \left(\log \frac{1}{1-|a|^2}\right)^{-1},$$

and for $n \geq 3$ we have

$$f_a^{(n)}(z) = \frac{1}{(1-\bar{a}z)^{n-1}} \left(\log \frac{1}{1-|a|^2}\right)^{-1} \left[A_n \log \frac{1}{1-\bar{a}z} + B_n\right],$$

where A_n and B_n are constants such that $1 \leq |A_n| \leq |B_n|$. Thus, in this case we have

$$\left|f_a^{(n)}(z)\right| \geq \frac{|A_n|}{(1-\bar{a}z)^{n-1}} \left(\log \frac{1}{1-|a|^2}\right)^{-1} \left[\log \frac{1}{1-\bar{a}z} + 1\right].$$

Since $f_a \in \mathcal{Z}$ and $I_{g,\varphi}^{(n)}$ is bounded, we have

$$\infty > \left\| I_{g,\varphi}^{(n)}\left(f_{\varphi(z)}\right) \right\|_{\mathcal{D}_\beta^p}^p$$

$$= \int_{\mathbb{D}} \left| f_{\varphi(z)}^{(n)}(\varphi(z)) g(z) \right|^p \left(\log \frac{1}{|z|}\right)^\beta dA(z)$$

$$\geq \int_{\mathbb{D}} \frac{|A_n|^p |g(z)|^p}{(1-|\varphi(z)|^2)^{p(n-1)}} \left(\log \frac{1}{1-|\varphi(z)|^2}\right)^{-p} \left(\log \frac{1}{1-|\varphi(z)|^2}+1\right)^{p} \left(\log \frac{1}{|z|}\right)^{\beta} dA(z)$$
$$\geq \int_{\mathbb{D}} \frac{|g(z)|^p}{(1-|\varphi(z)|^2)^{p(n-1)}} \left(\log \frac{1}{|z|}\right)^{\beta} dA(z).$$

Hence,

$$M = \int_{\mathbb{D}} \frac{|g(z)|^p}{(1-|\varphi(z)|^2)^{p(n-1)}} \left(\log \frac{1}{|z|}\right)^{\beta} dA(z) < \infty.$$

Of course, for $n = 1$,

$$\infty > \left\| I'_{g,\varphi}\left(f_{\varphi(z)}\right)\right\|^p_{\mathcal{D}^p_\beta}$$
$$= \int_{\mathbb{D}} \left| f'_{\varphi(z)}(\varphi(z)) g(z)\right|^p \left(\log \frac{1}{|z|}\right)^{\beta} dA(z)$$
$$= \int_{\mathbb{D}} \left(\log \frac{1}{1-|\varphi(z)|^2}\right)^{2p} \left(\log \frac{1}{1-|\varphi(z)|^2}\right)^{-p} |g(z)|^p \left(\log \frac{1}{|z|}\right)^{\beta} dA(z)$$
$$= \int_{\mathbb{D}} C|g(z)|^p \left(\log \frac{1}{|z|}\right)^{\beta} dA(z),$$

where C is a positive constant. Hence,

$$M = \int_{\mathbb{D}} |g(z)|^p \left(\log \frac{1}{|z|}\right)^{\beta} dA(z) < \infty.$$

For $n = 2$,

$$\infty > \left\| {I''_{g,\varphi}}^{(n)}\left(f_{\varphi(z)}\right)\right\|^p_{\mathcal{D}^p_\beta}$$
$$= \int_{\mathbb{D}} \left| f''_{\varphi(z)}(\varphi(z)) g(z)\right|^p \left(\log \frac{1}{|z|}\right)^{\beta} dA(z)$$
$$= \int_{\mathbb{D}} \frac{2|\varphi(z)|^p}{1-|\varphi(z)|^{2p}} \left(\log \frac{1}{1-|\varphi(z)|^2}\right)^{p} \left(\log \frac{1}{1-|\varphi(z)|^2}\right)^{-p} |g(z)|^p \left(\log \frac{1}{|z|}\right)^{\beta} dA(z)$$
$$\geq \int_{\mathbb{D}} \frac{|g(z)|^p}{(1-|\varphi(z)|^2)^p} \left(\log \frac{1}{|z|}\right)^{\beta} dA(z).$$

Hence,

$$M = \int_{\mathbb{D}} \frac{|g(z)|^p}{(1-|\varphi(z)|^2)^p} \left(\log \frac{1}{|z|}\right)^{\beta} dA(z) < \infty.$$

