On Some New Extensions of Inequalities of Hermite-Hadamard Type for Generalized Fractional Integrals

Huseyin Budak, Candan Can Bilisik and Mehmet Zeki Sarikaya

Sahand Communications in Mathematical Analysis

Print ISSN: 2322-5807
Online ISSN: 2423-3900
Volume: 19
Number: 2
Pages: 65-79

Sahand Commun. Math. Anal.
DOI: 10.22130/scma.2022.539417.992

Sahand Communications in Mathematical Analysis (SCMA) Vol. 19 No. 2 (2022), 65-79
http://scma.maragheh.ac.ir
DOI: 10.22130/scma.2022.539417.992

On Some New Extensions of Inequalities of Hermite-Hadamard Type for Generalized Fractional Integrals

Hüseyin Budak[1*], Candan Can Bilişik[2] and Mehmet Zeki Sarıkaya[3]

Abstract. In this paper, we establish some inequalities for generalized fractional integrals by utilizing the assumption that the second derivative of $\phi(x)=\varpi\left(\frac{\kappa_1\kappa_2}{\varkappa}\right)$ is bounded. We also prove again a Hermite-Hadamard type inequality obtained in [34] under the condition $\phi'\left(\kappa_1+\kappa_2-\varkappa\right)\geq\phi'(\varkappa)$ instead of harmonically convexity of ϖ. Moreover, some new inequalities for k-fractional integrals are given as special cases of main results.

1. Introduction

The Hermite-Hadamard inequality, which is the first fundamental result for convex mappings with a natural geometrical interpretation and many applications, has drawn attention much interest in elementary mathematics. A number of mathematicians have devoted their efforts to generalise, refine, counterpart and extend it for different classes of functions such as using convex mappings.

The inequalities discovered by C. Hermite and J. Hadamard for convex functions are considerable significant in the literature (see, e.g.,[24, p.137], [11]). These inequalities state that if $\varpi : I\to\mathbb{R}$ is a convex function on the interval I of real numbers and $\kappa_1,\kappa_2\in I$ with $\kappa_1<\kappa_2$, then

$$\varpi\left(\frac{\kappa_1+\kappa_2}{2}\right)\leq\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\varpi(\varkappa)d\varkappa\leq\frac{\varpi\left(\kappa_1\right)+\varpi\left(\kappa_2\right)}{2}. \tag{1.1}$$

Both inequalities hold in the reversed direction if ϖ is concave.

2020 *Mathematics Subject Classification.* 26D15, 26D10.

Key words and phrases. Hermite-Hadamard inequality, Convex function, Bounded function.

Received: 21 September 2021, Accepted: 06 March 2022.

* Corresponding author.

Iscan [16] gave the following definition of harmonically convex functions and related Hermite-Hadamard type inequality

Definition 1.1 ([16])**.** A function $\varpi : I \subseteq \mathbb{R}\backslash\{0\} \to \mathbb{R}$ is said to be harmonically convex function if the following inequality holds:

$$\varpi\left(\frac{\kappa_1\kappa_2}{\xi\kappa_1+(1-\xi)\kappa_2}\right) \le \xi\varpi(\kappa_2)+(1-\xi)\varpi(\kappa_1), \tag{1.2}$$

for all κ_1, κ_2 in I and ξ in $[0,1]$. If the inequality (1.2) holds in the reversed direction, then ϖ is called harmonically concave function.

Theorem 1.2 ([16])**.** *Let $\varpi : I \subseteq \mathbb{R}\backslash\{0\} \to \mathbb{R}$ be harmonically convex function and $\kappa_1, \kappa_2 \in I$ with $\kappa_1 < \kappa_2$. If $\varpi \in L([\kappa_1,\kappa_2])$, then the following double inequality holds:*

$$\varpi\left(\frac{2\kappa_1\kappa_2}{\kappa_1+\kappa_2}\right) \le \frac{\kappa_1\kappa_2}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\frac{\varpi(\varkappa)}{\varkappa^2}d\varkappa \le \frac{\varpi(\kappa_1)+\varpi(\kappa_2)}{2}. \tag{1.3}$$

In the following, we will give some necessary definitions and mathematical preliminaries of fractional calculus theory which are used further in this paper.

Definition 1.3 ([18])**.** Let $\varpi \in L_1[\kappa_1,\kappa_2]$. The Riemann-Liouville integrals $J_{\kappa_1+}^{\alpha}\varpi(\varkappa)$ and $J_{\kappa_2-}^{\alpha}\varpi(\varkappa)$ are defined by

$$J_{\kappa_1+}^{\alpha}\varpi(\varkappa) = \frac{1}{\Gamma(\alpha)}\int_{\kappa_1}^{\varkappa}(\varkappa-\xi)^{\alpha-1}\varpi(\xi)d\xi, \quad \varkappa > \kappa_1,$$

and

$$J_{\kappa_2-}^{\alpha}\varpi(\varkappa) = \frac{1}{\Gamma(\alpha)}\int_{\varkappa}^{\kappa_2}(\xi-\varkappa)^{\alpha-1}\varpi(\xi)d\xi, \quad \varkappa < \kappa_2,$$

respectively. Here, $\Gamma(\alpha)$ is the Gamma function and

$$J_{\kappa_1+}^{0}\varpi(\varkappa) = J_{\kappa_2-}^{0}\varpi(\varkappa) = \varpi(\varkappa).$$

Definition 1.4 ([22])**.** Let $\varpi \in L^1[\kappa_1,\kappa_2]$. Then k–fractional integrals of order $\alpha, k > 0$ with $\kappa_1 \ge 0$ are defined by

$$J_{\kappa_1+,k}^{\alpha}\varpi(\varkappa) = \frac{1}{k\Gamma_k(\alpha)}\int_{\kappa_1}^{\varkappa}(\varkappa-\xi)^{\frac{\alpha}{k}-1}\varpi(\xi)\,d\xi, \quad \varkappa > \kappa_1,$$

and

$$J_{\kappa_2-,k}^{\alpha}\varpi(\varkappa) = \frac{1}{k\Gamma_k(\alpha)}\int_{\varkappa}^{\kappa_2}(\xi-\varkappa)^{\frac{\alpha}{k}-1}\varpi(\xi)\,d\xi, \quad \kappa_2 > \varkappa,$$

where $\Gamma_k(\cdot)$ stands for the k-gamma function. For $k=1$, the k–fractional integrals yield Riemann–Liouville integrals. For $\alpha = k = 1$, the k–fractional integrals yield classical integrals.

Iscan [17] established the following inequalities of Hermite-Hadamard type for harmonically convex functions via Riemann-Liouville fractional integrals:

Theorem 1.5 ([17]). *Let $\varpi : I \subseteq (0,\infty) \to \mathbb{R}$ be function such that $\varpi \in L([\kappa_1,\kappa_2])$, where $\kappa_1,\kappa_2 \in I$ with $\kappa_1 < \kappa_2$. If ϖ is harmonically convex function on $[\kappa_1,\kappa_2]$, the following double inequalities hold for the fractional integrals:*

$$\varpi\left(\frac{2\kappa_1\kappa_2}{\kappa_1+\kappa_2}\right) \leq \frac{\Gamma(\alpha+1)}{2}\left(\frac{\kappa_1\kappa_2}{\kappa_2-\kappa_1}\right)^{\alpha} \times \left\{J^{\alpha}_{\frac{1}{\kappa_1}-}(\varpi\circ\psi)\left(\frac{1}{\kappa_2}\right)+J^{\alpha}_{\frac{1}{\kappa_2}+}(\varpi\circ\psi)\left(\frac{1}{\kappa_1}\right)\right\} \leq \frac{\varpi(\kappa_1)+\varpi(\kappa_2)}{2}, \tag{1.4}$$

where $\psi(\varkappa) = \frac{1}{\varkappa}$.

