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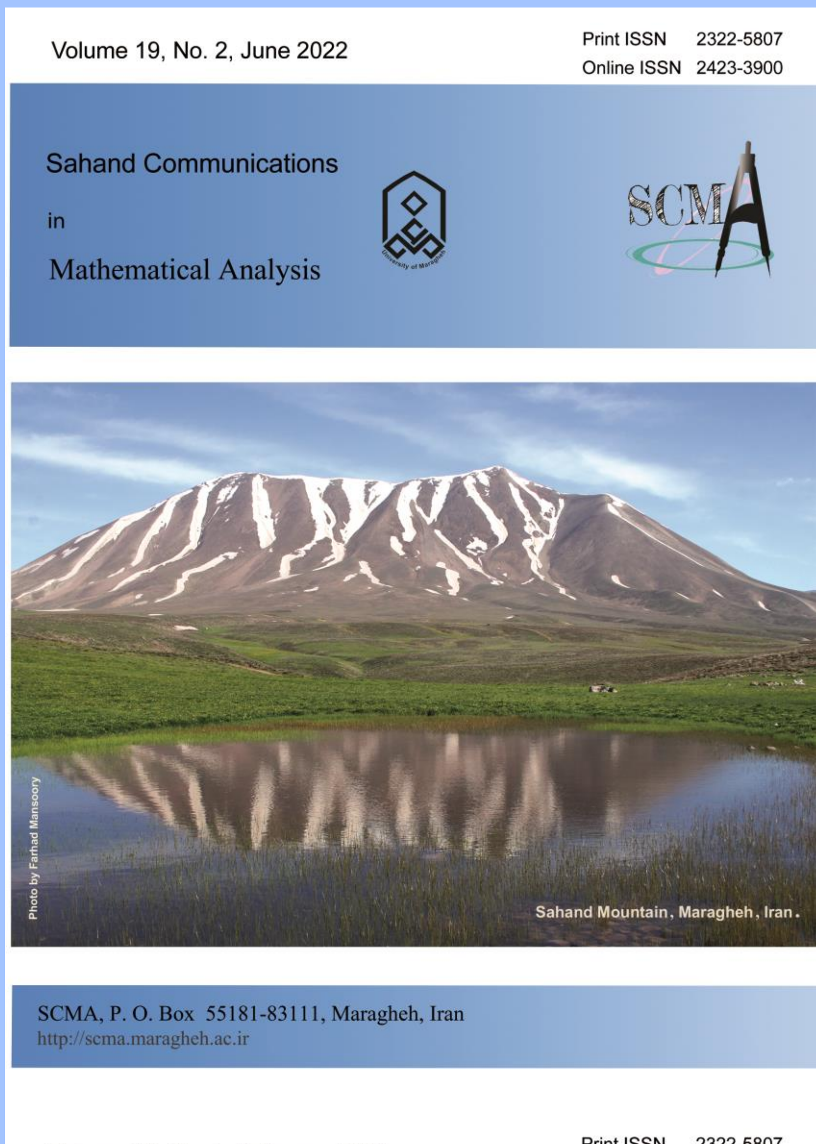
Sahand Communications in Mathematical Analysis

Print ISSN: 2322-5807
Online ISSN: 2423-3900
Volume: 19
Number: 2
Pages: 49-64

Sahand Commun. Math. Anal.
DOI: 10.22130/scma.2022.544742.1033



Sahand Communications in Mathematical Analysis (SCMA) Vol. 19 No. 2 (2022), 49-64
http://scma.maragheh.ac.ir
DOI: 10.22130/scma.2022.544742.1033

The Study of Felbin and BS Fuzzy Normed Linear Spaces

Farnaz Yaqub Azari[1]* and Ildar Sadeqi [2]

Abstract. In this paper, we first show that the induced topologies by Felbin and Bag-Samanta type fuzzy norms on a linear space X are equivalent. So all results in Felbin-fuzzy normed linear spaces are valid in Bag-Samanta fuzzy normed linear spaces and vice versa. Using this, we will be able to define a fuzzy norm on $FB(X,Y)$, the space of all fuzzy bounded linear operators from X into Y, where X and Y are fuzzy normed linear spaces.

1. Introduction

Kaleva and Seikkala [1] present the notion of a fuzzy metric space, which has a basic role in fuzzy nonlinear analysis. Based on the instructions in [1], Felbin [2] brings forward the notion of a fuzzy normed linear space by applying the intention of fuzzy distance to the linear space. Xiao and Zhu [3] investigated the linear topological structures and some basic properties of fuzzy normed linear spaces ($FNLS$). In [4] the fuzzy norm of a linear operator and the space of all fuzzy bounded linear operators are studied; consequently, its topological structure as well as completeness are given. In [5] three types of fuzzy topologies defined on fuzzy normed linear spaces. Finally, three essential theorems of uniform boundedness theorem, open mapping theorem, and closed graph theorem are examined on linear operators defined on fuzzy normed linear spaces equipped with fuzzy topologies. Also, Bag and Samanta [6] introduced another type of fuzzy norm on a linear space X and obtained some interesting results. In [7], they gave a comparative study of the

2020 *Mathematics Subject Classification.* 34B24, 34B27.
Key words and phrases. Fuzzy number, Fuzzy normed linear space (FNLS), Fuzzy bounded operator.
Received: 12 December 2021, Accepted: 07 May 2022.
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two types of fuzzy norms, Felbin fuzzy norm (F-fuzzy norm) and Bag-Samanta fuzzy norm (BS-fuzzy norm) and observed that Felbin's type fuzzy norm corresponds to a pair of which, one is a BS-fuzzy norm and the other is a BS-fuzzy antinorm. Also, Bag and Samanta [8], by introducing a fuzzy norm on $B(X,Y)$ (the set of all weakly fuzzy bounded linear operators from X into Y), have shown that $B(X,Y)$ is a complete fuzzy normed linear space. But it does not define a fuzzy norm on $FB(X,Y)$, the space of all fuzzy bounded linear spaces from X into Y. In 2011, Sadeqi and Azari [9] introduced the concept of gradual normed linear space ($GNLS$). They studied various properties of the space from both the algebraic and topological points of view. Further investigation in this direction has been occurred due to Ettefagh et. al. [10, 11], Choudhury and Debnath [12] and many others. Later, Sadeqi and Azari [13] have made an inductive survey of two kinds of norms on a linear space, gradual and fuzzy norms, and it is proved that they are equivalent in general. Later, Daraby et. al. [14, 15] redefined, the idea of Felbin's definition of fuzzy norm and studied various properties of its topological structure.
We divide this note into three sections. First, we will give some preliminary results. Then, we show that the topologies, induced by F-fuzzy norm and BS-fuzzy norm, are equivalent. Using this, we will be able to compare the results in Felbin and BS-fuzzy normed linear spaces and give the results in one type of fuzzy normed linear space, which are already proved, to the other type. Also, using the equality of the two types of norm and applying the method of Felbin's type norm, we introduce a BS-fuzzy norm on $FB(X,Y)$, where $FB(X,Y)$ becomes a fuzzy Banach linear space. Therefore, shift of results from one type of norm on $FB(X,Y)$ to the other can be derived.