$(ii) \Rightarrow (i)$. Assume that $M < \infty$. Then by Lemma 2.1 for $n \geq 2$, there exists a positive constant C_n such that

$$\begin{aligned}
\left\| I^{(n)}_{g,\varphi}(f) \right\|^p_{\mathcal{D}^p_\beta} &= \int_{\mathbb{D}} \left| f^{(n)}(\varphi(z)) \right|^p |g(z)|^p \left(\log \frac{1}{|z|} \right)^\beta dA(z) \\
&\leq \int_{\mathbb{D}} \frac{C_n^p \|f\|^p_{\mathcal{Z}}}{(1-|\varphi(z)|^2)^{p(n-1)}} |g(z)|^p \left(\log \frac{1}{|z|} \right)^\beta dA(z) \\
&= M C_n^p \|f\|^p_{\mathcal{Z}} \\
&< \infty.
\end{aligned}$$

So, $\left\| I^{(n)}_{g,\varphi}(f) \right\|^p_{\mathcal{D}^p_\beta} < \infty$ and this implies that $I^{(n)}_{g,\varphi}$ is bounded.

For $n = 1$, by Lemma 2.2,

$$\begin{aligned}
\left\| I'_{g,\varphi}(f) \right\|^p_{\mathcal{D}^p_\beta} &= \int_{\mathbb{D}} \left| f'(\varphi(z)) g(z) \right|^p \left(\log \frac{1}{|z|} \right)^\beta dA(z) \\
&\leq \int_{\mathbb{D}} \left(\log \frac{e}{1-|\varphi(z)|^2} \right)^p \|f\|_{\mathcal{Z}}{}^p |g(z)|^p \left(\log \frac{1}{|z|} \right)^\beta dA(z) \\
&\leq M C^p \|f\|_{\mathcal{Z}}{}^p \\
&< \infty,
\end{aligned}$$

where C is a positive constant and $M = \int_{\mathbb{D}} |g(z)|^p \left(\log \frac{1}{|z|} \right)^\beta dA(z) < \infty$. So, $I'_{g,\varphi}$ is bounded and the theorem is proved. □

Now, we investigate the compactness of $I^{(n)}_{g,\varphi} : \mathcal{Z} \longrightarrow \mathcal{D}^p_\beta$. For this investigation, we need the following lemma from [20]:

Lemma 2.4. *Let X and Y be two Banach spaces of analytic functions on $\mathbb{D}$. Suppose the following*

(i) *The point evaluation functions on Y are continuous.*
(ii) *The closed unit ball of X is compact subset of X in the topology of uniform convergence on compact sets.*
(iii) *$T : X \longrightarrow Y$ is continuous when X and Y are given the topology of uniform convergence on compact sets.*

Then, T is a compact operator if and only if, given a bounded sequence $\{f_n\}$ in X such that $f_n \longrightarrow 0$ uniformly on compact sets, the sequence $\{Tf_n\}$ converges to zero in the norm of Y.

For $X = \mathcal{Z}$ and $Y = \mathcal{D}^p_\beta$ the above lemma can be applied. So it follows that

Lemma 2.5. *Let $T : \mathcal{Z} \longrightarrow \mathcal{D}^p_\beta$ be a bounded operator. Then, T is a compact operator if and only if, given a bounded sequence $\{f_n\}$ in $\mathcal{Z}$*

such that $f_n \longrightarrow 0$ *uniformly on compact sets, then the sequence* $\{Tf_n\}$ *converges to zero in the norm of* $\mathcal{D}^p_\beta$.

Lemma 2.6. *Let* $g \in H(\mathbb{D})$, $n \in \mathbb{N}$, φ *be an analytic self-map of* $\mathbb{D}$, $0 < p < \infty$ *and* $\beta > -1$. *If* $\|\varphi\|_\infty < 1$ *and* $I^{(n)}_{g,\varphi} : \mathcal{Z} \longrightarrow \mathcal{D}^p_\beta$ *is bounded, then* $I^{(n)}_{g,\varphi} : \mathcal{Z} \longrightarrow \mathcal{D}^p_\beta$ *is compact*

Proof. Since $I^{(n)}_{g,\varphi} : \mathcal{Z} \longrightarrow \mathcal{D}^p_\beta$ is bounded, Theorem 2.3 implies that

$$\begin{aligned} M &= \int_{\mathbb{D}} \frac{|g(z)|^p}{(1-|\varphi(z)|^2)^{p(n-1)}} \left(\log \frac{1}{|z|}\right)^\beta dA(z) \\ &< \infty. \end{aligned}$$

