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Bijections on the Unit Ball of $B(H)$ Preserving *-Jordan Triple Product

Shirin Hejazian[1*] and Mozhdeh Safarizadeh[2]

Abstract. Let $\mathcal{B}_1$ denote the closed unit ball of $\mathcal{B}(H)$, the von Neumann algebra of all bounded linear operators on a complex Hilbert space H with $\dim H \geq 2$. Suppose that ϕ is a bijection on $\mathcal{B}_1$ (with no linearity assumption) satisfying

$$\phi(AB^*A) = \phi(A)\phi(B)^*\phi(A), \quad (A, B \in \mathcal{B}_1).$$

If I and $\mathbb{T}$ denote the identity operator on H and the unit circle in $\mathbb{C}$, respectively and if ϕ is continuous on $\{\lambda I : \lambda \in \mathbb{T}\}$, then we show that $\phi(I)$ is a unitary operator and $\phi(I)\phi$ extends to a linear or conjugate linear Jordan *-automorphism on $\mathcal{B}(H)$. As a consequence, there is either a unitary or an antiunitary operator U on H such that $\phi(A) = \phi(I)UAU^*$, $(A \in \mathcal{B}_1)$ or $\phi(A) = \phi(I)UA^*U^*$, $(A \in \mathcal{B}_1)$.

1. Introduction and Preliminaries

Let H be a complex Hilbert space of dimension ≥ 2 and let $\mathcal{B}(H)$ denote the von Neumann algebra of all bounded linear operators on H. Suppose I, $\mathcal{B}_1$ and $\mathbb{T}$ denote the identity operator on H, the closed unit ball of $\mathcal{B}(H)$ and the unit circle in the complex plane, respectively. We investigate the structure of a bijection $\phi : \mathcal{B}_1 \to \mathcal{B}_1$ which is continuous on $\{\lambda I : \lambda \in \mathbb{T}\}$ and preserves the *-Jordan triple product, that is

$$\phi(AB^*A) = \phi(A)\phi(B)^*\phi(B), \quad (A, B \in \mathcal{B}_1).$$

In studying this problem, we have been motivated by two different aspects.

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First, there are several results in the literature which derive the whole algebraic structure of a ring or algebra from different multiplicative structures on that ring or algebra. This subject was initiated by Martindale [15] who proved that a multiplicative bijection from a prime ring containing a nontrivial idempotent onto an arbitrary ring is additive, and consequently, the two rings are isomorphic. This result was followed extensively by operator algebraists who established several results showing that different multiplicative structures determine the algebraic structure of certain operator algebras. One of the first attempts in this direction was done by Hakeda [9] who established that if $\mathcal{M}$ is an AW^*-algebra without abelian direct summand, then every bijective *-Jordan multiplicative map from $\mathcal{M}$ onto another AW^*-algebra $\mathcal{N}$ which is uniformly continuous on each commutative C^*-subalgebra of the domain, is additive. Removing the assumption of continuity Hakeda and Saitô [10] proved that every Jordan multiplicative *-bijection from a C^*-algebra $\mathcal{A}$ containing a system of $n \times n$ $(n \geq 2)$ matrix units, onto an associative *-algebra is a Jordan isomorphism. Similar problems were considered by several authors who studied bijections between operator algebras $\mathcal{A}$ and $\mathcal{B}$ preserving different multiplicative structures such as Jordan product, Jordan triple product, skew Jordan product, and so on. Assuming certain conditions on $\mathcal{A}$ or $\mathcal{B}$, such mappings, generally, are proved to be Jordan isomorphisms, see e.g., [1, 2, 4, 5, 11–14, 16, 18, 19] and references therein.

Second, there are also results which investigate bijections between certain subsets of operator algebras that preserve a particular property or multiplication. Some of the most important subsets are the set of idempotents, the positive cone, the self-adjoint part, the unitary group and the set of effects of a C^*-algebra. The reader may find a reach overview on these results, until 2007, in [19]. Here we have a short look at some results which are more relevant to our purpose.

Let $\mathcal{A}$ be a von Neumann algebra with no abelian direct summand. Molnár [18] characterized all bijections ϕ from the positive cone (respectively, self-adjoint part) of $\mathcal{A}$ on to the positive cone (respectively, self-adjoint part) of a von Neumann algebra $\mathcal{B}$ satisfying $\phi(ABA) = \phi(A)\phi(B)\phi(A)$, $(A, B \in \mathcal{A})$. We recall that $(A, B) \mapsto ABA$ is called the Jordan triple product on $\mathcal{B}(H)$.

In a unital C^*-algebra $\mathcal{A}$, the set $\mathcal{E}(\mathcal{A}) = \{A \in \mathcal{A} : 0 \leq A \leq 1\}$ is called the set of effects of $\mathcal{A}$. The product $A \circ B := A^{1/2}BA^{1/2}$ is called the sequential product on $\mathcal{E}(\mathcal{A})$, (cf. [6–8] for a detailed study of effects). A bijection between the sets of effects of two C^*-algebras preserving the sequential product, is called a sequential isomorphism. Molnár [17] characterized sequential isomorphisms between the sets of effects of von

Neumann algebras. In particular, every sequential isomorphism between the sets of effects of two factors of dimension > 1 is extended to a *-isomprphism or to a *-antiisomorphism [17, Corollary 2].