2. Some Preliminary Results

In this section, we give some basic definitions and results in three kinds of well known fuzzy normed linear spaces, namely Felbin fuzzy normed linear space ($F-FNLS$), Bag-Samanta fuzzy normed linear space ($BS-FNLS$) and Bag-Samanta fuzzy antinormed linear space ($BS-FALS$). Throughout this note, by 0 we mean the origin of linear space.

2.1. Felbin Type Fuzzy Normed Linear Space.

Definition 2.1. A function $\eta : \mathbb{R} \to [0,1]$ is named a fuzzy real number, which $\alpha-$level set is explained by $[\eta]_\alpha$, i.e. $[\eta]_\alpha = \{t : \ \eta(t) \geq \alpha\}$, if it applies two principles:

(N1) There exists $t_0 \in \mathbb{R}$ such that $\eta(t_0) = 1$.

(N2) For every $\alpha \in (0,1]: \ [\eta]_\alpha = [\eta_\alpha^-, \ \eta_\alpha^+], \ -\infty < \eta_\alpha^- \leq \eta_\alpha^+ < +\infty$.

The collection of all fuzzy real numbers is illustrated by $\mathbb{R}(I)$. By (N2) in the upper definition of the fuzzy real number, it is obvious that, a level set of a fuzzy real number is an interval. In [18], Dubois and Prade suggested to call such a fuzzy real number as fuzzy interval. From now on "fuzzy real numbers" are called as "fuzzy intervals". While returning to conclusions about fuzzy real number in [2] the phrase fuzzy interval is composed within brackets after fuzzy real number to avoid any confusion, otherwise the fuzzy interval is used. If $\eta \in \mathbb{R}(I)$ and $\eta(t) = 0$ whenever $t < 0$, then η is named a non-negative fuzzy real number and $\mathbb{R}^*(I)$ adheres for the set of all non-negative fuzzy real numbers. The number $\overline{0}$ adheres for the fuzzy number gratifying $\overline{0}(t) = 1$ if $t = 0$ and $\overline{0}(t) = 0$ if $t \neq 0$. Clearly, $\overline{0} \in \mathbb{R}^*(I)$. The set of all real numbers can be installed in $\mathbb{R}(I)$ since if $r \in (-\infty, \infty)$, thus $\overline{r} \in \mathbb{R}(I)$ assures $\overline{r}(t) = \overline{0}(t - r)$. For $\eta \in \mathbb{R}(I)$, $r \in (0, \infty)$ and $\alpha \in (0, 1]$, $r \odot \eta$ is defined as $(r \odot \eta)(t) = \eta(t/r)$ and $0 \odot \eta$ is defined to be $\overline{0}$. Refer to references [16, 17, 25, 26] for more information on fuzzy real numbers.

Theorem 2.2 ([8]). *Suppose* $\{[a_\alpha, b_\alpha]; \ \alpha \in (0,1]\}$ *is a family of in volute bounded closed intervals. Let* $\eta : \mathbb{R} \to [0,1]$ *be a function defined by*

$$\eta(t) = \sup\{\alpha \in (0,1] : \ t \in [a_\alpha, b_\alpha]\}.$$

Then η *is a fuzzy real number [fuzzy interval].*

Theorem 2.3 ([1]). *Let* $[a_\alpha, b_\alpha]$, $0 < \alpha \leq 1$, *be a given dynasty of non-empty intervals. If*

(i) $[a_{\alpha_1}, b_{\alpha_1}] \supset [a_{\alpha_2}, b_{\alpha_2}]$ *for all* $0 < \alpha_1 \leq \alpha_2$.

(ii) $[\lim_{k\to\infty} a_{\alpha_k}, \lim_{k\to\infty} b_{\alpha_k}] = [a_\alpha, b_\alpha]$ *whenever* $\{\alpha_k\}$ *is an increasing sequence in* $(0,1]$ *converging to* α. *Then the dynasty* $[a_\alpha, b_\alpha]$ *demonstrates the* α*-level sets of a fuzzy interval* η *such that* $\eta(t) = \sup\{\alpha \in (0,1] : \ t \in [a_\alpha, b_\alpha]\}$ *and* $[\eta]_\alpha = [\eta_\alpha^-, \eta_\alpha^+] = [a_\alpha, b_\alpha]$.

Definition 2.4 ([3]). Suppose L and R (respectively, left norm and right norm) is symmetric and non-decreasing function from $[0,1] \times [0,1]$ into $[0,1]$ constructiveness $L(0,0) = 0$, $R(1,1) = 1$ and X be a real linear space. Then $\|.\|$ is named a fuzzy norm and $(X, \|.\|, L, R)$ a $F-FNLS$ if the function $\|.\|$ from X into $\mathbb{R}^*(I)$ satisfies the next principles, wherever $[\|x\|]_\alpha = [\|x\|_\alpha^-, \|x\|_\alpha^+]$ through $x \in X$ and $\alpha \in (0,1]$:

(A1) $\|x\| = \overline{0}$ iff $x = 0$,

(A2) $\|rx\| = |r| \odot \|x\|$ for every $x \in X$ and $r \in (-\infty, \infty)$,

(A3) For every $x, \ y \in X$:

(A3L) while $s \leq \|x\|_1^-$, $t \leq \|y\|_1^-$ and $s+t \leq \|x+y\|_1^-$, so $\|x+y\|(s+t) \geq L(\|x\|(s), \|y\|(t))$,

(A3R) while $s \geq \|x\|_1^-$, $t \geq \|y\|_1^-$ and $s+t \geq \|x+y\|_1^-$, hence $\|x+y\|(s+t) \leq R(\|x\|(s), \|y\|(t))$.