Let $\{f_k\}$ be a bounded sequence in $\mathcal{Z}$ converges to 0 uniformly on compact subsets of $\mathbb{D}$ as $k \longrightarrow \infty$. Then, Cauchy's estimate implies that $\{f^{(n)}_k\}$ for $n \in \mathbb{N}$ also converges to 0 uniformly on compact subsets of $\mathbb{D}$ as $k \longrightarrow \infty$. This implies that

$$\lim_{k \longrightarrow \infty} \sup_{z \in \varphi(\mathbb{D})} \left| f^{(n)}_k(z) \right| = 0.$$

So,

$$\begin{aligned} \left\| I^{(n)}_{g,\varphi} f_k \right\|^p_{\mathcal{D}^p_\beta} &= \int_{\mathbb{D}} \left| I^{(n)}_{g,\varphi} f_k(z) \right|^p \left(\log \frac{1}{|z|}\right)^\beta dA(z) \\ &= \int_{\mathbb{D}} \left| f^{(n)}_k(\varphi(z)) \right|^p |g(z)|^p \left(\log \frac{1}{|z|}\right)^\beta dA(z) \\ &\le M \sup_{z \in \varphi(\mathbb{D})} \left| f^{(n)}_k(w) \right|^p \longrightarrow 0. \end{aligned}$$

Hence, by Lemma 2.5, $I^{(n)}_{g,\varphi} : \mathcal{Z} \longrightarrow \mathcal{D}^p_\beta$ is compact. □

Theorem 2.7. *Let* $g \in H(\mathbb{D})$, $n \in \mathbb{N}$, φ *be an analytic self-map of* $\mathbb{D}$, $0 < p < \infty, \beta > -1$. *If* $\|\varphi\|_\infty = 1$, *then* $I^{(n)}_{g,\varphi} : \mathcal{Z} \longrightarrow \mathcal{D}^p_\beta$ *is compact if and only if it is bounded and*

$$\lim_{|\varphi(z)| \to 1} \int_{\mathbb{D}} \frac{|g(z)|^p}{(1-|\varphi(z)|^2)^{p(n-1)}} \left(\log \frac{1}{|z|}\right)^\beta dA(z) = 0. \tag{2.3}$$

Proof. Suppose that $I^{(n)}_{g,\varphi} : \mathcal{Z} \longrightarrow \mathcal{D}^p_\beta$ is compact. Consider the functions h and f_a as defined in (2.1) and (2.2). From the definition of f_a, it converges to zero uniformly on compact subsets of $\mathbb{D}$ as $|a| \to 1$ and calculations in Theorem 2.3 show that there exist constants C_n with

$|C_n| \geq 1$, such that for $n \geq 2$,

$$\left|f_a^{(n)}(a)\right| \geq \frac{|C_n|}{(1-|a|^2)^{n-1}}.$$

Let $\{z_k\}_{k\in\mathbb{N}}$ be a sequence in $\mathbb{D}$ such that $|\varphi(z_k)| \to 1$ as $k \to \infty$. Define the test functions $(f_k)_{k\in\mathbb{N}}$ by

$$\begin{aligned} f_k(z) &= f_{\varphi(z_k)}(z) \\ &= \frac{\overline{\varphi(z_k)}z-1}{\overline{\varphi(z_k)}}\left[\left(1+\log\frac{1}{(1-\overline{\varphi(z_k)}z)}\right)^2+1\right]\left(\log\frac{1}{1-|\varphi(z_k)|^2}\right)^{-1}. \end{aligned}$$

Then $f_k \in \mathcal{Z}$ and there exists $0 < C < \infty$ such that $\sup_{k\in\mathbb{N}} \|f_k\|_{\mathcal{Z}} < C$ and $\{f_k\}$ converges to zero uniformly on compact subsets of $\mathbb{D}$. Then by Lemma 2.5 we have $\|I_{g,\varphi}^{(n)}(f_k)\|_{\mathcal{D}_\beta^p}^p \longrightarrow 0$ as $k \to \infty$. Since for $n \geq 2$,

