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Abstract. In this study, we purpose to extend approximation properties of the (p, q)-Bernstein-Chlodowsky operators from real function spaces to fuzzy function spaces. Firstly, we define fuzzy (p, q)-Bernstein-Chlodowsky operators, and we give some auxiliary results. Later, we give a fuzzy Korovkin-type approximation theorem for these operators. Additionally, we investigate rate of convergence by using first order fuzzy modulus of continuity and Lipschitz-type fuzzy functions. Eventually, we give an estimate for fuzzy asymptotic expansions of the fuzzy (p, q)-Bernstein-Chlodowsky operators.

1. Introduction

With the rapid development of q-calculus, lately, approximation properties of q-generalization pertaining to various operators have been investigated. For more details, we refer the readers to [1, 5, 22, 23, 25, 35–38, 41]. In the recent years, together with the development of (p, q)-calculus in mathematics a lots of concepts have been extended to (p, q)-calculus. (p, q)-Generalizations of various operators have been obtained and investigated their approximation or other properties. For instance, we refer the readers to [21, 27–34].

The classical Bernstein-Chlodowsky operators were introduced as a generalization of the Bernstein polynomials [8]. In [7, 14, 15, 39], some approximation properties of Bernstein-Chlodowsky-type operators were investigated. Subsequently, the q-Bernstein-Chlodowsky operators were defined by Karslı and Gupta[13]. In [4], the (p, q)-Bernstein-Chlodowsky

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operators were defined and obtained their convergence both with a Korovkin-type approximation theorem and with a weighted Korovkin-type approximation theorem, and also their rate of convergences was studied both using the usual modulus of continuity and the weighted modulus of continuity. Also, the Chlodowsky-types of (p,q)-Bernstein-Stancu-Schurer operators were studied in [26, 40].

(p,q)-Bernstein-Chlodowsky[4] operators are defined by

$$C_n^{p,q}(f;x) = \sum_{k=0}^{n} f\left(\frac{b_n [k]_{p,q}}{p^{k-n}[n]_{p,q}}\right) \frac{p^{k(k-1)/2}}{p^{n(n-1)/2}} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} \left(\frac{x}{b_n}\right)^{n-k} \prod_{s=0}^{n-k-1} \left(p^s - q^s \frac{x}{b_n}\right),$$

where (b_n) is a non-negative non-decreasing sequence such that $\lim_{n\to\infty} b_n = \infty$ and $\lim_{n\to\infty} \frac{b_n}{[n]_{p,q}} = 0$, f is a real-valued function defined on the non-negative real numbers and $x \in [0,\infty)$. Here, (p,q)-integer $[i]_{p,q}$ is defined by

$$[i]_{p,q} := \frac{p^i - q^i}{p - q}, \quad i = 0, 1, 2, \ldots, \quad 0 < q < p \leq 1.$$

(p,q)-Factorial $[i]_{p,q}!$ and (p,q)-binomial coefficients are defined by

$$[i]_{p,q}! := [i]_{p,q} [i-1]_{p,q} \cdots [2]_{p,q} [1]_{p,q}, \qquad [0]_{p,q}! := 1,$$

and

$$\begin{bmatrix} j \\ i \end{bmatrix}_{p,q} := \frac{[j]_{p,q}!}{[i]_{p,q}! [j-i]_{p,q}!},$$

respectively.

It is clear that (p,q)-Bernstein-Chlodowsky operators are positive linear operators from $C[0,\alpha]$ into itself. For these operators, we have

$$C_n^{p,q}(1;x) = 1, \tag{1.1}$$

$$C_n^{p,q}(t;x) = x, \tag{1.2}$$

$$C_n^{p,q}(t^2;x) = \frac{q[n-1]_{p,q}}{[n]_{p,q}} x^2 + \frac{p^{n-1} b_n}{[n]_{p,q}} x. \tag{1.3}$$

Since $[n]_{p,q} = p^{n-1} + q[n-1]_{p,q}$, (1.3) can be reduced to

$$C_n^{p,q}(t^2;x) = \left(1 - \frac{p^{n-1}}{[n]_{p,q}}\right) x^2 + \frac{p^{n-1} b_n}{[n]_{p,q}} x. \tag{1.4}$$

It seems much more convenient to use (1.4) instead of (1.3) throughout this paper.

Fuzzy mathematics is a significant field of study in mathematics attracted attention. Fuzzy sets were introduced by Zadeh in [45]. Congxin and Ming defined fuzzy real numbers in [43]. Some basic concepts related to fuzzy numbers can be seen in the references [2, 3, 10–12, 19]. Significant conclusions in approximation theory have been recently generalized from real function spaces to fuzzy function spaces. The fuzzy Weierstrass approximation theorem was given by Gal[16, 18]. He gave also some quantitative estimates of the error of the approximation for the fuzzy Bernstein polynomials[17]. Gal et al. have studied on some approximation results such as in [3, 6, 9, 20]. Congxin and Danghang gave further a fuzzy Weierstrass approximation theorem in [42]. Anastassiou extended the well-known Korovkin theorem to fuzzy function spaces, and he applied its very important applications in operator theory in [2].

In this study, we purpose to extend the approximation properties of the (p,q)-Bernstein-Chlodowsky operators from real function spaces to fuzzy function spaces. For this aim, we define firstly fuzzy (p,q)-Bernstein-Chlodowsky operators, and we give a fuzzy Korovkin-type approximation theorem for these operators. Secondly, we investigate rate of convergence by using first order fuzzy modulus of continuity and Lipschitz-type fuzzy functions. Lastly, we give an estimate for fuzzy asymptotic expansions of the fuzzy (p,q)-Bernstein-Chlodowsky operators.

2. Preliminaries

Firstly, we recall some significant concepts of the fuzzy mathematics.

Let v be a function from $\mathbb{R}$ to $[0,1]$. If v satisfies the following properties

(i) There exists at least an element $a_0 \in \mathbb{R}$ satisfying $v(a_0) = 1$,

(ii) For each $\lambda \in [0,1]$ and $a, b \in \mathbb{R}$

$$v(\lambda a + (1-\lambda)b) \geq \min\{v(a), v(b)\},$$

(iii) There exists at least a neighborhood $U(a_0)$ satisfying $v(x) \leq v(a_0) + \varepsilon$ for all $x \in U(a_0)$,

(iv) The set $\overline{\{x \in \mathbb{R} : v(x) > 0\}}$ is compact in $\mathbb{R}$, where $\overline{A}$ is the closure of the set A,

then, v is called a fuzzy number, and by $\mathbb{R}_{\mathcal{F}}$ is denoted the set of all fuzzy numbers.