Over the years several papers devoted to fractional Hermite-Hadamard inequalities. One can refer to the references ([1]-[9], [12]-[15],[19]-[21], [23], [25]-[33]) for some of them.

Chen gave the following Lemma:

Lemma 1.6 ([10]). *Let $[\kappa_1,\kappa_2] \subset \mathbb{R}\setminus\{0\}$ be a real interval. A function $\varpi : [\kappa_1,\kappa_2] \to \mathbb{R}$ is harmonically convex, if and only if*

$$\phi(\varkappa) = \varpi\left(\frac{\kappa_1\kappa_2}{\varkappa}\right),$$

is convex on $[\kappa_1,\kappa_2]$.

Chen established following inequalities under the condition that second derivative of $\phi(\varkappa) = \varpi\left(\frac{\kappa_1\kappa_2}{\varkappa}\right)$ is bounded.

Theorem 1.7 ([10]). *Let $\varpi : [\kappa_1,\kappa_2] \subseteq (0,+\infty) \to \mathbb{R}$ be a positive, twice differentiable function with $\kappa_1 < \kappa_2$ and $\varpi \in L([\kappa_1,\kappa_2])$ .If ϕ'' is bounded in $[\kappa_1,\kappa_2]$, where $\phi(\varkappa) = \varpi\left(\frac{\kappa_1\kappa_2}{\varkappa}\right)$, then*

(1.5)

$$\frac{m\alpha}{2(\kappa_2-\kappa_1)^{\alpha}} \times \int_{\kappa_1}^{\frac{\kappa_1+\kappa_2}{2}}\left(\frac{\kappa_1+\kappa_2}{2}-\varkappa\right)^2\left[(\varkappa-\kappa_1)^{\alpha-1}+(\kappa_2-\varkappa)^{\alpha-1}\right]d\varkappa$$
$$\leq \frac{\Gamma(\alpha+1)}{2}\left(\frac{\kappa_1\kappa_2}{\kappa_2-\kappa_1}\right)^{\alpha} \times \left\{J^{\alpha}_{\frac{1}{\kappa_1}-}(\varpi\circ\psi)\left(\frac{1}{\kappa_2}\right)+J^{\alpha}_{\frac{1}{\kappa_2}+}(\varpi\circ\psi)\left(\frac{1}{\kappa_1}\right)\right\}-\varpi\left(\frac{2\kappa_1\kappa_2}{\kappa_1+\kappa_2}\right)$$
$$\leq \frac{M\alpha}{2(\kappa_2-\kappa_1)^{\alpha}}$$

$$\times \int_{\kappa_1}^{\frac{\kappa_1+\kappa_2}{2}} \left(\frac{\kappa_1+\kappa_2}{2} - \varkappa\right)^2 \left[(\varkappa-\kappa_1)^{\alpha-1} + (\kappa_2-\varkappa)^{\alpha-1}\right] d\varkappa,$$

and

(1.6)

$$\begin{aligned}
&\frac{m\alpha}{2(\kappa_2-\kappa_1)^{\alpha}} \times \int_{\kappa_1}^{\frac{\kappa_1+\kappa_2}{2}} (\varkappa-\kappa_1)(\kappa_2-\varkappa)\left[(\varkappa-\kappa_1)^{\alpha-1} + (\kappa_2-\varkappa)^{\alpha-1}\right] d\varkappa \\
&\leq \frac{\varpi(\kappa_1)+\varpi(\kappa_2)}{2} - \frac{\Gamma(\alpha+1)}{2}\left(\frac{\kappa_1\kappa_2}{\kappa_2-\kappa_1}\right)^{\alpha} \\
&\quad \times \left\{ J^{\alpha}_{\frac{1}{\kappa_1}-}(\varpi\circ\psi)\left(\frac{1}{\kappa_2}\right) + J^{\alpha}_{\frac{1}{\kappa_2}+}(\varpi\circ\psi)\left(\frac{1}{\kappa_1}\right) \right\} \\
&\leq \frac{M\alpha}{2(\kappa_2-\kappa_1)^{\alpha}} \\
&\quad \times \int_{\kappa_1}^{\frac{\kappa_1+\kappa_2}{2}} (\varkappa-\kappa_1)(\kappa_2-\varkappa)\left[(\varkappa-\kappa_1)^{\alpha-1} + (\kappa_2-\varkappa)^{\alpha-1}\right] d\varkappa,
\end{aligned}$$

where $m = \inf_{\xi\in[\kappa_1,\kappa_2]} \phi''(\xi)$, $M = \sup_{\xi\in[\kappa_1,\kappa_2]} \phi''(\xi)$.

Chen also proves the inequalities (1.4) under the condition $\phi'(\kappa_1+\kappa_2-\varkappa) \geq \phi'(\varkappa)$, $\phi(\varkappa) = \varpi\left(\frac{\kappa_1\kappa_2}{\varkappa}\right)$ instead of harmonically convexity of ϖ.

In this paper, we generalize Chen's results for generalized fractional integrals which are defined by Sarikaya and Ertuğral as follows:

Definition 1.8 ([29])**.** Let a function $\varphi : [0,\infty) \to [0,\infty)$ satisfying the following conditions:

$$\int_0^1 \frac{\varphi(\xi)}{\xi} d\xi < \infty.$$

The following left-sided and right–sided generalized fractional integral operators are defined, respectively, as follows:

$$ {}_{\kappa_1+}I_{\varphi}\varpi(\varkappa) = \int_{\kappa_1}^{\varkappa} \frac{\varphi(\varkappa-\xi)}{\varkappa-\xi}\varpi(\xi)d\xi, \quad \varkappa > \kappa_1, \tag{1.7}$$

$$ {}_{\kappa_2-}I_{\varphi}\varpi(\varkappa) = \int_{\varkappa}^{\kappa_2} \frac{\varphi(\xi-\varkappa)}{\xi-\varkappa}\varpi(\xi)d\xi, \quad \varkappa < \kappa_2. \tag{1.8}$$

Sarikaya and Ertuğral [29] also obtained the following Hermite-Hadamard inequality for convex functions:

Theorem 1.9 ([29])**.** *Let $\varpi : [\kappa_1,\kappa_2] \to \mathbb{R}$ be a convex function on $[\kappa_1,\kappa_2]$ with $\kappa_1 < \kappa_2$, then the following inequalities for fractional integral operators hold*

(1.9)

$$\varpi\left(\frac{\kappa_1+\kappa_2}{2}\right) \leq \frac{1}{2F(1)}\left[{}_{\kappa_1+}I_{\varphi}\varpi(\kappa_2) + {}_{\kappa_2-}I_{\varphi}\varpi(\kappa_1)\right] \leq \frac{\varpi(\kappa_1)+\varpi(\kappa_2)}{2},$$

where the mapping $F:[0,1]\to\mathbb{R}$ *is defined by*

$$F(\varkappa)=\int_0^{\varkappa}\frac{\varphi\left((\kappa_2-\kappa_1)\,\xi\right)}{\xi}d\xi.$$

On the other hand, Zhao et al. [34] obtained following Hermite-Hadamard inequality for harmonically convex functions:

Theorem 1.10 ([34])**.** *Let* $\varpi: I\subseteq(0,+\infty)\to\mathbb{R}$ *be a function such that* $\varpi\in L([\kappa_1,\kappa_2])$. *If* ϖ *is harmonically convex function on* $[\kappa_1,\kappa_2]$, *then following inequalities holds for the generalized fractional integrals:*

(1.10)
$$\varpi\left(\frac{2\kappa_1\kappa_2}{\kappa_1+\kappa_2}\right)\leq\frac{1}{2F(1)}\left[{}_{\frac{1}{\kappa_1}-}I_{\varphi}(\varpi\circ\psi)\left(\frac{1}{\kappa_2}\right)+{}_{\frac{1}{\kappa_2}+}I_{\varphi}(\varpi\circ\psi)\left(\frac{1}{\kappa_1}\right)\right]$$
$$\leq\frac{\varpi(\kappa_1)+\varpi(\kappa_2)}{2},$$

where

$$\psi(\varkappa)=\frac{1}{\varkappa},\qquad F(\varkappa)=\int_0^{\varkappa}\frac{\varphi\left(\frac{(\kappa_2-\kappa_1)}{\kappa_1\kappa_2}\xi\right)}{\xi}d\xi<+\infty. \tag{1.11}$$

2. Main Results

In this section, we give the following inequalities which give the upper and lower bounds for the left and right hand sides of the inequalities (1.10).

