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Corrigenda: “$\omega b-$Topological Vector Spaces”

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Abstract. In this corrigenda, we have pointed out that Example 2.7, Corollary 3.7 and Corollary 5.3 in the paper: *$\omega b-$ Topological Vector Spaces*, WSEAS Trans. Math. 19 (2020), $119-132$, by Latif are incorrect. We have also presented the corrected version of these results. Furthermore, we introduce and study some new classes of topological vector spaces.

1. Introduction

This corrigenda mainly concerns three results in the paper [2] by R.M. Latif. One of his examples, namely Example 2.7 is erroneous and some of his results, namely Corollary 3.7 and Corollary 5.3 are incorrect. In this paper, it is shown that Example 2.7 is wrong and Corollary 3.7 and Corollary 5.3 are refuted by a counterexample. Moreover, the correct version of Corollary 3.7 and Corollary 5.3 are presented. Most of our notation will be the same as in [2] unless explicitly mentioned. We denote the set of real numbers by $\mathbb{R}$.

We now state Example 2.7 in [2].

Example 1.1. Let τ be the usual topology on $X=\mathbb{R}$. Let $A=\mathbb{Q}$ be the set of rational numbers and $B=[0,\ 1)$. Then A and B are $\omega b-$open sets of X, but $A\cap B$ is not $\omega b-$open set.

In this example, $A\cap B$ is a $\omega b-$open set, because for each $x\in A\cap B$, take $D=(x-\epsilon,\ x+\epsilon)\cap\mathbb{Q}$ where ϵ is a sufficiently small positive real number. Then D is a $b-$open set of $\mathbb{R}$ such that $D-(A\cap B)$ is countable.

In fact, we have

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Remark 1.2. Let X and τ be as in Example 1.1. Then the family $\omega b - O(X)$ of all $\omega b-$open sets of X forms a topology on X. For this, it is enough to show that if A is a subset of $\mathbb{R}$, then $A \in \omega b - O(X)$. Let x be any element of A. Take $D = [(x-\epsilon,\ x+\epsilon) \cap \mathbb{Q}] \cup \{x\}$. Then D is $b-$open set of $\mathbb{R}$ such that $D - A$ is countable. Since $x \in A$ was arbitrary, it follows that A is $\omega b-$open. Thus, $\omega b - O(X)$ is the discrete topology on $\mathbb{R}$.

We now state Corollary 3.7 and Corollary 5.3 in [2], respectively.

Theorem 1.3. *Let A be any non-empty open subset of an $\omega b-$topological vector space X. Then the following statements are true:*

1) $x + A \subseteq Int[Cl(Int(x+A))]$ *for each* $x \in X$.
2) $\lambda A \subseteq Int[Cl(Int(\lambda A))]$ *for each non-zero scalar* λ.

Theorem 1.4. *Let A be any non-empty $\delta-$open set of a almost $\omega b-$topological vector space X. Then the following statements are true:*

1) $x + A \subseteq Cl(Int(x+A))$ *for each* $x \in X$.
2) $\lambda A \subseteq Cl(Int(\lambda A))$ *for each non-zero scalar* λ.

Theorem 1.3 is false which can be seen by the following example:

Example 1.5. Let $X = \mathbb{R}$, the set of real numbers and let τ be the topology on X generated by the base $\mathcal{B} = \{(a,\ b) \colon a,\ b \in \mathbb{R}\} \cup \{[c,\ d) \colon c,\ d \in \mathbb{R}_+\}$ where $\mathbb{R}_+$ denotes the set of positive real numbers. Then $(X_{(\mathbb{R})},\ \tau)$ is a $\omega b-$topological vector space because in [3], it is shown as an $s-$topological vector space and by definition of a $\omega b-$topological vector space, it follows that every $s-$topological vector space is a $\omega b-$topological vector space.

Now, let $A = [1,\ 2)$ and take $x = -2$. Then A is an open subset of X but $x + A = [-1,\ 0)$ is not contained in $Int[Cl(Int(x+A))] = (-1,\ 0)$. Similarly, for $\lambda = -1$, we can check that λA is not contained in $Int[Cl(Int(\lambda A))]$.

Remark 1.6. Since every $\omega b-$topological vector space is an almost $\omega b-$topological vector space, by the same space as in Example 1.5, it can be readily shown that Theorem 1.4 is incorrect.

Definition 1.7. A subset A of a topological space $(Y,\ \tau)$ is called $b-$open [1] if $A \subseteq Int(Cl(A)) \cup Cl(Int(A))$.