Let denote

$$[v]^r = \begin{cases} \{x \in \mathbb{R} : v(x) \geq r\}, & \text{if } r \in (0,1], \\ \overline{\{x \in \mathbb{R} : v(x) > 0\}}, & \text{if } r = 0, \end{cases}$$

which is a bounded and closed interval of $\mathbb{R}$ and has the representation

$$[v]^r = \left[v_-^r, v_+^r\right],$$

for all $r \in [0,1]$, where v_-^r and v_+^r denote the left and right endpoints of the interval $[v]^r$. If $r_1, r_2 \in [0,1]$ satisfying $r_1 \leq r_2$ then $[v]^{r_2} \subseteq [v]^{r_1}$.

For $v, w \in \mathbb{R}_{\mathcal{F}}$, $r \in [0,1]$ and $k \in \mathbb{R}$, the summation and the product with scalar on $\mathbb{R}_{\mathcal{F}}$ are defined uniquely by

$$[v \oplus w]^r = [v]^r + [w]^r, \qquad [k \odot v]^r = k\,[v]^r, \quad r \in [0,1],$$

where $[v]^r + [w]^r$ means the usual sum of two intervals as subsets of $\mathbb{R}$ and $k\,[v]^r$ means the usual product between a real scalar and a subset of $\mathbb{R}$, and also

$$\begin{array}{ll} (v \oplus w)_-^r = v_-^r + w_-^r, & (v \oplus w)_+^r = v_+^r + w_+^r, \\ (k \odot v)_-^r = k v_-^r, & (k \odot v)_+^r = k v_+^r. \end{array}$$

By $v \oplus w$ and $k \odot v$, $v, w \in \mathbb{R}_{\mathcal{F}}$, we denote the addition and the scalar product on $\mathbb{R}_{\mathcal{F}}$, respectively [19, 44] (see p.4 in [2]).

Let D be a non-negative real-valued function defined on $\mathbb{R}_{\mathcal{F}} \times \mathbb{R}_{\mathcal{F}}$ by

$$D(v,w) = \sup_{r \in [0,1]} \max \left\{ \left| v_-^r - w_-^r \right|, \left| v_+^r - w_+^r \right| \right\},$$

where $[v]^r = \left[v_-^r, v_+^r\right]$ and $[w]^r = \left[w_-^r, w_+^r\right] \subset \mathbb{R}$.

D is a complete metric on $\mathbb{R}_{\mathcal{F}}$, which satisfies following properties for all $z, v, w, e \in \mathbb{R}_{\mathcal{F}}$ and $k \in \mathbb{R}$:

$$D(z \oplus v, w \oplus v) = D(z,w), \tag{2.1}$$

$$D(k \odot z, k \odot w) = |k|\, D(z,w), \tag{2.2}$$

$$D(z \oplus w, v \oplus e) \leq D(z,v) + D(w,e). \tag{2.3}$$

A partial order on $\mathbb{R}_{\mathcal{F}}$ is denoted by $\preccurlyeq$ and defined by

$$z \preccurlyeq w \text{ iff } v_-^r \leq w_-^r, \qquad v_+^r \leq w_+^r, \tag{2.4}$$

for all $v, w \in \mathbb{R}_{\mathcal{F}}$ and $r \in [0,1]$. Here "$\leq$" is the partial order on $\mathbb{R}$ (see p.5 in [2]).

Some properties of fuzzy numbers are as in the following (see p.6 in [2]).

Let $k, l \in \mathbb{R}$, $w, z \in \mathbb{R}_{\mathcal{F}}$ and let $\widetilde{o} = \chi_{\{0\}}$ be the characteristic function of $\{0\}$. Then we have the following on $\mathbb{R}_{\mathcal{F}}$:

(i) The neutral element with regard to $\oplus$ is $\widetilde{o} \in \mathbb{R}_{\mathcal{F}}$,
(ii) For each $z \neq \widetilde{o}$, $z \in \mathbb{R}_{\mathcal{F}}$ has not an opposite in $\mathbb{R}_{\mathcal{F}}$,
(iii) If $k, l \geq 0$ or $k, l < 0$,

$$(k + l) \odot z = k \odot z \oplus l \odot z, \tag{2.5}$$

(For general $k, l \in \mathbb{R}$, (iii) is not true.)

(iv)

$$k \odot (w \oplus z) = k \odot w \oplus k \odot z, \tag{2.6}$$

(v)

$$k\odot(l\odot z)=(k.l)\odot z. \tag{2.7}$$

3. Construction of Fuzzy Operators

In this part, we define fuzzy (p,q)-Bernstein-Chlodowsky operators, and we give some auxiliary results.

Any function h from $[a,b]$ to $\mathbb{R}_{\mathcal{F}}$ is called a fuzzy-number-valued function, for convenience, a fuzzy-number-valued function h can be called fuzzy function defined on $[a,b]$, which has the following representation

$$[h(x)]^r=\left[h_-^r(x),h_+^r(x)\right],$$

for each $x\in[a,b]$ and $r\in[0,1]$, where $h_-^r(x)$ and $h_+^r(x)$ denote the left and right endpoints of $[h(x)]^r$. Also, h_-^r and h_+^r are real-valued functions defined on $[a,b]$ [19]. Similarly, a fuzzy algebraic polynomial of degree n has the following form

$$P_n(x)=\sum_{k=0}^{n}{}^{*}x^k\odot c_k,$$

where $c_k\in\mathbb{R}_{\mathcal{F}},\ k=0,1,\ldots,n$. Here, the sum $\sum\limits_{k=0}^{n}{}^{*}$ denotes the finite fuzzy summation(see p.82 in [2]).

Let g and h be fuzzy functions defined on $[a,b]$. The distance between g and h is defined by

$$\begin{aligned} D^*(g,h) &= \sup_{x\in[a,b]} D(g(x),h(x)) \\ &= \sup_{x\in[a,b]}\sup_{r\in[0,1]}\max\left\{\left|g_-^r(x)-h_-^r(x)\right|,\left|g_+^r(x)-h_+^r(x)\right|\right\}. \end{aligned}$$

A fuzzy function h defined on $[a,b]$ is called a fuzzy continuous at $x_0\in[a,b]$ if and only if h is sequential continuous at $x_0\in[a,b]$. If h is continuous for each $x_0\in[a,b]$, then h is called a fuzzy continuous function on $[a,b]$, that is, the fuzzy continuity of h is equivalent to the sequential continuity respect to the metric D. If h is a fuzzy continuous function on $[a,b]$, then the corresponding functions h_-^r and h_+^r are real-valued continuous functions defined on $[a,b]$ (see p.160 in [2] or [19]).