Let H be an infinite dimensional complex separable Hilbert space. Among characterizing several different preservers on the unitary group $\mathcal{U}$ of $\mathcal{B}(H)$, Molnár and Šemrl [21] gave the general form of a continuous bijection ϕ defined on $\mathcal{U}$ satisfying one of the following conditions

$$\phi(VWV) = \phi(V)\phi(W)\phi(V), \quad (V, W \in \mathcal{U}), \tag{1.1}$$

or

$$\phi(VW^{-1}V) = \phi(V)\phi(W)^{-1}\phi(V), \quad (V, W \in \mathcal{U}), \tag{1.2}$$

where $(U, V) \mapsto VW^{-1}V$ is called the inverted Jordan triple product on $\mathcal{U}$. The authors established that if a continuous bijection ϕ satisfies (1.1), then there exist $c \in \{-1, 1\}$ and a unitary or antiunitary $U \in \mathcal{B}(H)$ such that either $\phi(V) = cUVU^*$ $(V \in \mathcal{U})$ or $\phi(V) = cUV^*U^*$ $(V \in \mathcal{U})$ [21, Theorem 2.2]. As a consequence, the authors proved that a continuous bijection ϕ on $\mathcal{U}$ satisfying (1.2) is either of the form $\phi(V) = UVU'$ $(V \in \mathcal{U})$ or $\phi(V) = UV^*U'$ $(V \in \mathcal{U})$, where the pair U, U' are both unitary or both antiunitary operators on H [21, Theorem 2.1]. In the finite dimensional case, the general form of continuous endomorphisms on the unitary group $\mathcal{U}_n$ of all $n \times n$ complex matrices which satisfy (1.1), was given in [20].

The *-Jordan triple product on $\mathcal{B}(H)$ is defined by $(A, B) \mapsto AB^*A : \mathcal{B}(H)\times\mathcal{B}(H) \to \mathcal{B}(H)$. Note that, the restriction of the *-Jordan product to the unitary group of H, is the inverted Jordan triple product. Gao [3] studied mappings on $\mathcal{B}(H)$ preserving the *-Jordan triple product and proved that if H is an infinite dimensional (real or complex) Hilbert space, then a surjective map $\Phi : \mathcal{B}(H) \to \mathcal{B}(H)$ satisfies $\Phi(AB^*A) = \Phi(A)\Phi(B)^*\Phi(A)$ for all $A, B \in \mathcal{B}(H)$, if and only if one of the following conditions holds:

(i) Φ is a *-Jordan-triple homomorphism which vanishes on all rank one projections.
(ii) There exists a *-Jordan-triple multiplicative non-vanishing scalar function f on $\mathcal{B}(H)$, and unitaries or antiunitaries U and V on H such that $\Phi(A) = f(A)UAV^*$ for all $A \in \mathcal{B}(H)$ or $\Phi(A) = f(A)UA^*V^*$ for all $A \in \mathcal{B}(H)$.

Assume that H is a complex Hilbert space of dimension ≥ 2. In Section 2, we study a bijection ϕ on the unit ball $\mathcal{B}_1$ of $\mathcal{B}(H)$ satisfying

$$\phi(AB^*A) = \phi(A)\phi(B)^*\phi(A), \quad (A, B \in \mathcal{B}_1),$$

which is continuous on $\mathbb{T}I$. Note that, here the map ϕ is just defined on the closed unit ball of $\mathcal{B}(H)$. We are going to extend ϕ to a well behaved mapping on the hole of $\mathcal{B}(H)$. Here, no homogeneity is assumed so extending from the unit ball to the hole algebra is not trivial.

We establish that $\phi(I)$ is a unitary element and there exists a linear or conjugate linear Jordan *-automorphism Φ on $\mathcal{B}(H)$ such that $\phi(I)\Phi|_{\mathcal{B}_1} = \phi$. To be more precise,

$$\phi(A) = \phi(I)UAU^*, \quad (A \in \mathcal{B}_1) \quad \text{or} \quad \phi(A) = \phi(I)UA^*U^*, \quad (A \in \mathcal{B}_1),$$

where U is either a unitary or an antiunitary operator on H.

If $\dim H = 1$ the result does not hold. For instance, consider $\phi(z) = z|z|$ on the closed unit disk of $\mathbb{C}$.

As we will observe, our argument in proving complex homogeneity needs continuity of ϕ on $\mathbb{T}I$. It is natural to ask whether we can remove the continuity assumption. Actually, continuity assumption also arises in some well known results, see e.g. [20] and [21, Theorems 2.1 and 2.2], where the authors are dealing with different (continuous) triple endomorphisms on the unitary group of $\mathcal{B}(H)$, (see also [22, 23]). Removing the continuity assumption in each of these cases would be challenging.