Theorem 2.1. *Let* $\varpi:[\kappa_1,\kappa_2]\subseteq(0,+\infty)\to\mathbb{R}$ *be a positive, twice differentiable function with* $\kappa_1<\kappa_2$ *and* $\varpi\in L\left([\kappa_1,\kappa_2]\right)$*.If* ϕ'' *is bounded in* $[\kappa_1,\kappa_2]$, *where* $\phi(\varkappa)=\varpi\left(\frac{\kappa_1\kappa_2}{\varkappa}\right)$, *then*

(2.1)
$$\frac{m}{2F(1)}\int_{\kappa_1}^{\frac{\kappa_1+\kappa_2}{2}}\left(\frac{\kappa_1+\kappa_2}{2}-\varkappa\right)\left[\frac{\varphi\left(\frac{\varkappa-\kappa_1}{\kappa_1\kappa_2}\right)}{\varkappa-\kappa_1}+\frac{\varphi\left(\frac{\kappa_2-\varkappa}{\kappa_1\kappa_2}\right)}{\kappa_2-\varkappa}\right]d\varkappa$$
$$\leq\frac{1}{2F(1)}\left[{}_{\frac{1}{\kappa_1}-}I_{\alpha}(\varpi\circ\psi)\left(\frac{1}{\kappa_2}\right)+{}_{\frac{1}{\kappa_2}+}I_{\alpha}(\varpi\circ\psi)\left(\frac{1}{\kappa_1}\right)\right]$$
$$-\varpi\left(\frac{2\kappa_1\kappa_2}{\kappa_1+\kappa_2}\right)$$
$$\leq\frac{M}{2F(1)}\int_{\kappa_1}^{\frac{\kappa_1+\kappa_2}{2}}\left(\frac{\kappa_1+\kappa_2}{2}-\varkappa\right)\left[\frac{\varphi\left(\frac{\varkappa-\kappa_1}{\kappa_1\kappa_2}\right)}{\varkappa-\kappa_1}+\frac{\varphi\left(\frac{\kappa_2-\varkappa}{\kappa_1\kappa_2}\right)}{\kappa_2-\varkappa}\right]d\varkappa,$$

where $m = \inf_{\xi\in[\kappa_1,\kappa_2]} \phi''(\xi)$, $M = \sup_{\xi\in[\kappa_1,\kappa_2]} \phi''(\xi)$, ψ *and* F *are defined as in Theorem 1.10.*

Proof. By the Definition 1.8 and by using the changing variables, we get

$$\frac{1}{2F(1)}\left[{}_{\frac{1}{\kappa_1}-}I_\alpha(\varpi\circ\psi)\left(\frac{1}{\kappa_2}\right)+\ {}_{\frac{1}{\kappa_2}+}I_\alpha(\varpi\circ\psi)\left(\frac{1}{\kappa_1}\right)\right]$$

$$=\frac{1}{2F(1)}\left[\int_{\frac{1}{\kappa_2}}^{\frac{1}{\kappa_1}}\frac{\varphi\left(\varkappa-\frac{1}{\kappa_2}\right)}{\varkappa-\frac{1}{\kappa_2}}\varpi\left(\frac{1}{\varkappa}\right)d\varkappa+\int_{\frac{1}{\kappa_2}}^{\frac{1}{\kappa_1}}\frac{\varphi\left(\frac{1}{\kappa_1}-\varkappa\right)}{\frac{1}{\kappa_1}-\varkappa}\varpi\left(\frac{1}{\varkappa}\right)d\varkappa\right]$$

$$=\frac{1}{2F(1)}\left[\int_{\kappa_1}^{\kappa_2}\frac{\varphi\left(\frac{\varkappa-\kappa_1}{\kappa_1\kappa_2}\right)}{\varkappa-\kappa_1}\varpi\left(\frac{\kappa_1\kappa_2}{\varkappa}\right)du+\int_{\kappa_1}^{\kappa_2}\frac{\varphi\left(\frac{\kappa_2-\varkappa}{\kappa_1\kappa_2}\right)}{\kappa_2-\varkappa}\varpi\left(\frac{\kappa_1\kappa_2}{\varkappa}\right)d\varkappa\right]$$

$$=\frac{1}{2F(1)}\int_{\kappa_1}^{\kappa_2}\left[\frac{\varphi\left(\frac{\varkappa-\kappa_1}{\kappa_1\kappa_2}\right)}{\varkappa-\kappa_1}+\frac{\varphi\left(\frac{\kappa_2-\varkappa}{\kappa_1\kappa_2}\right)}{\kappa_2-\varkappa}\right]\varpi\left(\frac{\kappa_1\kappa_2}{\varkappa}\right)du$$

$$=\frac{1}{2F(1)}\int_{\kappa_1}^{\kappa_2}\left[\frac{\varphi\left(\frac{\varkappa-\kappa_1}{\kappa_1\kappa_2}\right)}{\varkappa-\kappa_1}+\frac{\varphi\left(\frac{\kappa_2-\varkappa}{\kappa_1\kappa_2}\right)}{\kappa_2-\varkappa}\right]\varpi\left(\frac{\kappa_1\kappa_2}{\kappa_1+\kappa_2-\varkappa}\right)d\varkappa,$$

i.e.

$$\frac{1}{2F(1)}\left[{}_{\frac{1}{\kappa_1}-}I_\alpha(\varpi\circ\psi)\left(\frac{1}{\kappa_2}\right)+\ {}_{\frac{1}{\kappa_2}+}I_\alpha(\varpi\circ\psi)\left(\frac{1}{\kappa_1}\right)\right]$$
$$=\frac{1}{4F(1)}\int_{\kappa_1}^{\kappa_2}\left[\frac{\varphi\left(\frac{\varkappa-\kappa_1}{\kappa_1\kappa_2}\right)}{\varkappa-\kappa_1}+\frac{\varphi\left(\frac{\kappa_2-\varkappa}{\kappa_1\kappa_2}\right)}{\kappa_2-\varkappa}\right]$$
$$\times\left[\varpi\left(\frac{\kappa_1\kappa_2}{\kappa_1+\kappa_2-\varkappa}\right)+\varpi\left(\frac{\kappa_1\kappa_2}{\varkappa}\right)\right]d\varkappa. \tag{2.2}$$