Let $C_{\mathcal{F}}[a,b]$ be the a space of all fuzzy continuous functions defined on $[a,b]$. $(C_{\mathcal{F}}[a,b],D^*)$ is a complete metric space(see [19]). The addition and scalar multiplication in $C_{\mathcal{F}}[a,b]$ are defined respectively by

$$\begin{aligned} (h\oplus l)(x) &= h(x)\oplus l(x), \\ (k\odot h)(x) &= k\odot h(x), \end{aligned}$$

for each $x \in [a,b]\,, k \in \mathbb{R},\ h,l \in C_{\mathcal{F}}[a,b]\,.$ Also, $\tilde{0}$ is a fuzzy-number-valued function defined on $[a,b]$ such that $\tilde{0}(x) = \tilde{o}$ for all $x \in [a,b]$, where $\tilde{o}$ is the neutral element with regard to $\oplus$ in $\mathbb{R}_{\mathcal{F}}$.

Considering the representation of $[h(x)]^r = \left[h_-^r(x), h_+^r(x)\right]$ and $[l(x)]^r = \left[l_-^r(x), l_+^r(x)\right]$ for each $x \in [a,b]$ and $r \in [0,1]\,,$ we have

$$(h \oplus l)_+^r = h_+^r + l_+^r, \qquad (h \oplus l)_-^r = h_-^r + l_-^r, \tag{3.1}$$

$$(k \odot h)_-^r = kh_-^r, \qquad (k \odot h)_+^r = kh_+^r, \quad \text{for } k \geq 0, \tag{3.2}$$

$$(k \odot h)_-^r = kh_+^r, \qquad (k \odot h)_+^r = kh_-^r, \quad \text{for } k < 0. \tag{3.3}$$

Lemma 3.1 ([19])**.** *For $k,m \in \mathbb{R}$ and $h,l,f,g \in C_{\mathcal{F}}[a,b]$, we have the following properties*

(i)

$$h \oplus l = l \oplus h,$$
$$(h \oplus l) \oplus g = h \oplus (l \oplus g)\,,$$
$$h \oplus \tilde{0} = \tilde{0} \oplus h,$$

(ii) *With respect to $\tilde{0}$ in $C_{\mathcal{F}}[a,b]$ for any $h \in C_{\mathcal{F}}[a,b]$ with $h([a,b]) \cap \mathbb{R}_{\mathcal{F}} \neq \emptyset$ has no opposite member regarding $\oplus$ in $C_{\mathcal{F}}[a,b]\,,$*

(iii) *For $k,m \in \mathbb{R}$ with $k,m \geq 0$ or $k,m < 0$*

$$(k+m) \odot h = k \odot h \oplus m \odot h,$$

(For general $k,m \in \mathbb{R}$, this property does not hold.)

(iv)

$$k \odot (h \oplus l) = k \odot h \oplus k \odot l,$$
$$k \odot (m \odot h) = (km) \odot h,$$

(v)

$$D^*(k \odot h, m \odot h) = |k-m|\, D^*(\tilde{0}, h),$$
$$D^*(h \oplus g, l \oplus g) = D^*(h,l),$$
$$D^*(h \oplus l, g \oplus f) = D^*(h,g) + D^*(l,f).$$

Also, any operator T from $C_{\mathcal{F}}[a,b]$ into itself has the following representation

$$[T(h)(x)]^r = \left[(T(h)(x))_-^r\,, (T(h)(x))_+^r\right].$$

Now, we construct fuzzy (p,q)-Bernstein-Chlodowsky operators.

Definition 3.2. Let $\alpha < b_n$ for each $n \in \mathbb{N}$ and let h be a fuzzy continuous function defined on $[0,\alpha]$. We define fuzzy (p,q)-Bernstein-Chlodowsky operators by

$$C_{n,p,q}^{\mathcal{F}}(h;x) = \sum_{k=0}^{n}{}^* h\left(\tau_{k,n}^{p,q}\right) \odot S_{k,n}^{p,q}(x)\,, \quad x \in [0,\alpha]\,,$$

where

$$\tau_{k,n}^{p,q} := \frac{b_n \left[k\right]_{p,q}}{p^{k-n} \left[n\right]_{p,q}},$$

and

$$S_{k,n}^{p,q}(x) = \frac{p^{k(k-1)/2}}{p^{n(n-1)/2}} \left[\begin{array}{c} n \\ k \end{array} \right]_{p,q} \left(\frac{x}{b_n}\right)^{n-k} \prod_{s=0}^{n-k-1} \left(p^s - q^s \frac{x}{b_n}\right),$$

for all $k,n \in \mathbb{N}$, $q \in (0,1]$ and $p \in (q,1]$.

It is obvious that $S_{k,n}^{p,q}(x) \geq 0$ for all $x \in [0,\alpha]$, $k,n \in \mathbb{N}$, $q \in (0,1]$ and $p \in (q,1]$, and the fuzzy (p,q)-Bernstein-Chlodowsky operators map $C_{\mathcal{F}}[0,\alpha]$ into itself.

Now, we give the connection between the (p,q)-Bernstein-Chlodowsky operators and the fuzzy (p,q)-Bernstein-Chlodowsky operators.

Lemma 3.3. *If $h \in C_{\mathcal{F}}[0,\alpha]$, then we have*

$$\left(C_{n,p,q}^{\mathcal{F}}(h;x)\right)_{\pm}^{r} = C_n^{p,q}(h_{\pm}^{r};x), \quad r \in [0,1], x \in [0,\alpha].$$

Proof. We have the following representation

$$\left[C_{n,p,q}^{\mathcal{F}}(h;x)\right]^{r} = \left[\left(C_{n,p,q}^{\mathcal{F}}(h;x)\right)_{-}^{r}, \left(C_{n,p,q}^{\mathcal{F}}(h;x)\right)_{+}^{r}\right],$$

for each $r \in [0,1]$, $x \in [0,\alpha]$. By the definition of fuzzy polynomial, we can see directly the following relation

$$\begin{aligned}\left(C_{n,p,q}^{\mathcal{F}}(h;x)\right)_{-}^{r} &= \left(\sum_{k=0}^{n}{}^{*} S_{k,n}^{p,q}(x) \odot h\left(\tau_{k,n}^{p,q}\right)\right)_{-}^{r} \\ &= \sum_{k=0}^{n} h_{-}^{r}\left(\tau_{k,n}^{p,q}\right) S_{k,n}^{p,q}(x) \\ &= C_n^{p,q}(h_{-}^{r};x).\end{aligned}$$

Similarly, we obtain

$$\left(C_{n,p,q}^{\mathcal{F}}(h;x)\right)_{+}^{r} = C_n^{p,q}(h_{+}^{r};x). \qquad \square$$

We recall the fuzzy linearity and the positivity of an operator from $C_{\mathcal{F}}[a,b]$ into itself.