2. Bijective *-Jordan Triple Product Preservers on the Unit Ball

Throughout this section H is a complex Hilbert space with $\dim(H) \geq 2$, and $\mathcal{B}(H)$ denotes the von Neumann algebra of all bounded linear operators on H. The closed unit disk in $\mathbb{C}$ is denoted by $\mathbb{D}$.

Recall that by a linear (respectively, conjugate linear) Jordan homomorphism on $\mathcal{B}(H)$ we mean a linear (respectively, conjugate linear) map $\Phi : \mathcal{B}(H) \to \mathcal{B}(H)$ satisfying $\Phi\left(A^2\right) = \Phi(A)^2$ for all $A \in \mathcal{B}(H)$, or equivalently, $\Phi(AB+BA) = \Phi(A)\Phi(B)+\Phi(B)\Phi(A)$ for all $A, B \in \mathcal{B}(H)$. As a consequence of a well known result due to Herstein, by primeness of $\mathcal{B}(H)$, every Jordan automorphism on $\mathcal{B}(H)$ is either an automorphism or an antiautomorphism. We also recall that an antiunitary operator on H is a bijective conjugate linear isometry on H.

Our aim is to prove the following theorem.

Theorem 2.1. *Let $\phi : \mathcal{B}_1 \longrightarrow \mathcal{B}_1$ be a bijection which is continuous on $\{\lambda I : \lambda \in \mathbb{T}\}$. Then ϕ satisfies*

$$\phi(AB^*A) = \phi(A)\phi(B)^*\phi(A), \quad (A, B \in \mathcal{B}_1), \tag{2.1}$$

if and only if $\phi(I)$ is a unitary operator and there exists a unitary or conjugate unitary on H such that

$$\phi(A) = \phi(I)UAU^*, \quad (A \in \mathcal{B}_1),$$

or

$$\phi(A) = \phi(I)UA^*U^*, \quad (A \in \mathcal{B}_1).$$

As the first step towards the proof of this theorem note that ϕ^{-1} also satisfies (2.1). To see this, suppose that $C, D \in \mathcal{B}_1$, then

$$\begin{aligned}\phi\left(\phi^{-1}(C)\phi^{-1}(D)^*\phi^{-1}(C)\right) &= C\left(\phi(\phi^{-1}(D))\right)^* C \\ &= CD^*C \\ &= \phi\left(\phi^{-1}(CD^*C)\right).\end{aligned}$$

Since ϕ is injective, we obtain

$$\phi^{-1}(CD^*C) = \phi^{-1}(C)\phi^{-1}(D)^*\phi^{-1}(C).$$

In what follows we assume that $\phi : \mathcal{B}_1 \to \mathcal{B}_1$ is a bijection satisfying (2.1). Notice that we do not use the continuity assumption on $\mathbb{T}I$ until Lemma 2.10.

Lemma 2.2. *The following assertions hold:*

(i) $\phi(0) = 0$.
(ii) $\phi(I)$ *is a unitary operator.*

Proof. (i) Since ϕ is surjective, there exists $A \in \mathcal{B}_1$ such that $\phi(A) = 0$. Thus

$$\begin{aligned}\phi(0) &= \phi(A0A) \\ &= \phi(A)\phi(0)^*\phi(A) \\ &= 0.\end{aligned}$$

(ii) Apply the same argument as in Step 2 of the proof of [3, Theorem 3.1]. □

Remark 2.3. Define $\psi(A) = \phi(I)^*\phi(A)$ for all $A \in \mathcal{B}_1$. Then $\psi : \mathcal{B}_1 \to \mathcal{B}_1$ satisfies (2.1), for

$$\begin{aligned}\psi(AB^*A) &= \phi(I)^*\phi(AB^*A) \\ &= \phi(I)^*\phi(A)\phi(B)^*\phi(A) \\ &= \phi(I)^*\phi(A)\phi(B)^*\phi(I)^*\phi(I)\phi(A) \\ &= \psi(A)\psi(B)^*\psi(A).\end{aligned}$$

Moreover, ψ is bijective, $\psi(I) = I$ and ψ^{-1} also satisfies the same condition.

In what follows we consider the mapping $\psi : \mathcal{B}_1 \to \mathcal{B}_1$ as in Remark 2.3.

Lemma 2.4. (i) *ψ is a *-mapping preserving the Jordan triple product on $\mathcal{B}_1$, that is $\psi(ABA) = \psi(A)\psi(B)\psi(A)$, $\forall A, B \in \mathcal{B}_1$.*
(ii) *$\psi\left(A^2\right) = \psi(A)^2$, $\forall A \in \mathcal{B}_1$. Consequently, ψ preserves positivity and $\psi(A^{1/2}) = \psi(A)^{1/2}$, $\forall A \geq 0$ in $\mathcal{B}_1$.*

Proof. (i) Let $A \in \mathcal{B}_1$ be an arbitrary element. Since $\psi(I) = I$, we have

$$\begin{aligned}\psi(A^*) &= \psi(IA^*I)\\ &= \psi(I)\psi(A)^*\psi(I)\\ &= \psi(A)^*.\end{aligned}$$

Therefore,

$$\begin{aligned}\psi(ABA) &= \psi(A(B^*)^*A)\\ &= \psi(A)\psi(B^*)^*\psi(A)\\ &= \psi(A)\psi(B)\psi(A), \quad \forall A, B \in \mathcal{B}_1\end{aligned}$$

(ii) Let $A \in \mathcal{B}_1$ be an arbitrary element. Then by (i)

$$\begin{aligned}\psi\left(A^2\right) &= \psi(AIA)\\ &= \psi(A)^2.\end{aligned}$$

The last assertion is now clear by (i). □

It is easily verified that ψ^{-1} also satisfies the conditions in Lemma 2.4. Thus ψ preserves positivity in both directions, that is $\psi(A) \geq 0 \Leftrightarrow A \geq 0$.