Then, it can be easily seen that we have the equalty

$$\frac{1}{2F(1)}\left[{}_{\frac{1}{\kappa_1}-}I_\alpha(\varpi\circ\psi)\left(\frac{1}{\kappa_2}\right)+{}_{\frac{1}{\kappa_2}+}I_\alpha(\varpi\circ\psi)\left(\frac{1}{\kappa_1}\right)\right]-\varpi\left(\frac{2\kappa_1\kappa_2}{\kappa_1+\kappa_2}\right)$$
$$=\frac{1}{4F(1)}\int_{\kappa_1}^{\kappa_2}\left[\frac{\varphi\left(\frac{\varkappa-\kappa_1}{\kappa_1\kappa_2}\right)}{\varkappa-\kappa_1}+\frac{\varphi\left(\frac{\kappa_2-\varkappa}{\kappa_1\kappa_2}\right)}{\kappa_2-\varkappa}\right]$$
$$\times\left[\varpi\left(\frac{\kappa_1\kappa_2}{\kappa_1+\kappa_2-\varkappa}\right)+\varpi\left(\frac{\kappa_1\kappa_2}{\varkappa}\right)-2f\left(\frac{2\kappa_1\kappa_2}{\kappa_1+\kappa_2}\right)\right]d\varkappa.$$

Since

$$\left[\frac{\varphi\left(\frac{\varkappa-\kappa_1}{\kappa_1\kappa_2}\right)}{\varkappa-\kappa_1}+\frac{\varphi\left(\frac{\kappa_2-\varkappa}{\kappa_1\kappa_2}\right)}{\kappa_2-\varkappa}\right]\left[\varpi\left(\frac{\kappa_1\kappa_2}{\kappa_1+\kappa_2-\varkappa}\right)+\varpi\left(\frac{\kappa_1\kappa_2}{\varkappa}\right)-2f\left(\frac{2\kappa_1\kappa_2}{\kappa_1+\kappa_2}\right)\right]$$

is symmetric about $\frac{\kappa_1+\kappa_2}{2}$, by using the the changing variables one has

$$\begin{aligned}
&\frac{1}{4F(1)}\int_{\kappa_1}^{\kappa_2}\left[\frac{\varphi\left(\frac{\varkappa-\kappa_1}{\kappa_1\kappa_2}\right)}{\varkappa-\kappa_1}+\frac{\varphi\left(\frac{\kappa_2-\varkappa}{\kappa_1\kappa_2}\right)}{\kappa_2-\varkappa}\right] \\
&\qquad\times\left[\varpi\left(\frac{\kappa_1\kappa_2}{\kappa_1+\kappa_2-\varkappa}\right)+\varpi\left(\frac{\kappa_1\kappa_2}{\varkappa}\right)-2f\left(\frac{2\kappa_1\kappa_2}{\kappa_1+\kappa_2}\right)\right]d\varkappa \\
&\quad=\frac{1}{2F(1)}\int_{\kappa_1}^{\frac{\kappa_1+\kappa_2}{2}}\left[\frac{\varphi\left(\frac{\varkappa-\kappa_1}{\kappa_1\kappa_2}\right)}{\varkappa-\kappa_1}+\frac{\varphi\left(\frac{\kappa_2-\varkappa}{\kappa_1\kappa_2}\right)}{\kappa_2-\varkappa}\right] \\
&\qquad\times\left[\varpi\left(\frac{\kappa_1\kappa_2}{\kappa_1+\kappa_2-\varkappa}\right)+\varpi\left(\frac{\kappa_1\kappa_2}{\varkappa}\right)-2f\left(\frac{2\kappa_1\kappa_2}{\kappa_1+\kappa_2}\right)\right]d\varkappa,
\end{aligned}$$

which gives

$$\begin{aligned}
(2.3)\quad &\frac{1}{2F(1)}\left[{}_{\frac{1}{\kappa_1}-}I_\alpha(\varpi\circ\psi)\left(\frac{1}{\kappa_2}\right)+{}_{\frac{1}{\kappa_2}+}I_\alpha(\varpi\circ\psi)\left(\frac{1}{\kappa_1}\right)\right] \\
&\quad-\varpi\left(\frac{2\kappa_1\kappa_2}{\kappa_1+\kappa_2}\right) \\
&\quad=\frac{1}{2F(1)}\int_{\kappa_1}^{\frac{\kappa_1+\kappa_2}{2}}\left[\frac{\varphi\left(\frac{\varkappa-\kappa_1}{\kappa_1\kappa_2}\right)}{\varkappa-\kappa_1}+\frac{\varphi\left(\frac{\kappa_2-\varkappa}{\kappa_1\kappa_2}\right)}{\kappa_2-\varkappa}\right] \\
&\qquad\times\left[\varpi\left(\frac{\kappa_1\kappa_2}{\kappa_1+\kappa_2-\varkappa}\right)+\varpi\left(\frac{\kappa_1\kappa_2}{\varkappa}\right)-2f\left(\frac{2\kappa_1\kappa_2}{\kappa_1+\kappa_2}\right)\right]d\varkappa.
\end{aligned}$$

On the other hand, we have the equalities

$$\begin{aligned}
\varpi\left(\frac{\kappa_1\kappa_2}{\kappa_1+\kappa_2-\varkappa}\right)-\varpi\left(\frac{2\kappa_1\kappa_2}{\kappa_1+\kappa_2}\right)&=\phi(\kappa_1+\kappa_2-\varkappa)-\phi\left(\frac{\kappa_1+\kappa_2}{2}\right) \\
&=\int_{\frac{\kappa_1+\kappa_2}{2}}^{\kappa_1+\kappa_2-\varkappa}\phi'(\xi)d\xi,
\end{aligned}$$

and

$$\begin{aligned}
\varpi\left(\frac{2\kappa_1\kappa_2}{\kappa_1+\kappa_2}\right)-\varpi\left(\frac{\kappa_1\kappa_2}{\varkappa}\right)&=\phi\left(\frac{\kappa_1+\kappa_2}{2}\right)-\phi(\varkappa) \\
&=\int_{\varkappa}^{\frac{\kappa_1+\kappa_2}{2}}\phi'(\xi)d\xi.
\end{aligned}$$

By these equalities, we get

$$(2.4)\qquad \varpi\left(\frac{\kappa_1\kappa_2}{\kappa_1+\kappa_2-\varkappa}\right)+\varpi\left(\frac{\kappa_1\kappa_2}{\varkappa}\right)-2f\left(\frac{2\kappa_1\kappa_2}{\kappa_1+\kappa_2}\right)$$

$$= \int_{\frac{\kappa_1+\kappa_2}{2}}^{\kappa_1+\kappa_2-\varkappa} \phi'(\xi)d\xi - \int_{\varkappa}^{\frac{\kappa_1+\kappa_2}{2}} \phi'(\xi)d\xi$$
$$= \int_{\varkappa}^{\frac{\kappa_1+\kappa_2}{2}} \phi'(\kappa_1+\kappa_2-\xi)d\xi - \int_{\varkappa}^{\frac{\kappa_1+\kappa_2}{2}} \phi'(\xi)d\xi$$
$$= \int_{\varkappa}^{\frac{\kappa_1+\kappa_2}{2}} \left(\phi'(\kappa_1+\kappa_2-\xi) - \phi'(\xi)\right) d\xi.$$

We also have

$$\phi'(\kappa_1+\kappa_2-\xi) - \phi'(\xi) = \int_{\xi}^{\kappa_1+\kappa_2-\xi} \phi''(z)dz. \tag{2.5}$$

Since ϕ'' is bounded with the constant m and M, we obtain the following inequality

$$m(\kappa_1+\kappa_2-2\xi) \le \int_{\xi}^{\kappa_1+\kappa_2-\xi} \phi''(z)dz \le M(\kappa_1+\kappa_2-2\xi). \tag{2.6}$$