Let T be an operator from $C_{\mathcal{F}}[a,b]$ into itself, such that

$$\begin{aligned} T(l \oplus h) &= T(l) \oplus T(h), \\ T(k \odot h) &= k \odot T(h), \end{aligned}$$

for each $k \in \mathbb{R}$, $h,l \in C_{\mathcal{F}}[a,b]$ then T is called a fuzzy linear operator. We say that a fuzzy linear operator T from $C_{\mathcal{F}}[a,b]$ into itself is positive, iff whenever $h,l \in C_{\mathcal{F}}[a,b]$ such that $h \preceq l$ then $T(h) \preceq T(l)$ iff

$(T(h))^r_+ \le (T(l))^r_+$ and $(T(h))^r_- \le (T(l))^r_-$ for all $r \in [0,1]$. Here, we have the representation

$$[T(h)(x)]^r = \left[(T(h)(x))^r_-, (T(h)(x))^r_+\right],$$

for all $x \in [a,b]$, $r \in [0,1]$ and also "$\preceq$" and "$\le$" are the partial orders on $C_{\mathcal{F}}[a,b]$ and $C[a,b]$, respectively (see p.100 in [2] or [19]).

Lemma 3.4. *The fuzzy (p,q)-Bernstein-Chlodowsky operators are fuzzy positive linear operators.*

Proof. Let h and g be fuzzy continuous functions defined on $[0,\alpha]$ and $k \in \mathbb{R}$.

By Lemma 3.3, we have

$$\left(C^{\mathcal{F}}_{n,p,q}(l \oplus h; x)\right)^r_\pm = C^{p,q}_n\left((l \oplus h)^r_\pm; x\right), \tag{3.4}$$

for each $x \in [0,\alpha]$, $r \in [0,1]$ respect to $-$ and $+$, respectively.

Since $C^{\mathcal{F}}_{n,p,q}(l \oplus h) \in C_{\mathcal{F}}[0,\alpha]$, considering the representation of $C^{\mathcal{F}}_{n,p,q}$, we have $\left(C^{\mathcal{F}}_{n,p,q}(l \oplus h)\right)^r_-, \left(C^{\mathcal{F}}_{n,p,q}(l \oplus h)\right)^r_+ \in C[0,\alpha]$.

Considering (3.1) and the fuzzy linearity of $C^{p,q}_n$, we can write

$$\begin{aligned} C^{p,q}_n\left((l \oplus h)^r_\pm; x\right) &= C^{p,q}_n\left(l^r_\pm + h^r_\pm; x\right) \\ &= C^{p,q}_n\left(l^r_\pm; x\right) + C^{p,q}_n\left(h^r_\pm; x\right). \end{aligned} \tag{3.5}$$

Applying (3.4) and (3.5), and considering Lemma 3.3, we obtain

$$\left(C^{\mathcal{F}}_{n,p,q}(l \oplus h; x)\right)^r_\pm = \left(C^{\mathcal{F}}_{n,p,q}(l; x)\right)^r_\pm + \left(C^{\mathcal{F}}_{n,p,q}(h; x)\right)^r_\pm, \tag{3.6}$$

for each $x \in [0,\alpha]$, $r \in [0,1]$ respect to $-$ and $+$, respectively.

Using (3.6) and considering the summation of the interval, we have

$$\begin{aligned} &\left[C^{\mathcal{F}}_{n,p,q}(l \oplus h; x)\right]^r \\ &= \left[\left(C^{\mathcal{F}}_{n,p,q}(l \oplus h; x)\right)^r_-, \left(C^{\mathcal{F}}_{n,p,q}(l \oplus h; x)\right)^r_+\right] \\ &= \left[\left(C^{\mathcal{F}}_{n,p,q}(l; x)\right)^r_- + \left(C^{\mathcal{F}}_{n,p,q}(h; x)\right)^r_-, \left(C^{\mathcal{F}}_{n,p,q}(l; x)\right)^r_+ + \left(C^{\mathcal{F}}_{n,p,q}(h; x)\right)^r_+\right] \\ &= \left[\left(C^{\mathcal{F}}_{n,p,q}(l; x)\right)^r_-, \left(C^{\mathcal{F}}_{n,p,q}(l; x)\right)^r_+\right] + \left[\left(C^{\mathcal{F}}_{n,p,q}(h; x)\right)^r_-, \left(C^{\mathcal{F}}_{n,p,q}(h; x)\right)^r_+\right] \\ &= \left[C^{\mathcal{F}}_{n,p,q}(l; x)\right]^r + \left[C^{\mathcal{F}}_{n,p,q}(h; x)\right]^r \\ &= \left[C^{\mathcal{F}}_{n,p,q}(l; x) \oplus C^{\mathcal{F}}_{n,p,q}(h; x)\right]^r \\ &= \left[\left(C^{\mathcal{F}}_{n,p,q}(l) \oplus C^{\mathcal{F}}_{n,p,q}(h); x\right)\right]^r, \end{aligned}$$

for each $x \in [0,\alpha]$, $r \in [0,1]$.

Thus

$$C^{\mathcal{F}}_{n,p,q}(l \oplus h) = C^{\mathcal{F}}_{n,p,q}(l) \oplus C^{\mathcal{F}}_{n,p,q}(h), \quad l, h \in C_{\mathcal{F}}[0,1].$$

Suppose that $k \ge 0$.

By Lemma 3.3, we can write

$$\left(C^{\mathcal{F}}_{n,p,q}\left(k\odot h;x\right)\right)^{r}_{\pm}=C^{p,q}_{n}\left(\left(k\odot h\right)^{r}_{\pm};x\right), \tag{3.7}$$

for each $x\in[0,\alpha]$, $r\in[0,1]$ respect to $-$ and $+$, respectively.