Lemma 2.5. (i) *ψ preserves projections in both directions.*

(ii) *ψ preserves orthogonality of projections in both directions.*

(iii) *ψ preserves the order of projections in both directions.*

(iv) *Let P, Q be orthogonal projections in $\mathcal{B}_1$. Then $\psi(P+Q) = \psi(P) + \psi(Q)$. Consequently, $\psi(I-P) = I - \psi(P)$ for each projection $P \in \mathcal{B}(H)$.*

Proof. (i) The result follows by Lemma 2.4.

(ii) Let P, Q be orthogonal projections in $\mathcal{B}(H)$. Then, by (i)

$$\begin{aligned}0 &= \psi(0)\\ &= \psi(PQP)\\ &= \psi(P)\psi(Q)\psi(P)\\ &= \psi(P)\psi(Q)\psi(Q)\psi(P).\end{aligned}$$

Thus, $\psi(P)\psi(Q)(\psi(P)\psi(Q))^* = 0$ and it follows that $\psi(P)\psi(Q) = 0$.

(iii) Let P, Q be projections in $\mathcal{B}(H)$ with $P \leq Q$. Then $PQ = P$ which implies that $PQP = P$. Therefore,

$$\psi(P)\psi(Q)\psi(P) = \psi(P),$$

and it follows that $\psi(P) \leq \psi(Q)$.

Since ψ^{-1} also satisfies (2.1) and Lemma 2.4, all the above assertions hold in both directions.

(iv) Let P, Q be orthogonal projections in $\mathcal{B}(H)$, then $P+Q$ is the least upper bound of $\{P, Q\}$ in the lattice of projections of $\mathcal{B}(H)$. Thus, we get from (iii) that $\psi(P+Q)=\psi(P)+\psi(Q)$. Now, using $I=P+(I-P)$ and $\psi(I)=I$, the last assertion follows easily. □

Lemma 2.6. *ψ preserves projections of rank 1 in both directions.*

Proof. Let $P \in \mathcal{B}_1$ be a projection of rank 1. Then, $\operatorname{rank}(\psi(P)) \geq 1$. If $\operatorname{rank}(\psi(P))>1$, then there are orthogonal rank 1 projections $Q_1, Q_2 \in \mathcal{B}_1$ such that $Q_1 \lneqq \psi(P)$, $Q_2 \lneqq \psi(P)$. Taking $P_i=\psi^{-1}(Q_i)$ $(i=1,2)$, we get by Lemma 2.5(ii)-(iii) that $P_1 P_2=P_2 P_1=0$ and $P_1 \lneqq P$, $P_2 \lneqq P$ which contradicts $\operatorname{rank}(P)=1$. □

Lemma 2.7. *If $A \in \mathcal{B}_1$ is self-adjoint and invertible with $A^{-1} \in \mathcal{B}_1$, then $\psi(A)$ is invertible and $\psi(A)^{-1}=\psi\left(A^{-1}\right)$.*

Proof. We proceed in two steps.

Step 1. Let $A \in \mathcal{B}_1$ be positive and invertible with $A^{-1} \in \mathcal{B}_1$. Then by Lemma 2.4 (ii), we get

$$\begin{aligned} \psi(A)^{1/2} \psi\left(A^{-1}\right) \psi(A)^{1/2} &= \psi\left(A^{1/2} A^{-1} A^{1/2}\right) \\ &= I, \end{aligned} \tag{2.2}$$

and so $\psi(A)^{1/2}$ is invertible. Therefore, $\psi(A)$ is invertible and (2.2) implies that $\psi\left(A^{-1}\right)=\psi(A)^{-1/2} \psi(A)^{-1/2}=\psi(A)^{-1}$.

Step 2. Let $A \in \mathcal{B}_1$ be an arbitrary self-adjoint and invertible element with $A^{-1} \in \mathcal{B}_1$. Then A^2 is positive and invertible and it follows from Step 1 that $\psi(A)^2=\psi\left(A^2\right)$ is invertible. Hence $\psi(A)$ is invertible and the same is true for $\psi\left(A^{-2}\right)$ and $\psi\left(A^{-1}\right)$. Moreover, it follows from Step 1 that $\psi\left(A^{-2}\right)=\psi\left(A^2\right)^{-1}=\psi(A)^{-2}$, by Lemma 2.4(ii). We have

$$I=\psi(A) \psi\left(A^{-2}\right) \psi(A),$$

and so $\psi(A)^{-1}=\psi\left(A^{-2}\right) \psi(A)=\psi(A) \psi\left(A^{-2}\right)$. Therefore,

$$\psi(A)=\psi(A)^{-1} \psi\left(A^2\right). \tag{2.3}$$

By multiplying both sides of (2.3) by $\psi\left(A^{-1}\right)$, we arrive at

$$\psi\left(A^{-1}\right) \psi(A) \psi\left(A^{-1}\right)=\psi\left(A^{-1}\right)\left(\psi(A)^{-1} \psi\left(A^2\right)\right) \psi\left(A^{-1}\right).$$