By the equalities (2.4) and (2.5), we have the inequality

$$m\int_{\varkappa}^{\frac{\kappa_1+\kappa_2}{2}} (\kappa_1+\kappa_2-2\xi)d\xi \le \int_{\varkappa}^{\frac{\kappa_1+\kappa_2}{2}} \left(\phi'(\kappa_1+\kappa_2-\xi) - \phi'(\xi)\right) d\xi$$
$$\le M\int_{\varkappa}^{\frac{\kappa_1+\kappa_2}{2}} (\kappa_1+\kappa_2-2\xi)d\xi.$$

That is

(2.7)
$$m\left(\frac{\kappa_1+\kappa_2}{2}-\varkappa\right)^2 \le \varpi\left(\frac{\kappa_1\kappa_2}{\kappa_1+\kappa_2-\varkappa}\right) + \varpi\left(\frac{\kappa_1\kappa_2}{\varkappa}\right) - 2f\left(\frac{2\kappa_1\kappa_2}{\kappa_1+\kappa_2}\right)$$
$$\le M\left(\frac{\kappa_1+\kappa_2}{2}-\varkappa\right)^2.$$

Multiplying the inequality (2.7) by $\frac{1}{2F(1)}\left[\frac{\varphi\left(\frac{\varkappa-\kappa_1}{\kappa_1\kappa_2}\right)}{\varkappa-\kappa_1} + \frac{\varphi\left(\frac{\kappa_2-\varkappa}{\kappa_1\kappa_2}\right)}{\kappa_2-\varkappa}\right]$ and integrating the obtained inequality with respect to $\varkappa$ on $\left[\kappa_1, \frac{\kappa_1+\kappa_2}{2}\right]$, we establish

$$\frac{m}{2F(1)}\int_{\kappa_1}^{\frac{\kappa_1+\kappa_2}{2}} \left(\frac{\kappa_1+\kappa_2}{2}-\varkappa\right)\left[\frac{\varphi\left(\frac{\varkappa-\kappa_1}{\kappa_1\kappa_2}\right)}{\varkappa-\kappa_1} + \frac{\varphi\left(\frac{\kappa_2-\varkappa}{\kappa_1\kappa_2}\right)}{\kappa_2-\varkappa}\right]d\varkappa$$
$$\le \frac{1}{2F(1)}\left[{}_{\frac{1}{\kappa_1}-}I_\alpha(\varpi\circ\psi)\left(\frac{1}{\kappa_2}\right) + {}_{\frac{1}{\kappa_2}+}I_\alpha(\varpi\circ\psi)\left(\frac{1}{\kappa_1}\right)\right] - \varpi\left(\frac{2\kappa_1\kappa_2}{\kappa_1+\kappa_2}\right)$$

$$\leq \frac{M}{2F(1)} \int_{\kappa_1}^{\frac{\kappa_1+\kappa_2}{2}} \left(\frac{\kappa_1+\kappa_2}{2} - \varkappa\right) \left[\frac{\varphi\left(\frac{\varkappa-\kappa_1}{\kappa_1\kappa_2}\right)}{\varkappa-\kappa_1} + \frac{\varphi\left(\frac{\kappa_2-\varkappa}{\kappa_1\kappa_2}\right)}{\kappa_2-\varkappa}\right] d\varkappa,$$

which completed the proof. □

Remark 2.2. If we choose $\varphi(\xi) = \frac{1}{\Gamma(\alpha)}\xi^\alpha$ in Theorem 2.1, then the inequalities (2.1) reduce to the inequalities (1.5).

Corollary 2.3. *If we choose $\varphi(\xi) = \frac{1}{k\Gamma_k(\alpha)}\xi^{\frac{\alpha}{k}},\ k > 0$ in Theorem 2.1, then we have the inequalities for k-fractional integrals*

$$\begin{aligned}
&\frac{m\alpha}{2k(\kappa_2-\kappa_1)^{\frac{\alpha}{k}}} \int_{\kappa_1}^{\frac{\kappa_1+\kappa_2}{2}} \left(\frac{\kappa_1+\kappa_2}{2} - \varkappa\right)^2 \left[(\varkappa-\kappa_1)^{\frac{\alpha}{k}-1} + (\kappa_2-\varkappa)^{\frac{\alpha}{k}-1}\right] d\varkappa \\
&\quad\leq \frac{\Gamma_k(\alpha+k)}{2}\left(\frac{\kappa_1\kappa_2}{\kappa_2-\kappa_1}\right)^{\frac{\alpha}{k}} \\
&\quad\times \left\{J^{\alpha}_{\frac{1}{\kappa_1}-,k}(\varpi\circ\psi)\left(\frac{1}{\kappa_2}\right) + J^{\alpha}_{\frac{1}{\kappa_2}+,k}(\varpi\circ\psi)\left(\frac{1}{\kappa_1}\right)\right\} - \varpi\left(\frac{2\kappa_1\kappa_2}{\kappa_1+\kappa_2}\right) \\
&\quad\leq \frac{M\alpha}{2k(\kappa_2-\kappa_1)^{\frac{\alpha}{k}}} \int_{\kappa_1}^{\frac{\kappa_1+\kappa_2}{2}} \left(\frac{\kappa_1+\kappa_2}{2} - \varkappa\right)^2 \left[(\varkappa-\kappa_1)^{\frac{\alpha}{k}-1} + (\kappa_2-\varkappa)^{\frac{\alpha}{k}-1}\right] d\varkappa.
\end{aligned}$$

Theorem 2.4. *Let $\varpi : [\kappa_1, \kappa_2] \subseteq (0, +\infty) \to \mathbb{R}$ be a positive, twice differentiable function with $\kappa_1 < \kappa_2$ and $\varpi \in L([\kappa_1, \kappa_2])$. If ϕ'' is bounded in $[\kappa_1, \kappa_2]$, where $\phi(\varkappa) = \varpi\left(\frac{\kappa_1\kappa_2}{\varkappa}\right)$, then*

(2.8)
$$\begin{aligned}
&\frac{m}{2F(1)} \int_{\kappa_1}^{\kappa_2} \left[\varphi\left(\frac{\varkappa-\kappa_1}{\kappa_1\kappa_2}\right)(\kappa_2-\varkappa) + (\varkappa-\kappa_1)\varphi\left(\frac{\kappa_2-\varkappa}{\kappa_1\kappa_2}\right)\right] d\varkappa \\
&\quad\leq \frac{\varpi(\kappa_1)+\varpi(\kappa_2)}{2} \\
&\quad\quad- \frac{1}{2F(1)}\left[{}_{\frac{1}{\kappa_1}-}I_\alpha(\varpi\circ\psi)\left(\frac{1}{\kappa_2}\right) + {}_{\frac{1}{\kappa_2}+}I_\alpha(\varpi\circ\psi)\left(\frac{1}{\kappa_1}\right)\right] \\
&\quad\leq \frac{M}{2F(1)} \int_{\kappa_1}^{\kappa_2} \left[\varphi\left(\frac{\varkappa-\kappa_1}{\kappa_1\kappa_2}\right)(\kappa_2-\varkappa) + (\varkappa-\kappa_1)\varphi\left(\frac{\kappa_2-\varkappa}{\kappa_1\kappa_2}\right)\right] d\varkappa,
\end{aligned}$$

where $m = \inf_{\xi\in[\kappa_1,\kappa_2]} \phi''(\xi),\ M = \sup_{\xi\in[\kappa_1,\kappa_2]} \phi''(\xi)$, ψ and F are defined as in Theorem 1.10.