Remark 2.5. Felbin [2] showed that if $L = \wedge(Min)$ and $R = \vee(Max)$ then Condition $(A3)$ in the above definition is equivalent to

$$\|x+y\| \leq \|x\| \oplus \|y\|.$$

Therefore, the definition of Felbin fuzzy norm can be easily restated as follows.

Definition 2.6. Let X be a real linear space. If the function $\|.\|$ from X into $\mathbb{R}^*(I)$ satisfies the next principles, where $[\|x\|]_\alpha = [\|x\|_\alpha^-, \|x\|_\alpha^+]$ for $x \in X$ and $\alpha \in (0,1]$:

(A1) $\|x\| = \overline{0}$ iff $x = 0$,
(A2) $\|rx\| = |r| \odot \|x\|$ for any $x \in X$ and $r \in (-\infty, \infty)$,
(A3) $\|x+y\| \preceq \|x\| \oplus \|y\|$ For all $x,\ y \in X$,
(A4) $x \neq 0 \Rightarrow \|x\|(t) = 0, \quad \forall t \leq 0$.

Then $\|.\|$ is called a F-fuzzy norm and $(X, \|.\|)$ a Felbin fuzzy normed linear space (abbreviated to $F-FNLS$).

Lemma 2.7. *Let X be a linear space and $\{[\|x\|_\alpha^-, \|x\|_\alpha^+] \ : \ \alpha \in (0,1]\}$ be a given family of nonempty nested bounded closed intervals for all $x \in X$ whose $\|.\|_\alpha^-$ and $\|.\|_\alpha^+$ are crisp norms on X and $\inf_{\alpha\in(0,1]} \|x\|_\alpha^- > 0$. If $[\lim\limits_{n\to\infty} \|x\|_{\alpha_n}^-, \lim\limits_{n\to\infty} \|x\|_{\alpha_n}^+] = [\|x\|_\alpha^-, \|x\|_\alpha^+]$ whenever $\{\alpha_n\}$ is an increasing sequence in $(0,1]$ converging to α, then X is a F-fuzzy normed linear space.*

Proof. We define the F-fuzzy norm map $\|.\|$ on X as:

$$\forall t \in \mathbb{R}, \quad \|x\|(t) = \sup\left\{\alpha \in (0,1] : \ t \in [\|x\|_\alpha^-, \|x\|_\alpha^+]\right\}.$$

By Theorem 2.3, it is easy to see that $\|x\|$ is a fuzzy real number [fuzzy interval]. Now we show that $(X, \|.\|)$ is a F-fuzzy normed linear space. For this we must show that the defined map satisfies the conditions of F-fuzzy norm. If $x = 0$ then $\|x\|_\alpha^- = \|x\|_\alpha^+ = 0$, forall $\alpha \in (0,1]$. Therefore, if $t \neq 0$, $\|x\|(t) = 0$ and if $t = 0$,

$$\|x\|(0) = \sup\left\{\alpha \in (0,1]; \quad 0 \in [\|x\|_\alpha^-, \|x\|_\alpha^+]\right\} = 1, \quad \text{hence} \quad \|x\| = \overline{0}.$$

Now if $\|x\| = \overline{0}$ then for all $\alpha \in (0,1]$, $\|x\|_\alpha^- = \|x\|_\alpha^+ = 0$. Therefore, by properties of crisp norms, $x = 0$. Hence, the defined map satisfies Condition $(A1)$. For Condition $(A2)$, if $x \in X$ and $r \neq 0$ then by the defined map and properties of crisp norms, for all $\alpha \in (0,1]$, we have

$\|rx\|_\alpha^\pm = |r|\|x\|_\alpha^\pm$. Therefore, $\|rf\| = |r| \odot \|f\|$. For Condition $(A3)$, by Remark 2.5, since

$$\|x+y\|_\alpha^- \le \|x\|_\alpha^- + \|y\|_\alpha^-$$
$$\|x+y\|_\alpha^+ \le \|x\|_\alpha^+ + \|y\|_\alpha^+,$$

we obtain that the defined map satisfies $(A3)$. Also, It is clear that $(A4)$ is equivalent to assumption $\inf_{\alpha\in(0,1]} \|x\|_\alpha^- > 0$. Therefore, $(X, \|.\|)$ is a fuzzy normed linear space. □

Theorem 2.8 ([23])**.** *Let $(X, \|.\|, L, R)$ be a $F-FNLS$ and $\lim_{a\to 0^+} R(a,a) = 0$. Then $(X, \|.\|, L, R)$ is a Hausdorff topological vector space, whose neighborhood base of origin θ is $\{N(\varepsilon, \alpha) : \varepsilon > 0,\ \alpha \in (0,1]\}$.*

To achieve the main conclusion, the provision of the following definitions appears necessary.

Suppose X and Y are topological vector spaces and $\Lambda : X \to Y$ is a linear operator. An operator Λ is named topological continuous if for any $x \in X$ and every neighborhood V of $\Lambda(x)$, there exists a neighborhood U of x such that $\Lambda(U) \subset V$.

Let $(X, \|.\|, L, R)$ be a $F-FNLS$, $\lim_{a\to 0^+} R(a,a) = 0$, $A \subseteq X$ and $x_0 \in X$. A point x_0 is called a point of closure of A if $\{x_0 + N(\alpha,\alpha)\} \bigcap A \neq \emptyset$ for every $\alpha \in (0,1]$; $\overline{A}$ denotes the set of all points of closure of A. The subset A is called a fuzzy closed set if $\overline{A} = A$. The subset A is called a fuzzy bounded set if for each $\alpha \in (0,1]$ there exists $M = M_\alpha > 0$ such that $A \subseteq N(M_\alpha, \alpha)$. A sequence $\{x_n\}_{n=1}^\infty \subseteq X$ converges to $x \in X$, denoted by $\lim_{n\to\infty} x_n = x$, if $\lim_{n\to\infty} \|x_n - x\|_\alpha^+ = 0$ for every $\alpha \in (0,1]$, and is called a Cauchy sequence if $\lim_{m,n\to\infty} \|x_m - x_n\|_\alpha^+ = 0$ for every $\alpha \in (0,1]$. A subset $A \subseteq X$ is said to be complete if every Cauchy sequence in A converges in A. $(X, \|.\|, L, R)$ is said to be fuzzy Banach space if every Cauchy sequence is convergent in X. Λ is said to be a fuzzy bounded operator if Λ maps fuzzy bounded sets into fuzzy bounded sets. Λ is fuzzy norm continuous if $\lim_{n\to\infty} x_n = x$ implies $\lim_{n\to\infty} \Lambda(x_n) = \Lambda(x)$.