The complement of a $b-$open set in a space X is called $b-$closed set. Given an space X, denote by $b - O(X)$, the class of $b-$open sets of X.

Definition 1.8. Given a subset A of an space X, the $b-$closure of A is the smallest $b-$closed set of X containing A and it is denoted by $b - Cl(A)$. The $b-$interior of A is the largest $b-$open set of X that is contained in A and it is denoted by $b - Int(A)$.

Definition 1.9. A function f from a topological space X to a topological space Y is called $b-$continuous if to every open set U of Y, $f^{-1}(U)$ is $b-$open set of X.

2. Corrected Versions

Definition 2.1. We call a pair $(L,\ \tau)$ b-topological vector space if

- L is a vector space over the field $\mathbb{F} \in \{\mathbb{R},\ \mathbb{C}\}$ where $\mathbb{F}$ is endowed with its ordinary topology.
- τ is a topology on L satisfying the following conditions:
  1) for each $x,\ y \in L$ and for each open neighborhood D of $x + y$, there exist $b-$open sets U and V in L containing x and y, respectively such that $U + V \subseteq D$, and
  2) for each $x \in L$, $\lambda \in \mathbb{F}$ and for each open neighborhood D of λx, there exist $b-$open sets U in $\mathbb{F}$ containing λ and V in L containing x such that $U.V \subseteq D$.

Definition 2.2. We call a pair $(L,\ \tau)$ almost b-topological vector space if:

- L is a vector space over the field $\mathbb{F} \in \{\mathbb{R},\ \mathbb{C}\}$ where $\mathbb{F}$ is endowed with its ordinary topology.
- τ is a topology on L satisfying the following conditions:
  1) for each $x,\ y \in L$ and for each regular open set D of L containing $x + y$, there exist $b-$open sets U and V in L containing x and y, respectively such that $U+V \subseteq D$, and
  2) for each $x \in L$, $\lambda \in \mathbb{F}$ and for each regular open set D in L containing λx, there exist $b-$open sets U in $\mathbb{F}$ containing λ and V in L containing x such that $U.V \subseteq D$.

It is obvious from the Definition 2.1 and Definition 2.2 that every b-topological vector space is a $\omega b-$topological vector space, and every almost b-topological vector space is an almost $\omega b-$topological vector space, but the converse is not true in general.

Theorem 2.3. *Let A be any non-empty open subset of a b-topological vector space $(L,\ \tau)$. Then the following statements are true:*

1) $x + A \subseteq Int(Cl(x + A)) \cup Cl(Int(x + A))$ *for each* $x \in L$.
2) $\lambda A \subseteq Int(Cl(\lambda A)) \cup Cl(Int(\lambda A))$ *for each non-zero scalar* λ.

Proof. 1) Let x be any element of L. Define $\ell_{-x} \colon L \to L$ by $\ell_{-x}(y) = -x+y$. Then it is easy to check that ℓ_{-x} is $b-$continuous. So, $\ell_{-x}^{-1}(A)$ is $b-$open set of L. That is, $\ell_x(A)$ is $b-$open set of L. Hence $\ell_x(A) \subseteq Int(Cl(\ell_x(A))) \cup Cl(Int(\ell_x(A)))$. This means that $x + A \subseteq Int(Cl(x + A)) \cup Cl(Int(x + A))$.

2) Similarly, is proved. □

Theorem 2.4. *Let A be any non-empty $\delta-$open set of an almost b-topological vector space $(L,\ \tau)$. Then the following statements are true:*

1) $x + A \subseteq Int(Cl(x + A)) \cup Cl(Int(x + A))$ *for each* $x \in L$.
2) $\lambda A \subseteq Int(Cl(\lambda A)) \cup Cl(Int(\lambda A))$ *for each non-zero scalar* λ.

Proof. Simple and thus, omitted. □

Example 2.5. There is a topology different from the usual topology of $\mathbb{R}$ that makes the real vector space $\mathbb{R}$ a b-topological vector space but not a topological vector space.

Proof. Consider the topology τ on the real vector space $\mathbb{R}$ consisting the sets $[0,\ \infty)$ and $\mathbb{R}$ with the empty set. We will prove that $(\mathbb{R},\ \tau)$ is a b-topological vector space.