Proof. By the equality (2.2), we can easily the equality

(2.9)
$$\begin{aligned}
&\frac{\varpi(\kappa_1)+\varpi(\kappa_2)}{2} \\
&\quad- \frac{1}{2F(1)}\left[{}_{\frac{1}{\kappa_1}-}I_\alpha(\varpi\circ\psi)\left(\frac{1}{\kappa_2}\right) + {}_{\frac{1}{\kappa_2}+}I_\alpha(\varpi\circ\psi)\left(\frac{1}{\kappa_1}\right)\right]
\end{aligned}$$

$$= \frac{1}{4F(1)} \int_{\kappa_1}^{\kappa_2} \left[\frac{\varphi\left(\frac{\varkappa-\kappa_1}{\kappa_1\kappa_2}\right)}{\varkappa-\kappa_1} + \frac{\varphi\left(\frac{\kappa_2-\varkappa}{\kappa_1\kappa_2}\right)}{\kappa_2-\varkappa} \right]$$
$$\times \left(\varpi(\kappa_1) + \varpi(\kappa_2) - \varpi\left(\frac{\kappa_1\kappa_2}{\kappa_1+\kappa_2-\varkappa}\right) - \varpi\left(\frac{\kappa_1\kappa_2}{\varkappa}\right) \right) d\varkappa.$$

Since

$$\left[\frac{\varphi\left(\frac{\varkappa-\kappa_1}{\kappa_1\kappa_2}\right)}{\varkappa-\kappa_1} + \frac{\varphi\left(\frac{\kappa_2-\varkappa}{\kappa_1\kappa_2}\right)}{\kappa_2-\varkappa} \right] \left(\varpi(\kappa_1) + \varpi(\kappa_2) - \varpi\left(\frac{\kappa_1\kappa_2}{\kappa_1+\kappa_2-\varkappa}\right) - \varpi\left(\frac{\kappa_1\kappa_2}{\varkappa}\right) \right)$$

is symmetric about $\frac{\kappa_1+\kappa_2}{2}$, we have

$$\frac{\varpi(\kappa_1)+\varpi(\kappa_2)}{2} - \frac{1}{2F(1)} \left[{}_{\frac{1}{\kappa_1}-}I_\alpha(\varpi\circ\psi)\left(\frac{1}{\kappa_2}\right) + {}_{\frac{1}{\kappa_2}+}I_\alpha(\varpi\circ\psi)\left(\frac{1}{\kappa_1}\right) \right]$$
$$= \frac{1}{2F(1)} \int_{\kappa_1}^{\frac{\kappa_1+\kappa_2}{2}} \left[\frac{\varphi\left(\frac{\varkappa-\kappa_1}{\kappa_1\kappa_2}\right)}{\varkappa-\kappa_1} + \frac{\varphi\left(\frac{\kappa_2-\varkappa}{\kappa_1\kappa_2}\right)}{\kappa_2-\varkappa} \right]$$
$$\times \left(\varpi(\kappa_1) + \varpi(\kappa_2) - \varpi\left(\frac{\kappa_1\kappa_2}{\kappa_1+\kappa_2-\varkappa}\right) - \varpi\left(\frac{\kappa_1\kappa_2}{\varkappa}\right) \right) d\varkappa.$$

By the equalities

$$\varpi(\kappa_1) - \varpi\left(\frac{\kappa_1\kappa_2}{\kappa_1+\kappa_2-\varkappa}\right) = \phi(\kappa_2) - \phi(\kappa_1+\kappa_2-\varkappa)$$
$$= \int_{\kappa_1+\kappa_2-\varkappa}^{\kappa_2} \phi'(\xi)d\xi,$$

and

$$\varpi(\kappa_2) - \varpi\left(\frac{\kappa_1\kappa_2}{\varkappa}\right) = \phi(\kappa_1) - \phi(\varkappa)$$
$$= -\int_{\kappa_1}^{\varkappa} \phi'(\xi)d\xi,$$

we have

$$\varpi(\kappa_1) + \varpi(\kappa_2) - \varpi\left(\frac{\kappa_1\kappa_2}{\kappa_1+\kappa_2-\varkappa}\right) - \varpi\left(\frac{\kappa_1\kappa_2}{\varkappa}\right) \tag{2.10}$$
$$= \int_{\kappa_1+\kappa_2-\varkappa}^{\kappa_2} \phi'(\xi)d\xi - \int_{\kappa_1}^{\varkappa} \phi'(\xi)d\xi.$$
$$= \int_{\varkappa}^{\kappa_1} \phi'(\kappa_1+\kappa_2-\xi)d\xi - \int_{\kappa_1}^{\varkappa} \phi'(\xi)d\xi$$
$$= \int_{\kappa_1}^{\varkappa} \left[\phi'(\kappa_1+\kappa_2-\xi) - \phi'(\xi)\right] d\xi.$$

On the other hand, integrating the inequality (2.6) with respect to ξ on $[\kappa_1, \varkappa]$, we have

$$m\int_{\kappa_1}^{u}(\kappa_1+\kappa_2-2\xi)d\xi \leq \int_{\kappa_1}^{u}\left(\phi'(\kappa_1+\kappa_2-\xi)-\phi'(\xi)\right)d\xi \leq M\int_{\kappa_1}^{u}(\kappa_1+\kappa_2-2\xi)d\xi.$$

From the equality (2.10), we get

(2.11)

$$\begin{aligned} m(\varkappa-\kappa_1)(\kappa_2-\varkappa) &\leq \varpi(\kappa_1)+\varpi(\kappa_2)-\varpi\left(\frac{\kappa_1\kappa_2}{\kappa_1+\kappa_2-\varkappa}\right)-\varpi\left(\frac{\kappa_1\kappa_2}{\varkappa}\right) \\ &\leq M(\varkappa-\kappa_1)(\kappa_2-\varkappa). \end{aligned}$$

Multiplying the inequality (2.11) by $\frac{1}{2F(1)}\left[\frac{\varphi\left(\frac{\varkappa-\kappa_1}{\kappa_1\kappa_2}\right)}{\varkappa-\kappa_1}+\frac{\varphi\left(\frac{\kappa_2-\varkappa}{\kappa_1\kappa_2}\right)}{\kappa_2-\varkappa}\right]$ and integrating the resultant inequality with respect to $\varkappa$ on $\left[\kappa_1, \frac{\kappa_1+\kappa_2}{2}\right]$, we establish

$$\begin{aligned} &\frac{m}{2F(1)}\int_{\kappa_1}^{\frac{\kappa_1+\kappa_2}{2}}\left[\varphi\left(\frac{\varkappa-\kappa_1}{\kappa_1\kappa_2}\right)(\kappa_2-\varkappa)+(\varkappa-\kappa_1)\varphi\left(\frac{\kappa_2-\varkappa}{\kappa_1\kappa_2}\right)\right]d\varkappa \\ &\leq \frac{\varpi(\kappa_1)+\varpi(\kappa_2)}{2}-\frac{1}{2F(1)}\left[{}_{\frac{1}{\kappa_1}-}I_{\alpha}(\varpi\circ\psi)\left(\frac{1}{\kappa_2}\right)+{}_{\frac{1}{\kappa_2}+}I_{\alpha}(\varpi\circ\psi)\left(\frac{1}{\kappa_1}\right)\right] \\ &\leq \frac{M}{2F(1)}\int_{\kappa_1}^{\frac{\kappa_1+\kappa_2}{2}}\left[\varphi\left(\frac{\varkappa-\kappa_1}{\kappa_1\kappa_2}\right)(\kappa_2-\varkappa)+(\varkappa-\kappa_1)\varphi\left(\frac{\kappa_2-\varkappa}{\kappa_1\kappa_2}\right)\right]d\varkappa, \end{aligned}$$

which completed the proof. □

Remark 2.5. If we choose $\varphi(\xi)=\frac{1}{\Gamma(\alpha)}\xi^{\alpha}$ in Theorem 2.4, then the inequalities (2.8) reduce to the inequalities (1.6).