$$\begin{aligned} \left\|I_{g,\varphi}^{(n)}(f_k)\right\|_{\mathcal{D}_\beta^p}^p &= \int_{\mathbb{D}} \left|f_k^{(n)}(\varphi(z_k))\right|^p |g(z_k)|^p \left(\log\frac{1}{|z_k|}\right)^\beta dA(z_k) \\ &\geq \int_{\mathbb{D}} \frac{|C_n|^p|g(z_k)|^p}{(1-|\varphi(z_k)|^2)^{p(n-1)}}\left(\log\frac{1}{|z_k|}\right)^\beta dA(z_k), \end{aligned}$$

we can write

$$\lim_{|\varphi(z)|\to 1}\int_{\mathbb{D}} \frac{|g(z)|^p}{(1-|\varphi(z)|^2)^{p(n-1)}}\left(\log\frac{1}{|z|}\right)^\beta dA(z) = 0.$$

For $n = 1$ we have

$$\begin{aligned} &\|I'_{g,\varphi}(f_k)\|_{\mathcal{D}_\beta^p}^p \\ &= \int_{\mathbb{D}} |f'_k(\varphi(z_k))g(z_k)|^p \left(\log\frac{1}{|z_k|}\right)^\beta dA(z_k) \\ &= \int_{\mathbb{D}} \left(\log\frac{1}{1-|\varphi(z_k)|^2}\right)^{2p}\left(\log\frac{1}{1-|\varphi(z_k)|^2}\right)^{-p}|g(z_k)|^p\left(\log\frac{1}{|z_k|}\right)^\beta dA(z_k) \\ &= \int_{\mathbb{D}} C|g(z_k)|^p\left(\log\frac{1}{|z_k|}\right)^\beta dA(z_k), \end{aligned}$$

where C is a positive constant. So,

$$\lim_{|\varphi(z)|\to 1}\int_{\mathbb{D}} |g(z)|^p\left(\log\frac{1}{|z|}\right)^\beta dA(z) = 0.$$

Conversely, let $\{f_k\}$ be a bounded sequence in $\mathcal{Z}$ converges to 0 uniformly on compact subsets of $\mathbb{D}$ as $k \longrightarrow \infty$. By (2.3), for every $\varepsilon > 0$

there is a $\rho \in (0,1)$ such that for $\rho < |\varphi(z)| < 1$ we have

$$\int_{|\varphi(z)|>\rho} \frac{|g(z)|^p}{(1-|\varphi(z)|^2)^{p(n-1)}} \left(\log \frac{1}{|z|}\right)^{\beta} dA(z) < \varepsilon. \tag{2.4}$$

Also, the uniform convergence of $\{f_k\}$ on compact subset of $\mathbb{D}$ together with Cauchy's estimate implies that $\{f_k^{(n)}\}$ for any $n \in \mathbb{N}$ also converges to 0 uniformly on compact subset of $\mathbb{D}$ as $k \longrightarrow \infty$. This implies that

$$\lim_{k\longrightarrow\infty} \sup_{|z|\le\rho} \left|f_k^{(n)}(z)\right| = 0. \tag{2.5}$$

For $n \geq 2$, by Lemma 2.1, (2.4) and (2.5), there exists a positive constant C_n such that

$$\begin{aligned}
\left\|I_{g,\varphi}^{(n)} f_k\right\|_{\mathcal{D}_\beta^p}^p &= \int_{\mathbb{D}} \left|\left(I_{g,\varphi}^{(n)} f_k(z)\right)'\right|^p \left(\log \frac{1}{|z|}\right)^{\beta} dA(z)\\
&= \int_{\mathbb{D}} \left|f_k^{(n)}(\varphi(z))g(z)\right|^p \left(\log \frac{1}{|z|}\right)^{\beta} dA(z)\\
&= \int_{|\varphi(z)|\le\rho} \left|f_k^{(n)}(\varphi(z))g(z)\right|^p \left(\log \frac{1}{|z|}\right)^{\beta} dA(z)\\
&\quad + \int_{\rho<|\varphi(z)|<1} \left|f_k^{(n)}(\varphi(z))\right|^p |g(z)|^p \left(\log \frac{1}{|z|}\right)^{\beta} dA(z)\\
&\le M \sup_{|\varphi(z)|\le\rho} \left|f_k^{(n)}(\varphi(z))\right|^p\\
&\quad + \int_{\rho<|\varphi(z)|<1} \frac{C_n^p \|f_k\|_{\mathcal{Z}}^p}{(1-|\varphi(z)|^2)^{p(n-1)}} |g(z)|^p \left(\log \frac{1}{|z|}\right)^{\beta} dA(z)\\
&\le M \sup_{|\varphi(z)|\le\rho} \left|f_k^{(n)}(\varphi(z))\right|^p + C_n^p \|f_k\|_{\mathcal{Z}}^p \varepsilon,
\end{aligned}$$