Let x and y be two real numbers. Let $x = y = 0$. Take $b-$open sets $U = [0,\ 1]$ and $V = [0,\ 1]$ of $\mathbb{R}$. Then $U + V \subseteq [0,\ \infty)$. If $x > 0$ and $y = -x$, we can take $b-$open sets $U = [x,\ \infty)$ and $V = [y,\ 0]$ of $\mathbb{R}$ that satisfy $U + V \subseteq [0,\ \infty)$. Suppose that $x + y$ is a positive real number. We have two cases:

Case (I). If x and y are positive real numbers, then there is nothing to prove.

Case (II). If one is positive and the other is negative real number, say $x > 0$ and $y < 0$, then take $b-$open sets $U = [x,\ \infty)$ and $V = [y,\ 0]$ of $\mathbb{R}$. We will find that $U + V \subseteq [0,\ \infty)$.

Next, let α be any real number as a scalar and x be any element of $\mathbb{R}$. We have the following cases:

Case (I). If $\alpha x = 0$, then we have the following possibilities:

1) $\alpha = 0,\ x = 0$
2) $\alpha > 0,\ x = 0$
3) $\alpha < 0,\ x = 0$
4) $\alpha = 0,\ x > 0$
5) $\alpha = 0,\ x < 0$

(1), (2) and (4) can be easily proved. If $\alpha < 0$ and $x = 0$, choose $b-$open sets $U = [\alpha,\ 0]$ of $\mathbb{F} = \mathbb{R}$ and $V = [-1,\ 0]$ of $L = \mathbb{R}$. Then $U.V \subseteq [0,\ \infty)$. If $\alpha = 0$ and $x < 0$, choose $b-$open sets $U = [-1,\ 0]$ of $\mathbb{R}$ (with respect to the usual topology of $\mathbb{R}$) and $V = [x,\ 0]$ of $\mathbb{R}$ (with respect to the given topology τ of $\mathbb{R}$). We will find that $U.V \subseteq [0,\ \infty)$.

Case (II). If $\alpha x > 0$, then we have the following possibilities:

1) both α and x are positive real numbers.
2) both α and x are negative real numbers.

If α and x are positive real numbers, then we have nothing to prove. If $\alpha < 0$ and $x < 0$, choose $b-$open sets $U = [\alpha,\ 0]$ of $\mathbb{R}$

(with respect to the usual topology) and $V = [x,\ 0]$ of $\mathbb{R}$ (with respect to the given topology). Then $U.V \subseteq [0,\ \infty)$.
Case (III). If αx is a negative real number, then we have nothing to prove. □

Example 2.6. Take the topology $\tau = \{\emptyset,\ (0,\ \infty),\ \mathbb{R}\}$ of $\mathbb{R}$. Then $(\mathbb{R},\ \tau)$ is not a b-topological vector space.

Example 2.7. Take the topology $\tau = \{\emptyset,\ \mathbb{R} - \mathbb{Q},\ \mathbb{R}\}$ of $\mathbb{R}$, where $\mathbb{Q}$ denotes the set of rational numbers. Then $(\mathbb{R},\ \tau)$ is a b-topological vector space.

Given spaces $(X,\ \tau_X)$ and $(Y,\ \tau_Y)$, call a function $f\colon X \to Y$ b-homeomorphism if it is bijective, $b-$continuous and for every $U \in \tau_X$, $f(U) \in b - O(Y)$.

Example 2.8. Let $X = (\mathbb{R},\ \Im)$, where $\Im = \{\emptyset,\ (0,\ \infty),\ \mathbb{R}\}$. Let $f\colon X \to X$ be defined as

$$f(x) = \begin{cases} 1, & \text{if } x > 0, \\ 0, & \text{otherwise.} \end{cases}$$

Then f is b-homeomorphism.

Theorem 2.9. *Let $(L,\ \tau)$ be a b-topological vector space. Then each translation $\ell_x\colon L \to L$ defined by $\ell_x(y) = x + y,\ x,\ y \in L$, is a b-homeomorphism.*

Proof. Let A be an open set of L. Let $C = \{y \in L\colon x + y \in A\}$. To prove this theorem, it is enough to show that C is $b-$open set of L. Let y be any element of C. Then there exist $U,\ V \in b - O(L)$ which satisfy

$$x \in U,\ y \in V \quad \text{and} \quad U + V \subseteq A.$$

In particular, $x + V \subseteq A$. This gives that $V \subseteq -x + A = C$. This indicates that $y \in b - Int(C)$. Hence $C \in b - O(L)$. It ends the proof. □

Theorem 2.10. *Let $(L,\ \tau)$ be a b-topological vector space and let α be any non-zero scalar. Then each dilation $D_\alpha\colon L \to L$ defined by $D_\alpha(x) = \alpha x$, is a b-homeomorphism.*