Corollary 2.6. *If we choose $\varphi(\xi)=\frac{1}{k\Gamma_k(\alpha)}\xi^{\frac{\alpha}{k}},\ k>0$ in Theorem 2.4, then we have the inequalities for k-fractional integrals*

$$\begin{aligned} &\frac{m\alpha}{2k(\kappa_2-\kappa_1)^{\frac{\alpha}{k}}}\int_{\kappa_1}^{\frac{\kappa_1+\kappa_2}{2}}(\varkappa-\kappa_1)(\kappa_2-\varkappa)\left[(\varkappa-\kappa_1)^{\frac{\alpha}{k}-1}+(\kappa_2-\varkappa)^{\frac{\alpha}{k}-1}\right]d\varkappa \\ &\leq \frac{\varpi(\kappa_1)+\varpi(\kappa_2)}{2} \\ &\quad -\frac{\Gamma(\alpha+1)}{2}\left(\frac{\kappa_1\kappa_2}{\kappa_2-\kappa_1}\right)^{\frac{\alpha}{k}}\left\{J^{\alpha}_{\frac{1}{\kappa_1}-,k}(\varpi\circ\psi)\left(\frac{1}{\kappa_2}\right)+J^{\alpha}_{\frac{1}{\kappa_2}+,k}(\varpi\circ\psi)\left(\frac{1}{\kappa_1}\right)\right\} \\ &\leq \frac{M\alpha}{2k(\kappa_2-\kappa_1)^{\frac{\alpha}{k}}}\int_{\kappa_1}^{\frac{\kappa_1+\kappa_2}{2}}(\varkappa-\kappa_1)(\kappa_2-\varkappa)\left[(\varkappa-\kappa_1)^{\frac{\alpha}{k}-1}+(\kappa_2-\varkappa)^{\frac{\alpha}{k}-1}\right]d\varkappa. \end{aligned}$$

Now, we prove the inequality (1.10) under the condition $\phi'(\kappa_1+\kappa_2-\varkappa)\geq\phi'(\varkappa)$, $\phi(\varkappa)=\varpi\left(\frac{\kappa_1\kappa_2}{\varkappa}\right)$ instead of harmonically convexity of ϖ.

Theorem 2.7. *Let $\varpi : [\kappa_1, \kappa_2] \subseteq (0, +\infty) \to \mathbb{R}$ be a positive, differentiable function with $\kappa_1 < \kappa_2$ and $\varpi \in L([\kappa_1, \kappa_2])$. If $\phi'(\kappa_1 + \kappa_2 - \varkappa) \geq \phi'(\varkappa)$, $\forall \varkappa \in \left[\kappa_1, \frac{\kappa_1+\kappa_2}{2}\right]$, where $\phi(\varkappa) = \varpi\left(\frac{\kappa_1\kappa_2}{\varkappa}\right)$ then for $\alpha > 0$ we have*

$$\varpi\left(\frac{2\kappa_1\kappa_2}{\kappa_1+\kappa_2}\right) \leq \frac{1}{2F(1)}\left[{}_{\frac{1}{\kappa_1}-}I_\alpha(\varpi\circ\psi)\left(\frac{1}{\kappa_2}\right) + {}_{\frac{1}{\kappa_2}+}I_\alpha(\varpi\circ\psi)\left(\frac{1}{\kappa_1}\right)\right] \leq \frac{\varpi(\kappa_1)+\varpi(\kappa_2)}{2}, \tag{2.12}$$

where ψ and F are defined as in Theorem 1.10.

Proof. From (2.3) and (2.4), one has

$$\begin{aligned}
&\frac{1}{2F(1)}\left[{}_{\frac{1}{\kappa_1}-}I_\alpha(\varpi\circ\psi)\left(\frac{1}{\kappa_2}\right) + {}_{\frac{1}{\kappa_2}+}I_\alpha(\varpi\circ\psi)\left(\frac{1}{\kappa_1}\right)\right] - \varpi\left(\frac{2\kappa_1\kappa_2}{\kappa_1+\kappa_2}\right)\\
&= \frac{1}{4F(1)}\int_{\kappa_1}^{\kappa_2}\left[\frac{\varphi\left(\frac{\varkappa-\kappa_1}{\kappa_1\kappa_2}\right)}{\varkappa-\kappa_1} + \frac{\varphi\left(\frac{\kappa_2-\varkappa}{\kappa_1\kappa_2}\right)}{\kappa_2-\varkappa}\right]\\
&\quad\times\left[\varpi\left(\frac{\kappa_1\kappa_2}{\kappa_1+\kappa_2-\varkappa}\right) + \varpi\left(\frac{\kappa_1\kappa_2}{\varkappa}\right) - 2f\left(\frac{2\kappa_1\kappa_2}{\kappa_1+\kappa_2}\right)\right]d\varkappa\\
&= \frac{1}{2F(1)}\int_{\kappa_1}^{\frac{\kappa_1+\kappa_2}{2}}\left[\frac{\varphi\left(\frac{u-\kappa_1}{\kappa_1\kappa_2}\right)}{\varkappa-\kappa_1} + \frac{\varphi\left(\frac{\kappa_2-u}{\kappa_1\kappa_2}\right)}{\kappa_2-\varkappa}\right]\\
&\quad\times\left(\int_{\varkappa}^{\frac{\kappa_1+\kappa_2}{2}}\left[\phi'(\kappa_1+\kappa_2-\xi) - \phi'(\xi)\right]d\xi\right)d\varkappa\\
&\geq 0,
\end{aligned}$$

which gives first inequality in (2.12). On the other hand, by the equalities (2.9) and (2.10), we have

$$\begin{aligned}
&\frac{\varpi(\kappa_1)+\varpi(\kappa_2)}{2} - \frac{1}{2F(1)}\left[{}_{\frac{1}{\kappa_1}-}I_\alpha(\varpi\circ\psi)\left(\frac{1}{\kappa_2}\right) + {}_{\frac{1}{\kappa_2}+}I_\alpha(\varpi\circ\psi)\left(\frac{1}{\kappa_1}\right)\right]\\
&= \frac{1}{2F(1)}\int_{\kappa_1}^{\frac{\kappa_1+\kappa_2}{2}}\left[\frac{\varphi\left(\frac{\varkappa-\kappa_1}{\kappa_1\kappa_2}\right)}{\varkappa-\kappa_1} + \frac{\varphi\left(\frac{\kappa_2-\varkappa}{\kappa_1\kappa_2}\right)}{\kappa_2-\varkappa}\right]\\
&\quad\times\left(\varpi(\kappa_1)+\varpi(\kappa_2) - \varpi\left(\frac{\kappa_1\kappa_2}{\kappa_1+\kappa_2-\varkappa}\right) - \varpi\left(\frac{\kappa_1\kappa_2}{\varkappa}\right)\right)du\\
&= \frac{1}{2F(1)}\int_{\kappa_1}^{\frac{\kappa_1+\kappa_2}{2}}\left[\frac{\varphi\left(\frac{\varkappa-\kappa_1}{\kappa_1\kappa_2}\right)}{\varkappa-\kappa_1} + \frac{\varphi\left(\frac{\kappa_2-\varkappa}{\kappa_1\kappa_2}\right)}{\kappa_2-\varkappa}\right]
\end{aligned}$$

$$\times \left(\int_{\kappa_1}^{\varkappa} \left[\phi'(\kappa_1 + \kappa_2 - \xi) - \phi'(\xi) \right] d\xi \right) d\varkappa$$
$$\geq 0.$$