2.2. **Bag and Samanta Type Fuzzy Normed Linear Space.**

Definition 2.9 ([6])**.** Suppose X is a a real linear space. A fuzzy subset N of $X \times \mathbb{R}$ is named a fuzzy norm on X if the addendum conditions, are applieded for every $x,\ y \in X$ and $c \in \mathbb{R}$:

(N1) $N(x,t) = 0;\ \forall t \in \mathbb{R}$ *with* $t \le 0$,
(N2) $N(x,t) = 1;\ \forall t \in \mathbb{R},\ t > 0$ if and only if $x = 0$,
(N3) $N(cx,t) = N(x, \frac{t}{|c|});\ \forall t \in \mathbb{R},\ t > 0$ and $c \neq 0$,
(N4) $N(x+y, t+s) \ge \min\{N(x,s), N(y,t)\}$; for every $x,\ y \in X$ for every $s,\ t \in \mathbb{R}$,

(N5) $N(x,.)$ is a non-decreasing mapping on $\mathbb{R}$, and $\lim\limits_{t\to\infty} N(x,t) = 1$.

The couple (X,N) is named to be a Bag-Samanta $FNLS$ (abbreviated to $BS-FNLS$).

The following assumptions of $BS-FNLS$ will be required later on.

(N6) $\forall t > 0, \quad N(x,\ t) > 0$ implies $x = 0$.
(N7) For $x \neq 0$, $N(x,.)$ is a continuous function of $\mathbb{R}$.
(N8) $\forall x \in X, \exists t = t(x) > 0 \quad s.t \ \ \forall s \geq t, N(x,s) = 1$

Theorem 2.10 ([8])**.** *Suppose (X,N) is a $BS-FNLS$ satisfying $(N6)$. For all $\alpha \in (0,\ 1)$, describe $\|x\|_\alpha = \inf\{t > 0 : N(x,\ t) \geq \alpha\}$, then $\{\|.\|_\alpha :\ \alpha \in (0,\ 1)\}$ is an ascending family of norms on X.*

Theorem 2.11 ([7])**.** *Suppose $\{\|x\|_\alpha :\ \ \alpha \in (0,1]\}$ is an ascending family of norms on linear space X. Now we describe a function $N' : X \times \mathbb{R} \to [0,1]$ as*

$$N'(x,t) = \begin{cases} \sup\{\alpha \in (0,1] :\ \|x\|_\alpha \leq t\} & \textit{when } (x,t) \neq (0,0) \\ 0 & \textit{when } (x,t) = (0,0) \\ & \textit{or } \{\alpha \in (0,1] :\ \|x\|_\alpha \leq t\} = \emptyset. \end{cases}$$

Then

a) *N' is a BS-fuzzy norm on X.*
b) *N' satisfies in condition $(N8)$.*

Remark 2.12. Let BS-fuzzy norm satisfy $(N8)$, then it can be easily verified that Theorem 2.10 remains true in case that $\alpha \in (0,1]$.

Theorem 2.13. *[24] Suppose (X,N) is a $BS-FNLS$ satisfying $(N7)$. Therewith (X,N) is a Hausdorff topological vector space, which neighborhood locally basis of origin 0 is $\{B(0,\alpha,t) :\ t > 0\ ,\ \alpha \in (0,1)\}$.*

In the rest of this section some definitions will be required.

Suppose (X,N) is a $BS-FNLS$and $G \subseteq X,\ x_0 \in X$ is named a point of closure of G if $\{B(0,\alpha,t) + x_0\} \cap G \neq \emptyset$, for every $\alpha \in (0,1),\ t > 0$. $\overline{G}$ denotes the closure of G. The set G is named a fuzzy closed set if $\overline{G} = G$. The set G is called a fuzzy bounded set if for any $\alpha \in (0,1)$, there exists $t = t(\alpha) > 0$ such that $G \subseteq B(0,\alpha,t(\alpha))$, it means for all $x \in G;\ N(x,t(\alpha)) > 1-\alpha$.

Suppose (X,N) is a $BS-FNLS$ satisfying $(N6)$; G is mentioned to be fuzzy bounded if for each $\alpha \in (0,1)$, there exists $t = t(\alpha) > 0$; $\|x\|_\alpha \leq t(\alpha)$. The sequence $\{x_n\}$ is called to be convergent if there exists an $x \in X$ such that $\lim\limits_{n\to\infty} N(x_n - x,t) = 1,\ \forall t > 0$, and we deduce $\lim\limits_{n\to\infty} x_n = x$. A sequence $\{x_n\}$ is mentioned to be Cauchy sequence if

$$\lim_{n\to\infty} N(x_{n+p} - x_n,t) = 1,\ \forall\ t > 0,\ p = 1,2,3,\ldots.$$

A subset $G \subseteq X$ is said to be fuzzy complete if every Cauchy sequence in G converges in G. Λ is said to be a fuzzy bounded operator if Λ maps fuzzy bounded sets into fuzzy bounded sets. A mapping $\Lambda : (X, N_1) \to (Y, N_2)$ is said to be fuzzy continuous at x_0, if for given $\epsilon > 0,\ \alpha \in (0,1),\ \exists\delta = \delta(\alpha, \epsilon) > 0,\ \beta = \beta(\alpha, \epsilon) \in (0,1)$ such that for all $x \in X$, $N_1(x - x_0, \delta) > \beta,\ \ then\ \ N_2(\Lambda(x) - \Lambda(x_0), \epsilon) > \alpha$.
If Λ is fuzzy continuous at each point of X, then Λ is said to be continuous on X.