Thus by Lemma 2.4(i)

$$\psi\left(A^{-1}\right)=\psi\left(A^{-1}\right)\left(\psi(A)^{-1} \psi\left(A^2\right)\right) \psi\left(A^{-1}\right),$$

and since $\psi\left(A^{-1}\right)$ is also invertible, using cancellation implies that

$$\psi(A)^{-1}\psi\left(A^2\right)\psi\left(A^{-1}\right)=I. \tag{2.4}$$

Multiplying (2.3) by $\psi\left(A^{-1}\right)$ from right and using (2.4) infers that $\psi(A)^{-1}=\psi\left(A^{-1}\right)$. □

Lemma 2.8. *The following conditions hold.*

(i) $\psi(-I)=-I$.

(ii) *If P and Q are orthogonal projections, then $\psi(P-Q)=\psi(P)-\psi(Q)$. As a consequence, $\psi(2P-I)=2\psi(P)-I$ for each projection P.*

(iii) *For each $\lambda\in\mathbb{D}$, $\psi(\lambda I)$ is a central element.*

(iv) *$\psi(iI)=\pm iI$, and $\psi(-A)=-\psi(A)$ for all $A\in\mathcal{B}_1$.*

Proof. (i) Using the fact that ψ is a *-map and Lemma 2.4(ii), an easy computation shows that $\frac{1}{2}\big(I+\psi(-I)\big)$ is a projection. Now, put $P_1=\psi^{-1}\left(\frac{1}{2}\big(I+\psi(-I)\big)\right)$ which by Lemma 2.5(i) is a projection, then

$$\begin{aligned}\psi(-P_1)&=\psi(P_1(-I)P_1)\\&=\psi(P_1)\psi(-I)\psi(P_1)\\&=\frac{1}{4}(I+\psi(-I))\psi(-I)(I+\psi(-I))\\&=\frac{1}{2}(\psi(-I)+I)\\&=\psi(P_1).\end{aligned}$$

By injectivity, $-P_1=P_1$ and so $P_1=0$. Therefore, $\psi(-I)=-I$.

(ii) By Lemma 2.5(iv) and part (i) above, we have

$$\begin{aligned}\psi(P-Q)&=\psi((P+iQ)(P+Q)(P+iQ))\\&=\psi(P+iQ)\psi(P+Q)\psi(P+iQ)\\&=\psi(P+iQ)\psi(P)\psi(P+iQ)+\psi(P+iQ)\psi(Q)\psi(P+iQ)\\&=\psi((P+iQ)P(P+iQ))+\psi((P+iQ)Q(P+iQ))\\&=\psi(P)+\psi(-Q)\\&=\psi(P)+\psi(Q(-I)Q)\\&=\psi(P)-\psi(Q).\end{aligned}$$

Now, it follows from the above computation and Lemma 2.5(iv) that

$$\psi(2P-I)=\psi(P-(I-P))$$

$$\begin{aligned} &= \psi(P) - \psi(I - P) \\ &= \psi(P) - (I - \psi(P)) \\ &= 2\psi(P) - I. \end{aligned}$$

(iii) Let P be a projection in $\mathcal{B}(H)$, then $S = 2P - I$ is a symmetry, that is $S = S^*$ and $S^2 = I$. Thus, for each $\lambda \in \mathbb{D}$

$$\psi(\lambda I) = \psi(S)\psi(\lambda I)\psi(S).$$

Multiplying the above equality by $\psi(S)$ from right and using Lemma 2.4(ii) implies that $\psi(S)$ and $\psi(\lambda I)$ commute, and by the last assertion in part (ii) above, $\psi(\lambda I)\psi(P) = \psi(P)\psi(\lambda I)$. Since P is arbitrary and ψ preserves projections in both directions, we get the result.

(iv) From (i) we have

$$\begin{aligned} -I &= \psi(-I) \\ &= \psi((iI)I(iI)) \\ &= \psi(iI)\psi(I)\psi(iI). \end{aligned} \tag{2.5}$$