Since $C^{\mathcal{F}}_{n,p,q}\left(k\odot h\right)\in C_{\mathcal{F}}\left[0,\alpha\right]$, considering the representation of $C^{\mathcal{F}}_{n,p,q}$, we have $\left(C^{\mathcal{F}}_{n,p,q}\left(k\odot h\right)\right)^{r}_{-},\left(C^{\mathcal{F}}_{n,p,q}\left(k\odot h\right)\right)^{r}_{+}\in C\left[0,\alpha\right]$.

Considering (3.2) and the linearity of $C^{p,q}_{n}$, we can write

$$\begin{aligned} C^{p,q}_{n}\left(\left(k\odot h\right)^{r}_{\pm};x\right) &= C^{p,q}_{n}\left(kh^{r}_{\pm};x\right) \\ &= kC^{p,q}_{n}\left(h^{r}_{\pm};x\right). \end{aligned} \tag{3.8}$$

Applying (3.7) and (3.8) and considering Lemma 3.3, we obtain

$$\begin{aligned} \left(C^{\mathcal{F}}_{n,p,q}\left(k\odot h;x\right)\right)^{r}_{\pm} &= k\left(C^{\mathcal{F}}_{n,p,q}\left(h;x\right)\right)^{r}_{\pm} \\ &= \left(\left(k\odot C^{\mathcal{F}}_{n,p,q}\right)\left(h;x\right)\right)^{r}_{\pm}, \end{aligned} \tag{3.9}$$

for each $x\in[0,\alpha]$, $r\in[0,1]$ respect to $-$ and $+$, respectively.

Using (3.9), we have

$$\begin{aligned} \left[C^{\mathcal{F}}_{n,p,q}\left(k\odot h;x\right)\right]^{r} &= \left[\left(C^{\mathcal{F}}_{n,p,q}\left(k\odot h;x\right)\right)^{r}_{-},\left(C^{\mathcal{F}}_{n,p,q}\left(k\odot h;x\right)\right)^{r}_{+}\right] \\ &= \left[\left(\left(k\odot C^{\mathcal{F}}_{n,p,q}\right)\left(h;x\right)\right)^{r}_{-},\left(\left(k\odot C^{\mathcal{F}}_{n,p,q}\right)\left(h;x\right)\right)^{r}_{+}\right] \\ &= \left[\left(k\odot C^{\mathcal{F}}_{n,p,q}\right)\left(h;x\right)\right]^{r}. \end{aligned}$$

Therefore

$$C^{\mathcal{F}}_{n,p,q}\left(k\odot h\right)=\left(k\odot C^{\mathcal{F}}_{n,p,q}\right)\left(h\right),\quad k\geq 0,h\in C_{\mathcal{F}}\left[0,\alpha\right].$$

For $k<0$, with the similar way to the case $k\geq 0$, we obtain

$$C^{\mathcal{F}}_{n,p,q}\left(k\odot h\right)=\left(k\odot C^{\mathcal{F}}_{n,p,q}\right)\left(h\right),\quad k<0,h\in C_{\mathcal{F}}\left[0,\alpha\right].$$

Let h and l be fuzzy continuous functions defined on $[0,\alpha]$ with $h\preceq l$, where "$\preceq$" is a partial order on $C_{\mathcal{F}}[0,\alpha]$. Then $h^{r}_{-}\leq l^{r}_{-}$ and $h^{r}_{+}\leq l^{r}_{+}$ for all $r\in[0,1]$, where "$\leq$" is a partial order on $C[0,\alpha]$.

Since $h^{r}_{-},h^{r}_{+},l^{r}_{-}$ and $l^{r}_{+}\in C\left[0,\alpha\right]$ and by the positivity of $C^{p,q}_{n}$, we have

$$C^{p,q}_{n}\left(h^{r}_{\pm}\right)\leq C^{p,q}_{n}\left(l^{r}_{\pm}\right),\quad r\in[0,1], \tag{3.10}$$

respect to $-$ and $+$, respectively.

Considering (3.10) and Lemma 3.3, we obtain

$$\left(C^{\mathcal{F}}_{n,p,q}(h)\right)^{r}_{\pm}\leq\left(C^{\mathcal{F}}_{n,p,q}(l)\right)^{r}_{\pm},\quad r\in[0,1],$$

respect to $-$ and $+$, respectively, that indicate

$$C^{\mathcal{F}}_{n,p,q}(h;x)\preceq C^{\mathcal{F}}_{n,p,q}(l;x),\quad x\in[0,\alpha],r\in[0,1],$$

which gives the positivity of $C^{\mathcal{F}}_{n,p,q}$. □

Now we consider a basic certain fuzzy continuous function illustrating the existence of the fuzzy (p,q)-Bernstein-Chlodowsky operators.

Example 3.5. Let $\alpha < b_n$ and define a function $g:\mathbb{R}\to\mathbb{R}_{\mathcal{F}}$ by

$$\begin{aligned} g(s) &= \chi_{[0,\alpha]} \\ &= \begin{cases} 1, & \text{if } s\in[0,\alpha], \\ 0, & \text{if } s\notin[0,\alpha]. \end{cases} \end{aligned}$$

It is clear that g is a basic fuzzy continuous function defined on $\mathbb{R}$.

For each $s\in\mathbb{R}$ and $r\in[0,1]$, we have

$$[g(s)]^r = \begin{cases} [0,\alpha], & \text{if } s\in[0,\alpha], \\ \emptyset, & \text{if } s\notin[0,\alpha]. \end{cases}$$

If we define $g_1:[0,\alpha]\to\mathbb{R}_{\mathcal{F}}$ by $g_1(s)=g(s)$ for each $s\in[0,\alpha]$, then

$$\begin{aligned} [g_1(s)]^r &= \left[(g_1(s))_-^r,(g_1(s))_+^r\right] \\ &= [0,\alpha], \end{aligned}$$

for each $s\in[0,\alpha]$, $r\in[0,1]$, that indicates

$$(g_1(s))_-^r = 0, \qquad (g_1(s))_+^r = \alpha,$$

for each $s\in[0,\alpha]$, $r\in[0,1]$ and $(g_1)_-^r,(g_1)_+^r\in C[0,\alpha]$, i.e. $g_1\in C_{\mathcal{F}}[0,\alpha]$.