Proof. Let A be an open set of L, let $C = \{x \in L\colon \alpha x \in A\}$ and y be any element of C. Then there exist $b-$open set U of $\mathbb{F}$ and V of L having properties

$$\alpha \in U,\ x \in V \quad \text{and} \quad U.V \subseteq A.$$

In particular, $\alpha V \subseteq A$. This gives that $V \subseteq C$. This indicates that $x \in b-Int(C)$. Hence $C \in b-O(L)$. Since $D_\alpha^{-1}(A) = \frac{1}{\alpha}A = C$, it follows that D_α is $b-$continuous. Moreover, it follows that $\alpha A \in b - O(L)$. Hence D_α is b-homeomorphism. □

Definition 2.11. An space X is called $b-$homogenous if for every pair of points x and y of X, there exists a b-homeomorphism $f\colon X \to X$ such that $f(x) = y$.

Theorem 2.12. *Every b-topological vector space is a $b-$homogenous space.*

Proof. Let $(L,\ \tau)$ be a b-topological vector space. Let x and y be two distinct points of L. By Theorem 2.9, ℓ_{-x+y} is b-homeomorphism. Also, $\ell_{-x+y}(x) = y$. Thus, L is $b-$homogenous. □

In the next theorem, we present characterizations of b-topological vector spaces which are a simple consequence of Theorem 2.9 and Theorem 2.10.

Theorem 2.13. *Let $(L,\ \tau)$ be a b-topological vector space, x an element of L and α a non-zero scalar. If A is any subset of L, then*

1) $b - Cl(x + A) \subseteq x + Cl(A)$,
2) $x + b - Cl(A) \subseteq Cl(x + A)$,
3) $x + Int(A) \subseteq b - Int(x + A)$,
4) $Int(x + A) \subseteq x + b - Int(A)$,
5) $b - Cl(\alpha.A) \subseteq \alpha.Cl(A)$,
6) $\alpha.b - Cl(A) \subseteq Cl(\alpha.A)$,
7) $\alpha.Int(A) \subseteq b - Int(\alpha.A)$, *and*
8) $Int(\alpha.A) \subseteq \alpha.b - Int(A)$.

Theorem 2.14. *Let $(L,\ \tau)$ be a b-topological vector space, x be an element of L and α be a non-zero scalar. If A is any open subset of L, then*

1) $x + A \in b - O(L)$, *and*
2) $\alpha A \in b - O(L)$.

Proof. Follows from Theorem 2.9 and Theorem 2.10. □

Example 2.15. There is a topology on the real vector space $\mathbb{R}$ that satisfies Theorem 2.14 but not a b-topological vector space.

Proof. Take the topology τ on $\mathbb{R}$ consisting $[-1,\ \infty)$ and $\mathbb{R}$ the only non-empty sets with the empty set. Then $x+A \in b-O(\mathbb{R})$ and $\alpha A \in b-O(\mathbb{R})$ for each $x \in \mathbb{R}$ and for each non-zero real number α, where $A \in \tau$ but $(\mathbb{R},\ \tau)$ is not a b-topological vector space. □

A subset A of a space X is called regularly open if $A = Int(Cl(A))$. Given an space X, denote by $RO(X)$, the class of regularly open sets of X.

Given spaces $(X,\ \tau_X)$ and $(Y,\ \tau_Y)$, call a function $f\colon X \to Y$ almost b-homeomorphism if it is bijective, for every $U \in RO(Y)$, $f^{-1}(U) \in b - O(X)$ and for every $V \in RO(X)$, $f(V) \in b - O(Y)$.

Example 2.16. There exists a function $f\colon X \to Y$ which is an almost b-homeomorphism but not a b-homeomorphism.

Proof. Let $X = (\mathbb{R},\ \Im)$, where $\Im = \{\emptyset,\ (0,\ \infty),\ \mathbb{R}\}$. Let $f\colon X \to X$ be defined as

$$f(x) = \begin{cases} 1, & \text{if } x > 0, \\ 0, & \text{otherwise.} \end{cases}$$

Then f is an almost b-homeomorphism but it is not a b-homeomorphism. □

Theorem 2.17. *Let $(L,\ \tau)$ be an almost b-topological vector space. Then each translation $\ell_x\colon L \to L$ defined by $\ell_x(y) = x + y,\ x,\ y \in L$, is an almost b-homeomorphism.*