2.3. **Bag and Samanta's Fuzzy Antinorm Linear Space.**

Definition 2.14 ([7])**.** Suppose X is a real linear space. A fuzzy subset N^* of $X \times \mathbb{R}$ is named a fuzzy antinorm on X if the addendum conditions, are applieded for every $x,\ y \in X$ and $c \in \mathbb{R}$:

(N^*1) $N^*(x,t) = 1\ \ ;\ \ \forall t \in \mathbb{R}$ with $t \leq 0$,
(N^*2) $N^*(x,t) = 0\ \ ;\ \ \forall t \in \mathbb{R},\ t > 0$ if and only if $x = 0$,
(N^*3) $N^*(cx,t) = N^*(x,\ \frac{t}{|c|}); \forall t \in \mathbb{R},\ t > 0$ and $c \neq 0$,
(N^*4) $N^*(x+y, t+s) \leq \max\{N^*(x,s), N^*(y,t)\}$; for all $x,\ y \in X$ for every $s,\ t \in \mathbb{R}$,
(N^*5) $N^*(x,.)$ is a non-increasing mapping on $\mathbb{R}$, and $\lim_{t\to\infty} N^*(x,t) = 0$.

The couple (X, N^*) is said to be a BS-fuzzy antinormed linear space (abbreviated in $BS - FALS$).

The following conditions on $BS - FALS$ will be required later on.

(N^*6) $\forall t > 0\ ,\ N^*(x,t) < 1\ \ implies\ \ x = 0$
(N^*7) For $x \neq 0$, $N^*(x,.)$ is a continous function of $\mathbb{R}$.

Remark 2.15 ([7])**.** N^* is a BS-fuzzy antinorm on X if and only if $1 - N^*$ is a BS-fuzzy norm on X.

Remark 2.16. Suppose (X, N^*) is a $BS - FALS$ satisfying (N^*7). By Theorem 2.11, it can be easily shown that (X, N^*) is a Hausdorff topological vector space, whose neighborhood locally base of origin 0 is $\{B^*(0, \alpha, t) : t > 0, \alpha \in (0,1)\}$, where

$$B^*(0, \alpha, t) = \{x : N^*(x,t) < \alpha\}.$$

Since the neighborhoods of the origin in both norms (BS-fuzzy norm, BS-fuzzy antinorm) are the same, the topologies induced by BS-fuzzy norm and BS-fuzzy antinorm are equivalent. Therefore, the definitions of fuzzy closed, fuzzy open and fuzzy bounded sets in $BS - FALS$ are the same as those definitions in $BS - FNLS$. Hence, considering the definition of antinorm, it is convenient to define the notion of convergence and Cauchy sequence in $BS - FALS$ as follows. The sequence $\{x_n\}$ is said to be convergent if there stands a point $x \in X$ such that

$\lim_{n\to\infty} N^*(x_n - x, t) = 0,\ \forall t > 0$, and we write $\lim_{n\to\infty} x_n = x$. A sequence $\{x_n\}$ is said to be Cauchy sequence if $\lim_{n\to\infty} N^*(x_{n+p} - x_n, t) = 0$ for all $t > 0$ and $p = 1, 2, 3, \ldots$.

Theorem 2.17 ([7]). *Suppose (X, N^*) is a BS-fuzzy antinorm on a linear space X satisfying $N^*(6)$. Define $\|x\|_\alpha = \inf\{t > 0 : \ N^*(x,\ t) \leq \alpha\}$, $\quad \alpha \in (0,\ 1)$, then $\{\|.\|^*_\alpha : \ \alpha \in (0,\ 1)\}$ is a descending family of norms on X.*

Theorem 2.18 ([7]). *Let $\{\|.\|^*_\alpha : \ \alpha \in (0, 1]\}$ be a descending family of norms on a linear space X. Now we define a function $N' : X \times \mathbb{R} \to [0, 1]$ as*

$$N'(x,t) = \begin{cases} \inf\{\alpha \in (0,1] : \ \|x\|^*_\alpha \leq t\} & \textit{when } (x,t) \neq (0,0); \\ 1 & \textit{when } (x,t) = (0,0) \\ & \textit{or } \{\alpha \in (0,1] : \ \|x\|^*_\alpha \leq t\} = \emptyset. \end{cases}$$

Then N' is a BS-fuzzy antinorm on X.

Remark 2.19. Since for each $x \neq 0, \|x\|^*_1 > 0$. So $\exists t = t(x) > 0$ such that $\|x\|^*_1 > t(x) > 0$ i.e., $\|x\|^*_\alpha > t(x)$, $\forall \alpha \in (0, 1]$. Thus $N'(x, t) = 1$.

3. The Main Results

In the following discussion we point out that the topologies induced by BS-fuzzy norm and F-fuzzy norm are equivalent. So all results in corresponding fuzzy normed linear spaces are the same.

Theorem 3.1 ([7]). *Let $(X, \|.\|)$ be a F-fuzzy normed linear space and $[\|x\|]_\alpha = [\|x\|^1_\alpha, \|x\|^2_\alpha], \alpha \in (0, 1]$. Let N and N^* be two functions in $X \times \mathbb{R}$ defined by*

$$N(x,t) = \begin{cases} \sup\{\alpha \in (0,1] : \ \|x\|^1_\alpha \leq t\} & \textit{when } (x,t) \neq (0,0); \\ 0 & \textit{when } (x,t) = (0,0) \\ & \textit{or } \{\alpha \in (0,1] : \ \|x\|_\alpha \leq t\} = \emptyset; \end{cases}$$

and

$$N^*(x,t) = \begin{cases} \inf\{\alpha \in (0,1] : \ \|x\|^2_\alpha \leq t\} & \textit{when } (x,t) \neq (0,0); \\ 1 & \textit{when } (x,t) = (0,0) \\ & \textit{or } \{\alpha \in (0,1] : \ \|x\|^*_\alpha \leq t\} = \emptyset. \end{cases}$$

Then N is a BS-fuzzy norm and N^* is a BS-fuzzy antinorm and they satisfy the following conditions:

(i) *N satisfies $(N6)$,*
(ii) *N^* satisfies (N^*6),*
(iii) *for each $x \neq 0, \exists r = r(x) > 0$ such that $N(x, t) = 1$ for all $t \geq r$,*

(iv) *for each* $x \neq 0, \exists t_1 = t_1(x) > 0$ *such that* $N(x,t_1) = 0$,

(v) $N^*(x,t) < 1 \Rightarrow N(x,t+) = 1$, *where* $N(x,t+) = lim_{s\downarrow t} N(x,s)$.

Also if we define

$$\|x\|'_\alpha = \inf\{t > 0 : N_1(x,t) \geq \alpha\}$$
$$\|x\|''_\alpha = \inf\{t > 0 : N_2(x,t) < \alpha\},$$

for all $\alpha \in (0,1]$, *then* $\|x\|'_\alpha$ *and* $\|x\|''_\alpha$ *are norms on* X *and* $\|x\|^1_\alpha = \|x\|'_\alpha$, $\|x\|^2_\alpha = \|x\|''_\alpha$, *for all* $\alpha \in (0,1]$.