Considering Lemma 3.3 and (1.1), we obtain

$$\begin{aligned} \left[C_{n,p,q}^{\mathcal{F}}(g_1(s);x)\right]^r &= \left[\left(C_{n,p,q}^{\mathcal{F}}(g_1(s);x)\right)_-^r,\left(C_{n,p,q}^{\mathcal{F}}(g_1(s);x)\right)_+^r\right] \\ &= \left[C_{n,}^{p,q}((g_1(s))_-^r;x),C_{n,}^{p,q}((g_1(s))_+^r;x)\right] \\ &= \left[C_{n,}^{p,q}(0;x),C_{n,}^{p,q}(\alpha;x)\right] \\ &= [0,\alpha] \\ &= [g_1(x)]^r, \end{aligned}$$

for each $s\in[0,\alpha]$, $r\in[0,1]$. Thus $C_{n,p,q}^{\mathcal{F}}(g_1;.)\in C_{\mathcal{F}}[0,\alpha]$.

Now, we recall the fuzzy Korovkin-type approximation theorem.

Theorem 3.6 (see p.103 in [2])**.** *Assume that there exists a corresponding sequence of positive linear operators* $\left\{\widetilde{T}_n\right\}_{n\in\mathbb{N}}$ *from* $C[a,b]$ *into itself for any sequence of fuzzy positive linear operators* $\{T_n\}_{n\in\mathbb{N}}$ *from* $C_{\mathcal{F}}[a,b]$ *into itself for* $[a,b]\subset\mathbb{R}$ *satisfying*

$$(T_n(h))_\pm^r = \left(\tilde{T}_n\left(h_\pm^r\right)\right), \tag{3.11}$$

for all $r\in[0,1]$ *and* $h\in C_{\mathcal{F}}[a,b]$, *respectively. Moreover suppose that* $\tilde{T}_n\left(t^i\right)$ *converge to* x^i *for* $i=0,1,2$ *uniformly, then* $D^*\left(T_n\left(h\right),h\right)$ *converges to* 0 *uniformly for any* $h\in C_{\mathcal{F}}[a,b]$, *i.e.* $T_n\left(h\right)$ *converges to* h *respect to* D^* *in the fuzzy sense.*

Now, we can give the following fuzzy Korovkin-type approximation theorem for the fuzzy (p,q)-Bernstein-Chlodowsky operators.

Theorem 3.7. *Let* (p_n) *and* (q_n) *be two arbitrary sequences such that* $q_n\in(0,1]$ *and* $p_n\in(q_n,1]$ *for each* $n\in\mathbb{N}$ *fulfilling following relations*

$$\lim_{n\to\infty}p_n=1=\lim_{n\to\infty}q_n,\qquad \lim_{n\to\infty}\left(p_n\right)^n<\infty. \tag{3.12}$$

Then $D^*\left(C^{\mathcal{F}}_{n,p_n,q_n}(h),h\right)$ *converges to zero uniformly for any* $h\in C_{\mathcal{F}}[0,\alpha]$, *i.e. the sequence of the fuzzy* (p,q)*-Bernstein-Chlodowsky operators* $C^{\mathcal{F}}_{n,p_n,q_n}(h)$ *converges to* h *respect to the metric* D^* *in the fuzzy sense.*

Proof. For the proof, we consider the fuzzy Korovkin-type approximation theorem (Theorem 3.6). By Lemma 3.3 and Lemma 3.4, the fuzzy (p,q)-Bernstein-Chlodowsky operators are positive linear operators fulfilling the assumption (3.11) that maps $C_{\mathcal{F}}[0,\alpha]$ into itself. From the proof of Theorem 3.1 in [4] and well-known Korovkin Theorem in [24], we have that $C_n^{p_n,q_n}\left(e_i\right)$ converge to e_i uniformly for $i=0,1,2$. Thus the hypotheses of the fuzzy Korovkin-type approximation theorem are verified, which completes the proof. □

4. Rate of Convergence

In this part, we give quantitative estimates by using the first order fuzzy modulus of continuity and Lipschitz-type fuzzy functions.

Now, we recall the concept of the first order fuzzy modulus continuity(see pp.160-161 in [2]).

The first order fuzzy modulus of continuity for any fuzzy function h defined on $[a,b]$ is defined by

$$\omega_1^{\mathcal{F}}\left(h;\mu\right):=\sup_{\substack{t,s\in[a,b]\\|t-s|\le\mu}}D\left(h\left(t\right),h\left(s\right)\right),\quad \mu>0.$$

Suppose that $\omega_1^{\mathcal{F}}\left(h;\mu\right)$, $\omega_1\left(h_-^r;\mu\right)$ and $\omega_1\left(h_+^r;\mu\right)$ are all finite for each $\mu>0$ and $r\in[0,1]$. Here, ω_1 is the usual first order modulus of continuity defined by

$$\omega_1\left(f;\mu\right)=\sup_{\substack{t,s\in[a,b]\\|t-s|\le\mu}}\left|f\left(t\right)-f\left(s\right)\right|,\quad \mu>0.$$

Then

$$\omega_1^{\mathcal{F}}\left(h;\mu\right)=\sup_{r\in[0,1]}\max\left\{\omega_1\left(h_-^r;\mu\right),\omega_1\left(h_+^r;\mu\right)\right\},\quad \mu>0.$$

Also, $\omega_1^{\mathcal{F}}(h;\mu)$ has the following properties

$$\omega_1^{\mathcal{F}}(h;\mu_1+\mu_2) \leq \omega_1^{\mathcal{F}}(h;\mu_1)+\omega_1^{\mathcal{F}}(h;\mu_2), \quad \mu_1,\mu_2>0,$$
$$\omega_1^{\mathcal{F}}(h;n\mu) \leq n\omega_1^{\mathcal{F}}(h;\mu), \ \mu>0, \quad n\in\mathbb{N},$$
$$\omega_1^{\mathcal{F}}(h;\kappa\mu) \leq (\kappa+1)\,\omega_1^{\mathcal{F}}(h;\mu), \quad \mu>0, \kappa>0,$$
$$h\in C_{\mathcal{F}}[a,b] \text{ implies } \lim_{\mu\to 0}\omega_1^{\mathcal{F}}(h;\mu)=0.$$

We can give rate of convergence of the fuzzy (p,q)-Bernstein-Chlodowsky operators with the help of the first order fuzzy modulus of continuity.