where M is defined in Theorem 2.3. So, $\left\|I_{g,\varphi}^{(n)} f_k\right\|_{\mathcal{D}_\beta^p} \longrightarrow 0$ and Lemma 2.5 implies that $I_{g,\varphi}^{(n)} : \mathcal{Z} \longrightarrow \mathcal{D}_\beta^p$ is compact.
For $n = 1$, by Lemma 2.2, (2.4) and (2.5),

$$\begin{aligned}
\left\|I'_{g,\varphi}(f_k)\right\|_{\mathcal{D}_\beta^p}^p &= \int_{\mathbb{D}} |f_k'(\varphi(z))g(z)|^p \left(\log \frac{1}{|z|}\right)^{\beta} dA(z)\\
&= \int_{|\varphi(z)|\le\rho} |f_k'(\varphi(z))g(z)|^p \left(\log \frac{1}{|z|}\right)^{\beta} dA(z)\\
&\quad + \int_{\rho<|\varphi(z)|<1} |f_k'(\varphi(z))g(z)|^p \left(\log \frac{1}{|z|}\right)^{\beta} dA(z)
\end{aligned}$$

$$
\begin{aligned}
&\le M \sup_{|\varphi(z)|\le\rho} |f_k'(\varphi(z))|^p \\
&\quad + \int_{\rho<|\varphi(z)|<1} \left(\log \frac{e}{1-|\varphi(z)|^2}\right)^p \|f\|_{\mathcal{Z}}^p |g(z)|^p \left(\log\frac{1}{|z|}\right)^\beta dA(z) \\
&\le M \sup_{|\varphi(z)|\le\rho} |f_k'(\varphi(z))|^p + C\|f\|_{\mathcal{Z}}^p.\varepsilon,
\end{aligned}
$$

where C is a positive constant. So, $\left\|I'_{g,\varphi}(f_k)\right\|_{\mathcal{D}_\beta^p}^p \longrightarrow 0$. The theorem is proved. □

3. The Essential Norm of the Operator $I_{g,\varphi}^{(n)} : \mathcal{Z} \longrightarrow \mathcal{D}_\beta^p$

In this section, we give an essential norm estimate for the generalized integration operator $I_{g,\varphi}^{(n)}$ from the Zygmund space into the weighted Dirichlet-type space. We denote the essential norm of $I_{g,\varphi}^{(n)} : \mathcal{Z} \longrightarrow \mathcal{D}_\beta^p$ by $\left\|I_{g,\varphi}^{(n)}\right\|_{e,\mathcal{Z}\longrightarrow\mathcal{D}_\beta^p}$. If $\|\varphi\|_\infty < 1$, then by Lemma 2.6, $I_{g,\varphi}^{(n)}$ is compact and this implies that $\left\|I_{g,\varphi}^{(n)}\right\|_{e,\mathcal{Z}\longrightarrow\mathcal{D}_\beta^p} = 0$. Now, we suppose that $\|\varphi\|_\infty = 1$ and give the main result of this section as follows:

Theorem 3.1. *Let $g \in H(\mathbb{D})$, $n \in \mathbb{N}$, φ be an analytic self-map of $\mathbb{D}$ with $\|\varphi\|_\infty = 1$, $0 < p < \infty$ and $\beta > -1$. If $I_{g,\varphi}^{(n)} : \mathcal{Z} \longrightarrow \mathcal{D}_\beta^p$ is bounded, then*

(3.1)

$$
\left\|I_{g,\varphi}^{(n)}\right\|_{e,\mathcal{Z}\longrightarrow\mathcal{D}_\beta^p}^p \sim \lim_{|\varphi(z)|\to 1} \int_{\mathbb{D}} \frac{|g(z)|^p}{(1-|\varphi(z)|^2)^{p(n-1)}} \left(\log\frac{1}{|z|}\right)^\beta dA(z).
$$