Remark 3.2. Condition (iv) is contrapositive of Condition (i), so there is no need to restate it.

Theorem 3.3 ([7])**.** *Let* N *be a* BS*-fuzzy norm and* N^* *be a* BS*-fuzzy antinorm on a linear space* X *satisfying Conditions (i)-(v) of Theorem 3.1. Then there exists a* F*-fuzzy norm on* X.

Theorem 3.4. *Let* N *be a* BS*-fuzzy norm on a linear space* X *satisfying* $(N6)$. *Then there exists a* F*-fuzzy norm on* X.

Proof. By Remark 2.15, $N^* = 1-N$ is a BS-fuzzy antinorm that satisfies (N^*6). Define $\|x\|^*_\alpha = \inf\{t > 0 : N^*(x,t) < \alpha\}$ and $\|x\|^1_\alpha = \|x\|^*_1$, for $0 < \alpha \leq 1$. From Theorem 2.17, we have $\{\| \ \|^*_\alpha \ : \ \alpha \in (0,1]\}$ is a descending family of crisp norms on X. Then $\|x\|^1_\alpha \leq \|x\|^*_\alpha$ and $\{[\|x\|^1_\alpha, \|x\|^*_\alpha] \ : \ \alpha \in (0,1]\}$ is a family of nonempty nested bounded closed intervals. Now we show that $[\lim_{n\to\infty} \|x\|^1_{\alpha_n}, \lim_{n\to\infty} \|x\|^*_{\alpha_n}] = [\|x\|^1_\alpha, \|x\|^*_\alpha]$, whenever $\{\alpha_n\}$ is an increasing sequence in $(0,1]$ converging to α. By the assumption $\|x\|^1_\alpha = \|x\|^*_1$, it is clear that $\lim_{n\to\infty} \|x\|^1_{\alpha_n} = \|x\|^1_\alpha$. Since $\|x\|^*_{\alpha_n} \leq \|x\|^*_\alpha$; $\forall n \in \mathbb{N}$, so $\lim_{n\to\infty} \|x\|^*_{\alpha_n} \leq \|x\|^*_\alpha$. Suppose $\lim_{n\to\infty} \|x\|^*_{\alpha_n} < \|x\|^*_\alpha$. Choose k such that $\lim_{n\to\infty} \|x\|^*_{\alpha_n} < k < \|x\|^*_\alpha$. Since $\|x\|^*_{\alpha_n}$ is a decreasing sequence and $\lim_{n\to\infty} \|x\|^*_{\alpha_n} < k$, so $\|x\|^*_{\alpha_n} < k$; $\forall n \in \mathbb{N}$. Then, by definition of $\|x\|^*_{\alpha_n}$, we have $\inf\{t > 0 : N(x,t) \geq \alpha_n\} \leq k$, for all $n \in \mathbb{N}$. So, $N(x,k) \geq \alpha_n$, for all $n \in \mathbb{N}$. Since $\{\alpha_n\}$ converges to α, $N(x,k) \geq \alpha$. Hence, $\inf\{t > 0 : N(x,t) \geq \alpha_n\} \leq k$. Therefore, $k < \|x\|^*_\alpha \leq k$ is a contradiction. So $\lim_{n\to\infty} \|x\|^*_{\alpha_n} = \|x\|^*_\alpha$. Now, by Lemma 2.7, X is a F-fuzzy normed linear space. □

Theorem 3.5. *Let* $\{x_n\}$ *be a sequence in* X. $\{x_n\}$ *is convergent w.r.t* F*-fuzzy norm if and only if it is convergent w.r.t* BS*-fuzzy norm satisfying* $(N6)$.

Proof. Let $(X, \|.\|)$ be a $F-FNLS$ and N^* be the fuzzy norm induced from the family $\{\| \ \|^2_\alpha \ : \ \alpha \in (0,1]\}$ where $[\|x\|]_\alpha = [\|x\|^1_\alpha, \|x\|^2_\alpha]$, $\alpha \in (0,1]$. Then by Theorem 3.1, N^* is a BS- fuzzy antinorm. Therefore,

$N = 1 - N^*$ is a BS-fuzzy norm. Now let $\{x_n\}$ be a sequence converging x in $F - FNLS$ $(X, \|.\|)$. So $\lim_{n\to\infty} \|x_n - x\| = 0$, i.e.

$$\lim_{n\to\infty} \|x_n - x\|_\alpha^2 = 0, \quad \forall\alpha \in (0, 1].$$

According to Theorem 3.1, it is clear that $\lim_{n\to\infty} N^*(x_n - x, t) = 0$, so $\lim_{n\to\infty} N(x_n - x, t) = 1$, $\forall t > 0$. Hence, $\{x_n\}$ converges to x w.r.t BS-fuzzy norm. Conversely, let N be a BS-fuzzy norm satisfying $(N6)$. By Remark 2.15, $N^* = 1 - N$ is a BS-fuzzy antinorm satisfying (N^*6). Let for $0 < \alpha \le 1$, $\|x\|_\alpha^* = \inf\{t > 0 \ : \ N^*(x, t) < \alpha\}$ and $\|x\|_\alpha^1 = \|x\|_1^*$. By Theorem 3.4, there exists a F-fuzzy norm, say, $\|.\|$ on X, such that

$$[\|x\|]_\alpha = [\|x\|_\alpha^1, \|x\|_\alpha^*], \quad x \in X, \alpha \in (0, 1].$$

Now let $\{x_n\}$ be a sequence in $(X, \|.\|)$, converging x w.r.t BS-fuzzy norm, then $\lim_{n\to\infty} N(x_n - x, t) = 1$, $\forall t > 0$. So $\lim_{n\to\infty} N^*(x_n - x, t) = 0$, $\forall t > 0$, therefore, $\lim_{n\to\infty} \|x_n - x\|_\alpha^* = 0$. Hence, $\lim_{n\to\infty} \|x_n - x\| = 0$, and $\{x_n\}$ converges to x w.r.t F-fuzzy norm. □

Remark 3.6. The above theorem results in the fact that F-fuzzy norm and BS-fuzzy norm are equivalent. Hence, all the results based on Felbin fuzzy normed linear space do hold for BS-fuzzy normed linear space and vice versa. So if we prove a result in any type of mentioned fuzzy normed linear space, it does automatically hold in the other one.