Thus

$$-I = \big(\psi(iI)\big)^2. \tag{2.6}$$

By (iii), $\psi(iI)$ is a central element of $\mathcal{B}(H)$ and hence is a scalar multiple of the identity operator I. Therefore by (2.6), $\psi(iI) = \pm iI$. Finally, let A be an arbitrary element in $\mathcal{B}_1$. Then

$$\begin{aligned} \psi(-A) &= \psi(iIAiI) \\ &= \psi(iI)\psi(A)\psi(iI) \\ &= -\psi(A), \end{aligned}$$

and the proof of (iv) is completed. □

Remark 2.9. By Lemma 2.8(iv), $\psi(iI) = \pm iI$. If $\psi(iI) = -iI$, let us define $\widetilde{\psi}(A) = \psi(A^*)$ for all $A \in \mathcal{B}_1$. Then $\widetilde{\psi}$ satisfies

$$\begin{aligned} \widetilde{\psi}(AB^*A) &= \psi(A^*BA^*) \\ &= \psi(A^*)\psi(B^*)^*\psi(A^*) \\ &= \widetilde{\psi}(A)\widetilde{\psi}(B)^*\widetilde{\psi}(A), \end{aligned}$$

for all $A, B \in \mathcal{B}_1$. Therefore, $\widetilde{\psi}$ is a *-bijection on $\mathcal{B}_1$ satisfying (2.1) and hence $\widetilde{\psi}(ABA) = \widetilde{\psi}(A)\widetilde{\psi}(B)\widetilde{\psi}(A)$ for all $A, B \in \mathcal{B}_1$ (see Lemma 2.4(i)). Moreover, $\widetilde{\psi}(I) = I$, $\widetilde{\psi}(iI) = iI$. Thus, in what follows, without loss of generality we may assume that $\psi(iI) = iI$. Hence, by Lemma 2.8(iv) we will have $\psi(-iI) = -iI$.

Lemma 2.10. *$\psi(\lambda A) = \lambda\psi(A)$ for all $\lambda \in \mathbb{D}$ and all $A \in \mathcal{B}_1$.*

Proof. By Lemma 2.8(iii), $\psi(\lambda I)$ is a central element for all $\lambda \in \mathbb{D}$. Hence, for each $\lambda \in \mathbb{D}$ there exists a unique scalar $f(\lambda) \in \mathbb{D}$ such that $\psi(\lambda I) = f(\lambda)I$. We show that $f : \mathbb{D} \to \mathbb{D}$ is the identity function. First of all note that, $f(0) = 0$, $f(1) = 1$, $f(-1) = -1$, $f(i) = i$ and $f(-i) = -i$, because $\psi(0) = 0$, $\psi(I) = I$, $\psi(-I) = -I$, $\psi(iI) = iI$, and $\psi(-iI) = -iI$. Now we proceed in some steps.

Step 1. *f is multiplicative on $\mathbb{D}$.*

For $\lambda, \mu \in \mathbb{D}$, let $\alpha, \beta \in \mathbb{D}$ satisfy $\alpha^2 = \lambda$ and $\beta^2 = \mu$. Thus, we get

$$\begin{aligned} f(\lambda\mu)I &= f(\lambda\mu)\psi(I) \\ &= \psi(\alpha^2\beta^2 I) \\ &= \psi\big((\alpha I)(\beta I)I(\beta I)(\alpha I)\big) \\ &= \psi(\alpha I)\psi(\beta I)\psi(I)\psi(\beta I)\psi(\alpha I) \\ &= \psi(\alpha^2 I)\psi(\beta^2 I) \quad \text{(Lemma 2.8(iii) and Lemma 2.4(ii))} \\ &= f(\lambda)f(\mu)I. \end{aligned}$$

Hence, f is multiplicative.

Step 2. $\psi(\lambda A) = f(\lambda)\psi(A)$, $(A \in \mathcal{B}_1,\ \lambda \in \mathbb{D})$.

Pick $\lambda \in \mathbb{D}$, and let $\alpha \in \mathbb{D}$ satisfy $\alpha^2 = \lambda$. Then, using Lemma 2.8(iii), for each $A \in \mathcal{B}_1$,

$$\begin{aligned} \psi(\lambda A) &= \psi(\alpha I A \alpha I) \\ &= \psi(\alpha I)^2\psi(A) \\ &= \psi(\lambda I)\psi(A) \\ &= f(\lambda)\psi(A). \end{aligned}$$

Step 3. *$f(\lambda) = \lambda$ for all $\lambda \in [0, 1]$.*

First of all note that since ψ is a bijection on $\mathcal{B}_1$ preserving positivity in both directions,, f maps $[0, 1]$ onto $[0, 1]$. Let x, y be orthogonal unit vectors in H. Then $P = x \otimes x$ and $Q = y \otimes y$ are orthogonal rank 1 projections in $\mathcal{B}(H)$. Suppose that $\lambda, \mu, \lambda + \mu \in [0, 1]$, $\alpha = \lambda^{1/2}$, $\beta = \mu^{1/2}$, and put $z = \alpha x + \beta y$. If $R = z \otimes z$, then by Step 2 and Lemma 2.5(iv)

$$\begin{aligned} f(\lambda + \mu)\psi(R) &= \psi((\lambda + \mu)R) \\ &= \psi\left(R(P + Q)R\right) \\ &= \psi(R)\psi(P + Q)\psi(R) \\ &= \psi(R)\left(\psi(P) + \psi(Q)\right)\psi(R) \\ &= \psi(RPR) + \psi(RQR) \\ &= f(\lambda)\psi(R) + f(\mu)\psi(R). \end{aligned}$$