Proof. We consider the functions h and f_a as defined in (2.1) and (2.2). The calculations in Theorem 2.3 show that there exist constants C_n with $|C_n| \ge 1$, such that for $n \ge 2$,

$$
|f_a^{(n)}(a)| \ge \frac{|C_n|}{(1-|a|^2)^{n-1}}.
$$

Also, f_a converges to zero uniformly on compact subsets of $\mathbb{D}$ as $|a| \to 1$. Let $\{z_k\}$ be a sequence in $\mathbb{D}$ such that $|\varphi(z_k)| \to 1$ as $k \to \infty$. Define the test functions $(f_k)_{k\in\mathbb{N}}$ by

$$
\begin{aligned}
f_k(z) &= f_{\varphi(z_k)}(z) \\
&= \frac{\overline{\varphi(z_k)}z - 1}{\overline{\varphi(z_k)}} \left[\left(1 + \log\frac{1}{(1-\overline{\varphi(z_k)}z)}\right)^2 + 1\right] \left(\log\frac{1}{1-|\varphi(z_k)|^2}\right)^{-1}.
\end{aligned}
$$

Then $f_k \in \mathcal{Z}$, $\sup_{k\in\mathbb{N}} \|f_k\|_{\mathcal{Z}} < C$ and $\{f_k\}$ converges to zero uniformly on compact subsets of $\mathbb{D}$. Then by Lemma 2.5, for every compact operator

$S : \mathcal{Z} \longrightarrow \mathcal{D}^p_\beta$, we have $\lim_{k\to\infty} \|Sf_k\|_{\mathcal{D}^p_\beta} = 0$. Then, for $n \geq 2$,

$$\begin{aligned} C\left\|I^{(n)}_{g,\varphi} - S\right\| &\geq \lim_{k\to\infty} \sup \left\|\left(I^{(n)}_{g,\varphi} - S\right) f_k\right\|_{\mathcal{D}^p_\beta} \\ &\geq \lim_{k\to\infty} \sup \left(\left\|I^{(n)}_{g,\varphi} f_k\right\|_{\mathcal{D}^p_\beta} - \|Sf_k\|_{\mathcal{D}^p_\beta}\right) \\ &\geq \lim_{k\to\infty} \sup \left\|I^{(n)}_{g,\varphi} f_k\right\|_{\mathcal{D}^p_\beta} \\ &\geq \lim_{k\to\infty} \sup \left(\int_{\mathbb{D}} |f^{(n)}_k(\varphi(z_k))|^p |g(z_k)|^p \left(\log \frac{1}{|z_k|}\right)^\beta dA(z_k)\right)^{\frac{1}{p}} \\ &\geq \lim_{k\to\infty} \sup \left(\int_{\mathbb{D}} \frac{C_n |g(z_k)|^p}{(1 - |\varphi(z_k)|^2)^{p(n-1)}} \left(\log \frac{1}{|z_k|}\right)^\beta dA(z_k)\right)^{\frac{1}{p}}. \end{aligned}$$

Thus,

(3.2)

$$\left\|I^{(n)}_{g,\varphi}\right\|^p_{e,\mathcal{Z}\longrightarrow\mathcal{D}^p_\beta} \geq \frac{C_n}{C} \lim_{|\varphi(z)|\to 1} \int_{\mathbb{D}} \frac{|g(z)|^p}{(1 - |\varphi(z)|^2)^{p(n-1)}} \left(\log \frac{1}{|z|}\right)^\beta dA(z).$$

For $n = 1$, similarly we have

$$\begin{aligned} &C\left\|I'_{g,\varphi} - S\right\| \\ &\geq \lim_{k\to\infty} \sup \left\|\left(I'_{g,\varphi} - S\right) f_k\right\|_{\mathcal{D}^p_\beta} \\ &\geq \lim_{k\to\infty} \sup \left\|I'_{g,\varphi} f_k\right\|_{\mathcal{D}^p_\beta} \\ &\geq \lim_{k\to\infty} \sup \left(\int_{\mathbb{D}} |f'_k(\varphi(z_k))|^p |g(z_k)|^p \left(\log \frac{1}{|z_k|}\right)^\beta dA(z_k)\right)^{\frac{1}{p}} \\ &= \left(\int_{\mathbb{D}} \left(\log \frac{1}{1 - |\varphi(z_k)|^2}\right)^{2p} \left(\log \frac{1}{1 - |\varphi(z_k)|^2}\right)^{-p} |g(z_k)|^p \left(\log \frac{1}{|z_k|}\right)^\beta dA(z_k)\right)^{\frac{1}{p}} \\ &= \left(\int_{\mathbb{D}} C_1 |g(z_k)|^p \left(\log \frac{1}{|z_k|}\right)^\beta dA(z_k)\right)^{\frac{1}{p}}, \end{aligned}$$

where C_1 is a positive constant. So,

$$\left\|I'_{g,\varphi}\right\|^p_{e,\mathcal{Z}\longrightarrow\mathcal{D}^p_\beta} \geq \frac{C_1}{C} \lim_{|\varphi(z)|\to 1} \int_{\mathbb{D}} |g(z)|^p \left(\log \frac{1}{|z|}\right)^\beta dA(z).$$