In the following, we list some common results which are posed separately in $BS - FNLS$ and $F - FNLS$.

Theorem 3.7.
a) *Every finite dimension $BS - FNLS$ whose norm satisfies $(N6)$ condition is complete [7].*
b) *Every finite dimension $F - FNLS$ is complete[3].*

Theorem 3.8.
a) *In finite dimension $BS - FNLS$ satisfies $(N6)$, a subset A is compact if and only if A is fuzzy closed and bounded[7].*
b) *In finite dimension F-fuzzy normed linear space, a subset A is compact if and only if A is fuzzy closed and bounded[3].*

Theorem 3.9.
a) *Let T be a linear operator from (X_1, N_1) into (X_2, N_2) where N_1 satisfies $N(6)$. Then T is a fuzzy norm continuous if and only if T is a topological continuous[23].*
b) *Let T be a linear operator from $(X_1, \|.\|)$ into $(X_2, \|.\|)$. Then T is a fuzzy norm continuous if and only if T is a topological continuous[24].*

Remark 3.10. In the following, we give some results in $BS - FNLS$ which have not been proved in $F - FNLS$, before. Since we showed that $BS - FNLS$ and $F - FNLS$ are equivalent, so the same results are valid in $F - FNLS$.

Theorem 3.11 ([24])**.** *A BS-fuzzy normed space (X, N) in which every Cauchy sequence has a convergent sub-sequence is complete.*

Theorem 3.12 ([19])**.** *(Riesz) Let V and W be two subspace fuzzy normed linear space (X, N) satisfying $N(6)$ and $N(7)$ of which W is closed and is a proper subset of V. Then for any $\theta \in (0, 1)$ and for each $\alpha \in (0, 1)$, $\exists y_\alpha \in V \setminus W$ such that $N(y_\alpha, 1) \geq \alpha$ and $N(y_\alpha - W, \theta) \leq \alpha$, $\forall w \in W$.*

Theorem 3.13 ([21])**.** *Let $T : (X, N_1) \to (Y, N_2)$ be a linear operator. Then T is fuzzy compact if and only if it maps every fuzzy bounded sequence $\{x_n\}$ in X onto a sequence $\{T(x_n)\}$ in Y which has a fuzzy convergent sub sequence.*

Lemma 3.14 ([21])**.** *Let $T : (X, N_1) \to (Y, N_2)$ be a fuzzy compact operator, where (X, N_1) and (Y, N_2) are BS-fuzzy normed linear spaces satisfying $(N6)$. Then $T : (X, \|.\|^1_\alpha) \to (Y, \|.\|^2_\alpha)$ is an ordinary compact operator for all $\alpha \in (0, 1)$.*

Theorem 3.15 ([21])**.** *Let (X, N_1) and (Y, N_2) be two BS-fuzzy normed linear spaces. Then the set of all fuzzy compact linear operators from X into Y is a linear subspace of $F'(X, Y)$.*

Remark 3.16. In the following, we give some results in $F - FNLS$ which have not been proved in $BS - FNLS$ before. Since we showed that $BS - FNLS$ and $F - FNLS$ are equivalent, so the some results valid in $BS - FNLS$.

Theorem 3.17 ([3])**.** *Let $(X, \|.\|)$ be a complete F-fuzzy linear space.*

(1) *If $\{V_n\}$ is a sequence of fuzzy open dense subsets in X, then $\cap_{n=1}^{\infty} V_n$ is dense in X.*
(2) *$(X, \|.\|)$ is of the second category.*

Theorem 3.18 ([27])**.** *Suppose $(X, \|.\|)$ is a $F - FNLS$. Then*

(1) *For each $\alpha \in (0, 1]$, $\|.\|^+_\alpha$ is continuous mapping from X into $\mathbb{R}$ at $x \in X$.*
(2) *For each $\alpha \in (0, 1]$, $\|.\|^-_\alpha$ is continuous mapping from X into $\mathbb{R}$ at $x \in X$.*

Theorem 3.19 ([22])**.** *Let $(X, \|.\|)$ be a $F - FNLS$. Suppose T be a fuzzy compact operator on X and I be the identity operator. If $A \subset X$ is a fuzzy closed and fuzzy bounded set then $(I - T)(A)$ is fuzzy closed in X.*

4. Felbin and BS-Fuzzy Bounded and Continuous Operator

Bag and Samanta in [6], were not able to define the BS-fuzzy norm on fuzzy bounded operator in $L(X_1, X_2)$, so they applied the notion of weak boundedness to define BS-fuzzy norm. But, using Theorem 4.1, we are able to define the norm of an operator in $BS - FNLS$.

Theorem 4.1 ([4])**.** *Let $(X_i, \|\ \|)$ be $F - FNLS$, $B(X_1, X_2)$ be the set of all fuzzy bounded linear operators from X_1 into X_2 and $\forall \in X_1$, $\sup_{\alpha\in(0,1]} \|x\|_\alpha < +\infty$. Then $(B(X_1, X_2), \|.\|_B)$ is a $F - FNLS$ where*

$$\|T\|_B(t) = \lim_{\alpha\to 0^+} \sup_{\|x\|^+_\alpha=1} \|T(x)\|(t).$$

Theorem 4.2 ([4])**.** *Let $(X_i, \|\ \|)$ be F fuzzy normed linear spaces, $F - FNLS$, and $\sup_{\alpha\in(0,1]} \|x\|_\alpha < +\infty$, $\forall x \in X_1$. If $(X_2, \|.\|)$ is a complete space, so is $(B(X_1, X_2), \|.\|_B)$.*

Corollary 4.3. *Let (X_i, N_i) be $BS - FNLS$ such that (X_1, N_1) satisfying $N(6)$ and $N(8)$. Then $(B(X_1, X_2), N_B)$ is a BS-fuzzy normed linear spaces.*