Theorem 4.1. *If* $h\in C_{\mathcal{F}}[0,\alpha]$ *then*

$$D^*\left(C_{n,p,q}^{\mathcal{F}}(h;.),h\right) \leq 2\omega_1^{\mathcal{F}}\left(h;\sqrt{\frac{p^{n-1}\alpha\left(b_n-\alpha\right)}{[n]_{p,q}}}\right).$$

Proof. Since $\sum\limits_{k=0}^{n} S_{k,n}^{p,q}(x)=1$, by the definition of fuzzy polynomials, we can write

$$h(x) = \left[\sum_{k=0}^{n} S_{k,n}^{p,q}(x)\right]\odot h(x) = \sum_{k=0}^{n}{}^{*}\left[S_{k,n}^{p,q}(x)\odot h(x)\right]. \tag{4.1}$$

By (4.1) and considering the metric D on $\mathbb{R}_{\mathcal{F}}$, (2.5) and (2.6), we get

$$D\left(C_{n,p,q}^{\mathcal{F}}(h;x),h(x)\right) = D\left(\sum_{k=0}^{n}{}^{*}S_{k,n}^{p,q}(x)\odot h\left(\tau_{k,n}^{p,q}\right),\sum_{k=0}^{n}{}^{*}S_{k,n}^{p,q}(x)\odot h(x)\right) \leq \sum_{k=0}^{n} D\left(h\left(\tau_{k,n}^{p,q}\right),h(x)\right)S_{k,n}^{p,q}(x). \tag{4.2}$$

Considering the properties of the first order fuzzy modulus of continuity, we can write

$$D\left(h\left(\tau_{k,n}^{p,q}\right),h(x)\right) \leq \omega_1^{\mathcal{F}}\left(h;\left|\tau_{k,n}^{p,q}-x\right|\right) \leq \left(1+\frac{1}{\mu}\left|\tau_{k,n}^{p,q}-x\right|\right)\omega_1^{\mathcal{F}}(h;\mu). \tag{4.3}$$

Applying (4.3) to (4.2) and taking Cauchy-Schwarz inequality into account, we obtain

$$D\left(C_{n,p,q}^{\mathcal{F}}(h;x),h(x)\right) \leq \left(1+\frac{1}{\mu}\left(C_n^{p,q}\left((t-x)^2;x\right)\right)^{1/2}\right)\omega_1^{\mathcal{F}}(h;\mu). \tag{4.4}$$

On the other hand, for each $x \in [0,\alpha]$, using (1.1), (1.2) and (1.4), we have

$$C_n^{p,q}\left((t-x)^2;x\right) = \frac{p^{n-1}}{[n]_{p,q}}\left(-x^2+b_n x\right) := \phi(x),$$

where $\phi(x)$ is a non-negative decreasing function regard to x on the interval $[0,\alpha]$.

Since $\max_{x\in[0,\alpha]} \phi(x) = \phi(\alpha)$, we obtain

$$C_n^{p,q}\left((t-x)^2;x\right) \leq \frac{p^{n-1}\alpha(b_n-\alpha)}{[n]_{p,q}}. \tag{4.5}$$

Applying (4.5) to (4.4), we get

$$D\left(C^{\mathcal{F}}_{n,p,q}(h;x), h(x)\right) \leq \left(1+\frac{1}{\mu}\left(\frac{p^{n-1}\alpha(b_n-\alpha)}{[n]_{p,q}}\right)^{1/2}\right)\omega_1^{\mathcal{F}}(h;\mu).$$

Selecting $\mu = \sqrt{\frac{p^{n-1}\alpha(b_n-\alpha)}{[n]_{p,q}}}$ and taking supremum on the right hand side of the last inequality, we reach the desired result. □

We recall Lipschitz-type fuzzy functions (see p.144 in [2]).

Let h be a fuzzy continuous function defined on $[a,b]$. If h satisfies the following property

$$D(h(t), h(s)) \leq M|t-s|^{\gamma}, \quad \gamma\in(0,1],\ M>0,\ s,t\in[0,1],$$

then h is called a Lipschitz-type fuzzy function.

Now, we give rate of convergence of the fuzzy algebraic (p,q)-Bernstein operators with the help of Lipschitz-type fuzzy functions.

Theorem 4.2. *If $h \in C_{\mathcal{F}}[0,\alpha]$ is any fuzzy Lipschitz-type function, then*

$$D^*\left(C^{\mathcal{F}}_{n,p,q}(h;.), h\right) \leq M\left(\sqrt{\frac{p^{n-1}\alpha(b_n-\alpha)}{[n]_{p,q}}}\right)^{\gamma}, \quad M>0.$$

Proof. Since h is a fuzzy Lipschitz-type function, we have

$$D(h(t), h(s)) \leq M|t-s|^{\gamma}, \quad \gamma\in(0,1],\ M>0,\ s,t\in[0,1]. \tag{4.6}$$

Considering (4.6) and (4.2) in the proof of Theorem 4.1, we can write

$$\begin{aligned} D\left(C^{\mathcal{F}}_{n,p,q}(h;x), h(x)\right) &\leq \sum_{k=0}^{n} D\left(h\left(\xi^{p,q}_{n,k}\right), h(x)\right) S^{p,q}_{k,n}(x) \\ &\leq M\sum_{k=0}^{n}\left|\xi^{p,q}_{n,k}-x\right|^{\gamma} S^{p,q}_{k,n}(x). \end{aligned} \tag{4.7}$$

Applying Hölder inequality to (4.7), we get

$$D\left(C_{n,p,q}^{\mathcal{F}}(h;x),h(x)\right)\leq M\left(\sum_{k=0}^{n}\left(\xi_{n,k}^{p,q}-x\right)^{2}S_{k,n}^{p,q}(x)\right)^{\gamma/2} \tag{4.8}$$

$$=M\left[\left(C_{n}^{p,q}\left((t-x)^{2}\right)(x)\right)^{1/2}\right]^{\gamma}.$$

In last inequality, by (4.5) and taking supremum on the right hand side for each $x\in[0,\alpha]$, we get the desired result. □

5. Fuzzy Asymptotic Expansions

Any fuzzy function h defined on $[a,b]$ is called differentiable at $x\in[a,b]$ if there exists $h^{'}(x)\in\mathbb{R}_{\mathcal{F}}$ such that

$$h^{'}(x)=\lim_{t\to 0}\frac{h(x+t)-h(x)}{t}.$$

The $k-$times derivatives $h^{(k)}(x)$, $k\in\mathbb{N}$ of h is defined similarly (see [44] or p.7 in [2]).

Let by $C_{\mathcal{F}}^{2}[0,\alpha]$ be denoted a space of all fuzzy continuous functions defined on $[0,\alpha]$ that is $k-$times differentiable continuously.

Now, we give an estimate for fuzzy asymptotic expansions of the fuzzy (p,q)-Bernstein-Chlodowsky operators.