Therefore, we have found the lower estimate of $\left\|I^{(n)}_{g,\varphi}\right\|_{e,\mathcal{Z}\longrightarrow\mathcal{D}^p_\beta}$.

Now, we shall find the upper estimate of $\left\|I^{(n)}_{g,\varphi}\right\|_{e,\mathcal{Z}\longrightarrow\mathcal{D}^p_\beta}$. From boundedness of $I^{(n)}_{g,\varphi}$, there exists $0 < C < \infty$ such that for every $f \in \mathcal{Z}$,

$\left\|I_{g,\varphi}^{(n)}f\right\|_{\mathcal{D}_\beta^p} \le C\|f\|_{\mathcal{Z}}$. For a fixed $\rho \in (0,1)$, we have

$$\begin{aligned}&\int_{\mathbb{D}} \frac{|g(z)|^p}{(1-|\rho\varphi(z)|^2)^{p(n-1)}}\left(\log\frac{1}{|z|}\right)^\beta dA(z)\\ &\le \int_{\mathbb{D}} \frac{|g(z)|^p}{(1-|\varphi(z)|^2)^{p(n-1)}}\left(\log\frac{1}{|z|}\right)^\beta \, dA(z)\\ &< \infty.\end{aligned}$$

Thus, from Theorem 2.3, $I_{g,\rho\varphi}^{(n)}$ is bounded and from $|\rho\varphi(z)| < \rho < 1$, by Lemma 2.6, $I_{g,\rho\varphi}^{(n)}$ is compact. This implies that

(3.3)

$$\begin{aligned}&\left\|I_{g,\varphi}^{(n)}\right\|_{e,\mathcal{Z}\longrightarrow\mathcal{D}_\beta^p}\\ &\le \left\|I_{g,\varphi}^{(n)} - I_{g,\rho\varphi}^{(n)}\right\|\\ &= \sup_{\|f\|_{\mathcal{Z}}\le 1}\left\|\left(I_{g,\varphi}^{(n)} - I_{g,\rho\varphi}^{(n)}\right)f\right\|_{\mathcal{D}_\beta^p}\\ &= \sup_{\|f\|_{\mathcal{Z}}\le 1}\left(\int_{\mathbb{D}}\left|f^{(n)}(\varphi(z)) - f^{(n)}(\rho\varphi(z))\right|^p |g(z)|^p\left(\log\frac{1}{|z|}\right)^\beta dA(z)\right)^{\frac{1}{p}}\\ &\le \sup_{\|f\|_{\mathcal{Z}}\le 1}\left(\int_{|\varphi(z)|\le r}\left|f^{(n)}(\varphi(z)) - f^{(n)}(\rho\varphi(z))\right|^p |g(z)|^p\left(\log\frac{1}{|z|}\right)^\beta dA(z)\right)^{\frac{1}{p}}\\ &\quad + \sup_{\|f\|_{\mathcal{Z}}\le 1}\left(\int_{r<|\varphi(z)|<1}\left|f^{(n)}(\varphi(z)) - f^{(n)}(\rho\varphi(z))\right|^p |g(z)|^p\left(\log\frac{1}{|z|}\right)^\beta dA(z)\right)^{\frac{1}{p}}\\ &= I_1 + I_2.\end{aligned}$$

Taking $f(z) = \frac{z^{n+1}}{(n+1)!} \in \mathcal{Z}$, boundedness of $I_{g,\varphi}^{(n)}$ implies that,

$$\int_{|\varphi(z)|\le r} |\varphi(z)|^p|g(z)|^p\left(\log\frac{1}{|z|}\right)^\beta dA(z) < \infty. \tag{3.4}$$

For every $\varepsilon > 0$ we can choose ρ close to 1 enough, such that for $|\varphi(z)| \le r$,

$$\left|f^{(n)}(\varphi(z)) - f^{(n)}(\rho\varphi(z))\right| < \varepsilon^{\frac{1}{p}}|\varphi(z)|.$$

So, there exists a positive constant C such that

$$\begin{aligned}I_1 &\le \sup_{\|f\|_{\mathcal{Z}}\le 1}\left(\int_{|\varphi(z)|\le r}\varepsilon|\varphi(z)|^p|g(z)|^p\left(\log\frac{1}{|z|}\right)^\beta \, dA(z)\right)^{\frac{1}{p}}\\ &< C\varepsilon,\end{aligned} \tag{3.5}$$

where in the above last inequality, we have used (3.4).