Proof. Let A be a regularly open set of L. Let $C = \{y \in L\colon x + y \in A\}$. Let y be any element of C. Then we have $U,\ V \in b - O(L)$ with the properties

$$x \in U,\ y \in V \quad \text{and} \quad U + V \subseteq A.$$

In particular, $x + V \subseteq A$. This gives that $V \subseteq -x + A = C$. This indicates that $y \in b - Int(C)$. Hence $C \in b - O(L)$. This ends the proof. □

Theorem 2.18. *Let $(L,\ \tau)$ be an almost b-topological vector space and α be any non-zero scalar. Then each dilation $D_\alpha\colon L \to L$ defined by $D_\alpha(x) = \alpha x$, is an almost b-homeomorphism.*

Proof. Let A be any regularly open set of L. Let $C = \{x \in L\colon \alpha x \in A\}$. Let y be any element of C. Then there exist $b-$open set U of $\mathbb{F}$ and V of L having properties

$$\alpha \in U,\ x \in V \quad \text{and} \quad U.V \subseteq A.$$

In particular, $\alpha V \subseteq A$. This gives that $V \subseteq C$. This indicates that $x \in b - Int(C)$. Hence $C \in b - O(L)$. Consequently, D_α is an almost b-homeomorphism. □

Definition 2.19. An space X is called almost $b-$homogenous if for every pair of points x and y of X, there exists an almost b-homeomorphism $f\colon X \to X$ such that $f(x) = y$.

Theorem 2.20. *Every almost b-topological vector space is an almost $b-$homogenous space.*

Proof. Follows from Theorem 2.17. □

A function $f\colon X \to Y$ from a topological space X to a topological space Y is called $b-$continuous if for every open set U of Y, $f^{-1}(U) \in b - O(X)$.

Definition 2.21. Let L be a vector space over the field $\mathbb{F} \in \{\mathbb{R},\ \mathbb{C}\}$ and τ be a topology of L such that $(L,\ +,\ \tau)$ is a paratopological group. We say that $(L,\ \tau)$ is

1) quasi-b-topological vector space if the function $H_r \colon L \to L$ defined by $H_r(x) = rx$, with $r \in \mathbb{R}^+$ (the set of non-negative real numbers), is $b-$continuous.
2) super quasi-b-topological vector space if the function $H_r \colon L \to L$ defined by $H_r(x) = rx$, with $r \in \mathbb{R}$, is $b-$continuous.
3) para-b-topological vector space if for each neighborhood U of rx, with $x \in L$ and $r \in \mathbb{R}^+$, there exist a $b-$open set V of L and an $\epsilon > 0$ such that $[r,\ r+\epsilon[.V \subseteq U$.

Theorem 2.22. *Let $(L,\ \Im)$ be a nearly b-topological vector space. Then every $\sigma \in N_0(L)$, the family of neighborhoods of 0 in L, is absorbing.*

Proof. Let x be any element of L and σ be an open neighborhood of 0 in L. Then there exists an open neighborhood U of 0 in $\mathbb{F}$ such that $\{\lambda x \colon \lambda \in U\} \subseteq \sigma$. We can choose a sufficiently small positive number ϵ such that $\{\lambda x \colon |\lambda| \leq \epsilon\} \subseteq \sigma$ which shows that σ is absorbing. □

Example 2.23. There are topologies of $\mathbb{R}$ that make the real vector space $\mathbb{R}$ a b-topological vector space but not a nearly b-topological vector space.

Proof. By Theorem 2.14, we can readily check that Example 2.5 and Example 2.7 are not nearly b-topological vector spaces. □

Theorem 2.24. *Let $(L,\ \tau)$ be a nearly b-topological vector space and C be a convex subset of L. If for every $U \in N_0(L)$, $\alpha U \in \tau$ for each non-zero scalar α then every $U \in N_0(L)$ contains a balanced neighborhood of* 0.

Proof. Take any $\nabla \in N_0(L)$. Then there exist an open neighborhood U of 0 in $\mathbb{F}$, and $V \in N_0(L)$ such that

$$U.V \subseteq \nabla.$$

Choose a sufficiently small positive number ϵ such that $D = \{\lambda \in \mathbb{F} \colon |\lambda| \leq \epsilon\} \subseteq U$, and set

$$\sigma = \cup_{\lambda \in D} \lambda V.$$

By given condition, $\sigma \in N_0(L)$. Now, for any $\mu \in \mathbb{F}$ with $|\mu| \leq 1$, we have

$$\mu\sigma = \cup_{\lambda \in D} \mu(\lambda V) = \cup_{\nu \in D} \nu V,$$

where $\nu = \mu.\lambda$.

This ends the proof.

□

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