Therefore,

$$f(\lambda+\mu)=f(\lambda)+f(\mu),\quad \forall\lambda,\mu\in[0,1] \text{ with } \lambda+\mu\in[0,1]. \tag{2.7}$$

Let $n\geq 2$ be an integer. We show that $f(\frac{1}{n})=\frac{1}{n}$. It follows from (2.7) that

$$\begin{aligned} 1&=f(1)\\ &=f\left(\frac{n-1}{n}+\frac{1}{n}\right)\\ &=f\left(\frac{n-1}{n}\right)+f\left(\frac{1}{n}\right)\\ &=f\left(\frac{n-2}{n}\right)+2f\left(\frac{1}{n}\right)\\ &\ \ \vdots\\ &=nf\left(\frac{1}{n}\right). \end{aligned} \tag{2.8}$$

Therefore, $f\left(\frac{1}{n}\right)=\frac{1}{n}$ for all integers $n\geq 1$. A similar argument shows that $f\left(\dfrac{m}{n}\right)=mf\left(\dfrac{1}{n}\right)=\dfrac{m}{n}$ for all integers m,n satisfying $1\leq m\leq n$. Thus,

$$f(r)=r,\quad (r\in[0,1]\cap\mathbb{Q}). \tag{2.9}$$

If $0\leq\lambda\leq\mu\leq 1$, then $\mu-\lambda\in[0,1]$ and by (2.7) we have $f(\mu)=f(\mu-\lambda+\lambda)=f(\mu-\lambda)+f(\lambda)$. Therefore,

$$f(\mu-\lambda)=f(\mu)-f(\lambda),\quad (\lambda,\mu,\mu-\lambda\in[0,1]). \tag{2.10}$$

Moreover,

$$\begin{aligned} (f(\mu)-f(\lambda))I&=f(\mu-\lambda)I\\ &=\psi((\mu-\lambda)I)\\ &>0, \end{aligned}$$

for all $\lambda,\mu\in[0,1]$ with $\mu-\lambda\in(0,1]$. It follows that f is strictly increasing on $[0,1/2]$. Now, applying (2.9) and using monotonicity of f, an easy argument shows that

$$f(\lambda)=\lambda,\quad (\lambda\in[0,1]). \tag{2.11}$$

Step 4. *$f(\lambda)=\lambda$ for all $\lambda\in\mathbb{D}$.* (This is the point that we use the continuity assumption)

Since ψ is a *-map, $f(\overline{\lambda})=\overline{f(\lambda)}$ for each $\lambda\in\mathbb{D}$. By Steps 1 and 3 and the fact that ψ is a *-map, we have

$$|\lambda|^2=f(|\lambda|)^2$$

$$
\begin{aligned}
&= f(|\lambda|^2)\\
&= f(\lambda\overline{\lambda})\\
&= f(\lambda)\overline{f(\lambda)}\\
&= |f(\lambda)|^2, \quad (\lambda \in \mathbb{D}).
\end{aligned}
$$

This means that f preserves the absolute value. Since ψ is continuous on $\mathbb{T}I$, $f|_{\mathbb{T}} : \mathbb{T} \to \mathbb{T}$ is a character (a continuous $\mathbb{T}$-valued group homomorphism) of the circle group $\mathbb{T}$. Therefore, by the well known characterization of characters on $\mathbb{T}$ there exists an integer n such that $f(\lambda) = \lambda^n$ ($\lambda \in \mathbb{T}$). Since f is not identically 1, we have $n \neq 0$. Suppose $n \neq 1$. There exists $\lambda \neq 1$ in $\mathbb{T}$ such that $\lambda^n = 1$. Thus by Step 2, for each $A \in \mathcal{B}_1$ we have $\psi(\lambda A) = f(\lambda)\psi(A) = \lambda^n\psi(A) = \psi(A)$. Since ψ is injective, we infer that $\lambda = 1$, a contradiction. Thus, $n = 1$ and $f(\lambda) = \lambda$ for all $\lambda \in \mathbb{T}$. Now, if $\lambda \in \mathbb{D}$, then $\lambda = |\lambda|e^{i\theta}$ for some real number θ. It follows from Step 1, (2.11) and the previous argument that $f(\lambda) = f(|\lambda|e^{i\theta}) = f(|\lambda|)f(e^{i\theta}) = |\lambda|e^{i\theta} = \lambda$ and we are done. □

Lemma 2.11. *If $A, B, A+B \in \mathcal{B}_1$, then $\psi(A+B) = \psi(A) + \psi(B)$.*

Proof. Suppose that P is a rank 1 projection in $\mathcal{B}(H)$. Then $P = x \otimes x$ for some unit vector $x \in H$. We have

$$
\begin{aligned}
\psi(P)\psi(A+B)\psi(P) &= \psi(P(A+B)P)\\
&= \langle (A+B)(x), x\rangle \psi(P), \quad \text{(by Lemma 2.10)}\\
&= \langle A(x), x\rangle\psi(P) + \langle B(x), x\rangle\psi(P)\\
&= \psi(PAP) + \psi(PBP)\\
&= \psi(P)\psi(A)\psi(P) + \psi(P)\psi(B)\psi(P)\\
&= \psi(P)\big(\psi(A) + \psi(B)\big)\psi(P).
\end{aligned}
$$