Proof. Take an operator $T \in B(X_1, X_2)$. By Theorem 3.4, $(X_i, \|.\|_i)$ is a F-fuzzy normed linear space and $(X_1, \|.\|_1)$ satisfies in

$$\sup_{\alpha\in(0,1]} \|x\|_\alpha < +\infty, \quad \forall x \in X_1.$$

Since it is clear that condition $\sup_{\alpha\in(0,1]} \|x\|_\alpha < +\infty$ and $(N6)$ are equivalent, using Theorem 4.1, T is an F-fuzzy bounded operator and $(B(X_1, X_2), \|.\|_B)$ is an F-fuzzy normed linear space where

$$\|T\|_B(t) = \lim_{\alpha\to 0^+} \sup_{\|x\|^+_\alpha=1} \|T(x)\|(t).$$

Therefore, by Theorems 3.4 and 3.1, we define BS-fuzzy norm on $B(X_1, X_2)$ as follows:

$$N_B(T,t) = \begin{cases} \sup\{\alpha \in (0,1] : \ \|T\|^1_{B_\alpha} \le t\} & \text{when } (T,t) \neq 0; \\ 0 & \text{when } (T,t) = 0 \\ & \text{or } \{\alpha \in (0,1] : \ \|T\|^1_{B_\alpha} \le t\} = \emptyset. \end{cases}$$

Thus $(B(X_1, X_2), N_B)$ is a BS-fuzzy normed linear spaces. □

Corollary 4.4. *Let (X_i, N_i) be BS-fuzzy normed linear spaces such that (X_1, N_1) satisfying $N(6)$ and $N(8)$. If (X_2, N_2) is a complete space, so is $(B(X_1, X_2), N_B)$.*

Definition 4.5 ([6])**.** Let (X, N_1) and (Y, N_2) be two BS-fuzzy normed linear spaces and T be an operator from (X, N_1) to (Y, N_2). T is said to be strongly BS-fuzzy continuous at x_0, if for each $\epsilon > 0$, $\exists \delta > 0$

such that for all $x \in X,\ N_2(T(x) - T(x_0), \epsilon) \geq N_1(x - x_0, \delta)$. If T is BS-strongly fuzzy continuous at each point of X, then T is said to be strongly BS-fuzzy continuous on X.

The operator T is said to be:

(i) weakly BS-fuzzy continuous at x_0, if for given $\epsilon > 0,\ \alpha \in (0,1),\ \exists \delta = \delta(\alpha, \epsilon) > 0,\ \beta = \beta(\alpha, \epsilon) \in (0,1)$ such that for all $x \in X,\ N_1(x - x_0, \delta) > \alpha,\ \ \textit{then}\ \ N_2(T(x) - T(x_0), \epsilon) > \alpha$. If T is weakly BS-fuzzy continuous at each point of X, then T is said to be BS-weakly fuzzy continuous on X.

(ii) strongly BS-fuzzy bounded on X if and only if $\exists M > 0,\ \forall x \in X, \forall s \in \mathbb{R},\ N_2(T(x), s) \geq N_1(x, s/M)$.

(iii) weakly BS-fuzzy bounded on X if and only if for any $\alpha \in (0,1),\ \exists M_\alpha > 0$ such that $\forall x \in X, \forall t \in \mathbb{R},\ N_1(x, t/M_\alpha) \geq \alpha \Rightarrow N_2(T(x), t) \geq \alpha$.

Definition 4.6 ([8])**.** Let $(X, \|.\|)$ and $(Y, \|.\|')$ be two F-fuzzy normed linear spaces. An operator $T : X \to Y$ is said to be

(i) strongly F-fuzzy continuous at $x_0 \in X$ if for a given $\epsilon > 0$, $\exists \delta > 0$ such that $\|T(x) - T(x_0)\|'^{2}_{\alpha} < \epsilon$ whenever $\|x - x_0\|^1_\alpha < \delta,\ \ \forall \alpha \in (0,1]$.

(ii) weakly F-fuzzy continuous at $x_0 \in X$ if for a given $\epsilon > 0$, $\exists \delta \in \mathbb{R}^*$, $\delta > 0$ such that

$$\|T(x) - T(x_0)\|'^{1}_{\alpha} < \epsilon \ \text{ whenever } \ \|x - x_0\|^2_\alpha < \delta^2_\alpha,$$

$$\|T(x) - T(x_0)\|'^{2}_{\alpha} < \epsilon \ \text{ whenever } \ \|x - x_0\|^1_\alpha < \delta^1_\alpha,$$

where $[\delta]_\alpha = [\delta^1_\alpha, \delta^2_\alpha],\ \ \alpha \in (0,1]$.

(iii) strongly F-fuzzy bounded if there exists a real number $k > 0$ such that $\|T(x)\|' \oslash \|x\| \preceq \bar{k},\ \forall x (\neq \theta) \in X$.

(iv) weakly F-fuzzy bounded if there exists a fuzzy interval $\eta \in \mathbb{R}^*, \eta \succ \bar{0}$ such that $\|T(x)\|' \oslash \|x\| \preceq \eta,\ \forall x (\neq \theta) \in X$.

Also, if T is strongly F-fuzzy continuous at all points of X, then T is said to be strongly F-fuzzy continuous on X.

Theorem 4.7 ([6, 8])**.** *Let $T : (X, N_1) \to (Y, N_2)$ be a linear operator where (X, N_1) and (Y, N_2) are BS (Felbin)fuzzy normed linear spaces. If T is strongly fuzzy continuous, then T is weakly fuzzy continuous, but not conversely.*

Theorem 4.8 ([6, 8])**.** *Let $T : (X, N_1) \to (Y, N_2)$ be a linear operator where (X, N_1) and (Y, N_2) are BS (Felbin) fuzzy normed linear spaces. Then T is strongly (weakly) fuzzy continuous if and only if T is strongly (weakly) fuzzy bounded.*

Theorem 4.9 ([6])**.** *Let $T : (X, N_1) \to (Y, N_2)$ be a linear operator. Then T is strongly fuzzy bounded if and only if T is uniformly bounded with respect to α-norms of N_1, N_2.*

Theorem 4.10. *Let $T : X \to Y$ be a linear operator. Then X, Y are BS-fuzzy normed linear spaces and T is strongly BS-fuzzy bounded if and only if X, Y are F-fuzzy normed linear spaces and T is strongly F-fuzzy bounded.*

Proof. By Theorem 4.9, T is strongly BS-fuzzy bounded if and only if T is uniformly bounded with respect to α-norms of N_1, N_2. It is clear that being strongly F-fuzzy bounded is equivalent to being uniformly bounded with respect to α-norms of N_1, N_2. Therefore, T is strongly BS-fuzzy bounded if and only if T is strongly F-fuzzy bounded. □

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