There exist a positive constant C and using Lemma 2.1, a positive constant C_n such that

$$
\begin{aligned}
I_2 &= \sup_{\|f\|_{\mathcal{Z}}\le 1}\left(\int_{r<|\varphi(z)|<1}\left|f^{(n)}(\varphi(z))-f^{(n)}(\rho\varphi(z))\right|^p|g(z)|^p\left(\log\frac{1}{|z|}\right)^{\beta}dA(z)\right)^{\frac{1}{p}}\\
&\le C\sup_{\|f\|_{\mathcal{Z}}\le 1}\left(\int_{r<|\varphi(z)|<1}\left(\left|f^{(n)}(\varphi(z))\right|^p+\left|f^{(n)}(\rho\varphi(z))\right|^p\right)|g(z)|^p\left(\log\frac{1}{|z|}\right)^{\beta}dA(z)\right)^{\frac{1}{p}}\\
&= C\sup_{\|f\|_{\mathcal{Z}}\le 1}\left(\int_{r<|\varphi(z)|<1}\left|f^{(n)}(\varphi(z))\right|^p|g(z)|^p\left(\log\frac{1}{|z|}\right)^{\beta}dA(z)\right.\\
&\quad\left.+\int_{r<|\varphi(z)|<1}\left|f^{(n)}(\rho\varphi(z))\right|^p|g(z)|^p\left(\log\frac{1}{|z|}\right)^{\beta}dA(z)\right)^{\frac{1}{p}}\\
&\le C\sup_{\|f\|_{\mathcal{Z}}\le 1}\left(\int_{r<|\varphi(z)|<1}\frac{C_n^p\|f\|_{\mathcal{Z}}^p}{(1-|\varphi(z)|^2)^{p(n-1)}}|g(z)|^p\left(\log\frac{1}{|z|}\right)^{\beta}dA(z)\right.\\
&\quad\left.+\int_{r<|\varphi(z)|<1}\frac{C_n^p\|f\|_{\mathcal{Z}}^p}{(1-|\rho\varphi(z)|^2)^{p(n-1)}}|g(z)|^p\left(\log\frac{1}{|z|}\right)^{\beta}dA(z)\right)^{\frac{1}{p}}\\
&= CC_n\|f\|_{\mathcal{Z}}\left(\int_{r<|\varphi(z)|<1}\left(\frac{|g(z)|^p}{(1-|\varphi(z)|^2)^{p(n-1)}}+\frac{|g(z)|^p}{(1-|\rho\varphi(z)|^2)^{p(n-1)}}\right)\left(\log\frac{1}{|z|}\right)^{\beta}dA(z)\right)^{\frac{1}{p}}.
\end{aligned}
$$

Letting $\rho\to 1$, the above inequality implies that

$$
I_2\le 2CC_n\|f\|_{\mathcal{Z}}\left(\left(\int_{r<|\varphi(z)|<1}\left(\frac{|g(z)|^p}{(1-|\varphi(z)|^2)^{p(n-1)}}\left(\log\frac{1}{|z|}\right)^{\beta}dA(z)\right)^{\frac{1}{p}}\right)\right).
$$

Then by (3.5) and above inequality, letting $r\to 1$ in (3.3), we have

(3.6)
$$
\left\|I_{g,\varphi}^{(n)}\right\|_{e,\mathcal{Z}\longrightarrow\mathcal{D}_\beta^p}^p\le\lim_{|\varphi(z)|\to 1}\int_{\mathbb{D}}\frac{2CC_n\|f\|_{\mathcal{Z}}|g(z)|^p}{(1-|\varphi(z)|^2)^{p(n-1)}}\left(\log\frac{1}{|z|}\right)^{\beta}dA(z).
$$

Thus, (3.2) and (3.6) imply that (3.1) holds. □

Acknowledgment. The authors would like to express their sincere gratitude to the referees for all the valuable comments, which led to improvement of this article.

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