This gives second inequality in (2.12) and completes the proof. □

References

1. M.U. Awan, M.A. Noor, T.S. Du and K.I. Noor, *New refinements of fractional Hermite-Hadamard inequality*, Rev. R. Acad. Cienc. Exactas Fıs. Nat. Ser. A Mat., 113 (2019), pp. 21-29.
2. H. Budak, M.Z. Sarikaya and M.K. Yildiz, *Hermite-Hadamard type inequalities for F-convex function involving fractional integrals*, Filomat, 32 (2018), pp. 5509-5518.
3. H. Budak, *On refinements of Hermite-Hadamard type inequalities for Riemann-Liouville fractional integral operators*, Int. J. Optim. Control. Theor. Appl., 9 (2019), pp. 41-48.
4. H. Budak, *On Fejer type inequalities for convex mappings utilizing fractional integrals of a function with respect to another function*, Results Math., 74 (2019), Art. 29.
5. H. Budak, H. Kara, M.Z. Sarikaya and M.E. Kiriş, *New extensions of the Hermite-Hadamard inequalities involving Riemann-Liouville fractional integrals*, Miskolc Math. Notes, 21 (2020), pp. 665-678.
6. H. Budak, F. Ertugral and M.Z. Sarikaya, *New generalization of Hermite-Hadamard type inequalities via generalized fractional integrals*, An. Univ. Craiova Ser. Mat. Inform., 47 (2020), pp. 369-386.
7. H. Budak, E. Pehlivan and P. Kosem, *On new extensions of Hermite-Hadamard inequalities for generalized fractional integrals*, Sahand Commun. Math. Anal., 18 (2021), pp. 73-88.
8. F.X. Chen, *Extensions of the Hermite-Hadamard inequality for convex functions via fractional integrals* J. Math. Inequal, 10 (2016), pp. 75-81.
9. F.X. Chen, *On the generalization of some Hermite-Hadamard Inequalities for functions with convex absolute values of the second derivatives via fractional integrals*, Ukrainian Math. J., 12 (2019), pp. 1953-1965.
10. F.X. Chen, *Extensions of the Hermite-Hadamard inequality for harmonically convex functions via fractional integrals*, Appl. Math. Comput., 268 (2015), pp. 121-128.
11. S.S. Dragomir and C.E.M. Pearce, *Selected topics on Hermite-Hadamard inequalities and applications*, RGMIA Monographs, Victoria University, 2000. Online: `http://www.sta.vu.edu.au/RGMIA/monographs/hermite_hadamard.html`.

12. S.S. Dragomir, P. Cerone and A. Sofo, *Some remarks on the midpoint rule in numerical integration*, Stud. Univ. Babeş-Bolyai Math., XLV (2000), pp. 63-74.
13. S.S. Dragomir, P. Cerone and A. Sofo, *Some remarks on the trapezoid rule in numerical integration*, Indian J. Pure Appl. Math., 31 (2000), pp. 475-494.
14. A. Gozpinar, E. Set and S.S. Dragomir, *Some generalized Hermite-Hadamard type inequalities involving fractional integral operator for functions whose second derivatives in absolute value are s-convex,* Acta Math. Univ. Comenian., 88 (2019), pp. 87-100.
15. S.R. Hwang and K.L. Tseng, *New Hermite-Hadamard-type inequalities for fractional integrals and their applications*, Rev. R. Acad. Cienc. Exactas Fıs. Nat. Ser. A Mat., 112 (2018), pp. 1211-1223.
16. I. Iscan, *Hermite-Hadamard type inequalities for harmonically convex functions*, Hacet. J. Math. Stat. 43 (2014), pp. 935-942.
17. I. Iscan and S. Wu, *Hermite-Hadamard type inequalities for harmonically convex functions via fractional integrals*, Appl. Math. Comp., 238 (2014), pp. 237-244.
18. A.A. Kilbas, H.M. Srivastava and J.J. Trujillo, *Theory and Applications of Fractional Differential Equations*, North-Holland Mathematics Studies, 204, Elsevier Sci. B.V., Amsterdam, 2006.
19. K. Liu, J. Wang and D. O'Regan, *On the Hermite-Hadamard type inequality for ψ-Riemann-Liouville fractional integrals via convex functions*, J. Inequal. Appl., 2019 (2017), Art. 27.
20. A.R. Khan, H. Nasir, and S.S. Shirazi, *Weighted Cebysev type inequalities for double integrals and application*, Sahand Commun. Math. Anal., 18 (2021), pp. 59-72.
21. P.O. Mohammed and M.Z. Sarikaya, *Hermite-Hadamard type inequalities for F-convex function involving fractional integrals*, J. Inequal. Appl., 2018 (2018), Art. 359.
22. S. Mubeen and G.M. Habibullah, *k-Fractional integrals and application*, Int. J. Contemp. Math. Sciences, 7 (2012), pp.89-94.
23. N. Minculete and F-C. Mitroi, *Fejer-type inequalities*, Aust. J. Math. Anal. Appl., 9 (2012), Art. 12.
24. J.E. Pečarić, F. Proschan and Y.L. Tong, *Convex functions, partial orderings and statistical applications*, Academic Press, Boston, 1992.
25. S. Qaisar, M. Iqbal, S. Hussain, S. Butt, and M.A. Meraj, *New inequalities on Hermite-Hadamard utilizing fractional integrals,* Kragujevac J. Math., 42 (2018), pp. 15-27.

26. K. Qiu and J.R. Wang, *A fractional integral identity and its application to fractional Hermite-Hadamard type inequalities*, J. Interdiscip. Math., 21 (2018), pp. 1-16.
27. M.Z. Sarikaya and N. Aktan, *On the generalization some integral inequalities and their applications,* Math. Comput. Modelling, 54 (2011), pp. 2175-2182.
28. M.Z. Sarikaya and H. Yildirim, *On Hermite-Hadamard type inequalities for Riemann-Liouville fractional integrals*, Miskolc Mathematical Notes, 17 (2016), pp.1049-1059.
29. M.Z. Sarikaya and F. Ertuğral, *On the generalized Hermite-Hadamard inequalities*, An. Univ. Craiova Ser. Mat. Inform., 47 (2020), pp. 193-213..
30. M.Z. Sarikaya, *On Fejer type inequalities via fractional integrals*, J. Interdiscip. Math., 21 (2018), pp. 143-155.
31. M.Z. Sarikaya, E. Set, H. Yaldiz and N. Basak, *Hermite - Hadamard's inequalities for fractional integrals and related fractional inequalities*, Math. Comput. Modelling, 57 (2013), pp. 2403-2407.
32. E. Set, A. Akdemir and B. Çelik, *On generalization of Fejer type inequalities via fractional integral operator*, Filomat, 32 (2018), pp. 5537-5547.
33. T. Tunç, S. Sönmezoğlu and M.Z. Sarikaya, *On integral inequalities of Hermite-Hadamard type via Green function and applications*, Appl. Appl. Math., 14 (2019), pp. 452-462.
34. D. Zhao, M.A. Ali, A. Kashuri and H. Budak, *Generalized fractional integral inequalities of Hermite-Hadamard type for harmonically convex functions*, Adv. Difference Equ., 2020 (2020), pp. 1-14.

[1] Department of Mathematics, Faculty of Science and Arts, Düzce University, Düzce, Turkey.

Email address: hsyn.budak@gmail.com

[2] Department of Mathematics, Faculty of Science and Arts, Düzce University, Düzce, Turkey.

Email address: candancanbilisik@gmail.com

[3] Department of Mathematics, Faculty of Science and Arts, Düzce University, Düzce, Turkey.

Email address: sarikayamz@gmail.com