Since P is an arbitrary rank 1 projection, it follows from Lemma 2.6 that

$$\psi(A+B) = \psi(A) + \psi(B),$$

whenever $A, B, A+B \in \mathcal{B}_1$. □

Remark 2.12. Let $\mathcal{E}(H) = \{A \in \mathcal{B}(H) : 0 \leq A \leq I\}$ denote the set of all effects in $\mathcal{B}(H)$. Since every positive element $A \in \mathcal{B}(H)$ satisfies $0 \leq A \leq \|A\|I$, $\mathcal{E}(H)$ is the positive part of $\mathcal{B}_1$. Thus by Lemma 2.4(ii) and the fact that ψ maps $\mathcal{B}_1$ onto itself, ψ preserves effects. The same is true for ψ^{-1}. So ψ maps $\mathcal{E}(H)$ onto itself. Moreover by Lemma 2.4, $\psi' := \psi|_{\mathcal{E}(H)}$ is a sequential isomorphism on $\mathcal{E}(H)$; that is ψ' is a bijection on $\mathcal{E}(H)$ satisfying $\psi'(A^{1/2}BA^{1/2}) = \psi'(A)^{1/2}\psi'(B)\psi'(A)^{1/2}$ for all $A, B \in \mathcal{E}(H)$. Therefore, by [17, Corollary 2], there is a *-automorphism or a *-antiautomorphism $\Phi : \mathcal{B}(H) \to \mathcal{B}(H)$ such that $\Phi|_{\mathcal{E}(H)} = \psi' = \psi|_{\mathcal{E}(H)}$.

Now, we are ready to establish our main result.

Proof. (*Theorem* 2.1). From Remark 2.9, we recognize two cases.
Case 1. Assume that $\psi(iI) = iI$. Let Φ be as in Remark 2.12. Thus, $\Phi|_{\mathcal{E}(H)} = \psi|_{\mathcal{E}(H)}$. We show that $\Phi = \psi$ on $\mathcal{B}_1$. Let $A \in \mathcal{B}_1$ be an arbitrary element, then $A = A_1 - A_2 + i(B_1 - B_2)$, where $A_i, B_i,\ i = 1, 2$, are positive elements in $\mathcal{B}_1$, and also $A_1 - A_2, B_1 - B_2 \in \mathcal{B}_1$. By Lemmas 2.10 and 2.11,

$$\begin{aligned}\psi(A) &= \psi(A_1 - A_2) + i\psi(B_1 - B_2)\\ &= \psi(A_1) - \psi(A_2) + i(\psi(B_1) - \psi(B_2)).\end{aligned}$$

Since $A_i, B_i,\ i = 1, 2$, are effects, it follows from Remark 2.12 that $\psi(A) = \Phi(A)$, that is $\Phi|_{\mathcal{B}_1} = \psi$. As Φ is either a *-automorphism or a *-antiautomorphism, by [19, Theorem A.8] there is either a unitary or an antiunitary operator U on H such that
(2.12)

$$\Phi(A) = UAU^*, \quad (A \in \mathcal{B}(H)) \quad \text{or} \quad \Phi(A) = UA^*U^*, \quad (A \in \mathcal{B}(H)),$$

respectively. Using Remark 2.3, we get $\phi = \phi(I)\psi = \phi(I)\Phi|_{\mathcal{B}_1}$ and so
(2.13)

$$\phi(A) = \phi(I)UAU^*, \quad (A \in \mathcal{B}_1) \quad \text{or} \quad \phi(A) = \phi(I)UA^*U^*, \quad (A \in \mathcal{B}_1),$$

where U is a unitary or an antiunitary operator on H, respectively.
Case 2. Assume that $\psi(iI) = -iI$, and let $\widetilde{\psi}$ be as in Remark 2.9. Then $\widetilde{\psi}$ is also continuous on $\mathbb{T}I$ and $\widetilde{\psi}(iI) = iI$. As in the previous case, there is a *-automorphism or a *-antiautomorphism $\widetilde{\Phi}$ on $\mathcal{B}(H)$ such that $\widetilde{\Phi}|_{\mathcal{B}_1} = \widetilde{\psi}$. Thus, by applying (2.12) to $\widetilde{\Phi}$, there exists either a unitary or an antiunitary U on H such that

$$\widetilde{\psi}(A) = UAU^*, \quad (A \in \mathcal{B}_1) \quad \text{or} \quad \widetilde{\psi}(A) = UA^*U^*, \quad (A \in \mathcal{B}_1),$$

respectively. Since $\psi(A) = \widetilde{\psi}(A^*)$ $(A \in \mathcal{B}_1)$, it follows from Remark 2.3 that

$$\phi(A) = \phi(I)UA^*U^*, \quad (A \in \mathcal{B}_1) \quad \text{or} \quad \phi(A) = \phi(I)UAU^*, \quad (A \in \mathcal{B}_1),$$

where U is unitary or antiunitary, respectively. □

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