Theorem 5.1. *If any $h\in C_{\mathcal{F}}^{2}[0,\alpha]$ then*

$$D^{*}\left(C_{n,p,q}^{\mathcal{F}}(h;.),h\oplus\frac{1}{2}C_{n}^{p,q}(\sigma;.)\odot h''\right)$$

$$\leq D\left(\omega_{1}^{\mathcal{F}}\left(h;\sqrt{\frac{p^{n-1}\alpha(b_{n}-\alpha)}{[n]_{p,q}}}\right),\tilde{o}\right)+\frac{p^{n-1}\alpha(b_{n}-\alpha)}{2[n]_{p,q}}D^{*}\left(h'',\tilde{0}\right),$$

where $\tilde{o}$ is the neutral element respect to $\oplus$ in $\mathbb{R}_{\mathcal{F}}$, $\tilde{0}$ is a fuzzy function defined on $[0,\alpha]$ such that $\tilde{0}(x)=\tilde{o}$ for all $x\in[0,\alpha]$ and $\sigma(t)=(t-x)^{2}$ for each $t\in[0,\alpha]$.

Proof. Since $h\in C_{\mathcal{F}}^{2}[0,\alpha]$, from the representation

$$[h(x)]^{r}=\left[h_{-}^{r}(x),h_{+}^{r}(x)\right]$$

for all $x\in[0,\alpha]$, it is obvious that h_{-}^{r} and h_{+}^{r} are in $C^{2}[0,\alpha]$, and also $\left\|(h'')_{-}^{r}\right\|$ and $\left\|(h'')_{+}^{r}\right\|$ are bound in $\mathbb{R}$.

By considering the classical (p,q)-Bernstein-Chlodowsky operators, we can write

$$\left|C_{n}^{p,q}(h_{\pm}^{r};x)-h_{\pm}^{r}(x)-\frac{1}{2}\left(h''(x)\right)_{\pm}^{r}C_{n}^{p,q}\left((t-x)^{2};x\right)\right| \tag{5.1}$$

$$\leq \left| C_n^{p,q}(h_\pm^r;x) - h_\pm^r(x) \right| + \frac{1}{2} \left\| (h'')_\pm^r \right\| \left| C_n^{p,q}\left((t-x)^2 ; x \right) \right|,$$

respect to $-$ and $+$, respectively.

On the other hand, since h_-^r and h_+^r are in $C^2[0,\alpha]$, they are also in $C[0,\alpha]$. By Theorem 3.2 in [4], we have

$$\left| C_n^{p,q}(h_\pm^r;x) - h_\pm^r(x) \right| \leq 2\omega_1 \left(h_\pm^r, \sqrt{\frac{p^{n-1}\alpha(b_n-\alpha)}{[n]_{p,q}}} \right), \tag{5.2}$$

for each $x \in [0,\alpha]$ respect to $-$ and $+$, respectively and applying (5.2) and (4.5) to (5.1), we obtain

$$\begin{aligned} &\left| C_n^{p,q}(h_\pm^r;x) - h_\pm^r(x) - \frac{1}{2}\left(h''(x)\right)_\pm^r C_n^{p,q}\left((t-x)^2;x\right) \right| \\ &\leq \left| C_n^{p,q}(h_\pm^r;x) - h_\pm^r(x) \right| + \frac{1}{2} \left\| (h'')_\pm^r \right\| \left| C_n^{p,q}\left((t-x)^2 ; x \right) \right| \\ &\leq 2\omega_1 \left(h_\pm^r, \sqrt{\frac{p^{n-1}\alpha(b_n-\alpha)}{[n]_{p,q}}} \right) + \frac{p^{n-1}\alpha(b_n-\alpha)}{2[n]_{p,q}} \left\| (h'')_\pm^r \right\|, \end{aligned} \tag{5.3}$$

for each $x \in [0,\alpha]$ respect to $-$ and $+$, respectively.

Consequently, considering the definition of D and the representation of fuzzy functions, we can write

$$\begin{aligned} &D\left(C_{n,p,q}^{\mathcal{F}}(h;x), h(x) \oplus \frac{1}{2} C_n^{p,q}\left((t-x)^2;x\right) \odot h''(x) \right) \\ &= D\Big(\left[\left(C_{n,p,q}^{\mathcal{F}}(h;x)\right)_-^r, \left(C_{n,p,q}^{\mathcal{F}}(h;x)\right)_+^r \right], \\ &\quad \left[h_-^r(x), h_+^r(x) \right] + \frac{1}{2} C_n^{p,q}\left((t-x)^2;x\right) \left[\left(h''(x)\right)_-^r, \left(h''(x)\right)_+^r \right] \Big) \\ &= \sup_{r\in[0,1]} \max \Big\{ \left| C_n^{p,q}(h_-^r;x) - h_-^r(x) - \frac{1}{2}\left(h''(x)\right)_-^r C_n^{p,q}\left((t-x)^2;x\right) \right|, \\ &\quad \left| C_n^{p,q}(h_+^r;x) - h_+^r(x) - \frac{1}{2}\left(h''(x)\right)_+^r C_n^{p,q}\left((t-x)^2;x\right) \right| \Big\}, \end{aligned}$$

for each $x \in [0,\alpha]$ and $r \in [0,1]$. Using (5.3) in the last equality, we get

$$\begin{aligned} &D\left(C_{n,p,q}^{\mathcal{F}}(h;x), h(x) \oplus \frac{1}{2} C_n^{p,q}\left((t-x)^2;x\right) \odot h''(x) \right) \\ &\leq \sup_{r\in[0,1]} \max \Big\{ 2\omega_1 \left(h_-^r; \sqrt{\frac{p^{n-1}\alpha(b_n-\alpha)}{[n]_{p,q}}} \right) + \frac{p^{n-1}\alpha(b_n-\alpha)}{2[n]_{p,q}} \left\| (h'')_-^r \right\|, \\ &\quad 2\omega_1 \left(h_+^r; \sqrt{\frac{p^{n-1}\alpha(b_n-\alpha)}{[n]_{p,q}}} \right) + \frac{p^{n-1}\alpha(b_n-\alpha)}{2[n]_{p,q}} \left\| (h'')_+^r \right\| \Big\}, \end{aligned}$$

for each $x \in [0,\alpha]$ and $r \in [0,1]$. By considering the property of supremum, we obtain

$$D\left(C^{\mathcal{F}}_{n,p,q}(h;x), h(x) \oplus \frac{1}{2} C^{p,q}_{n}\left((t-x)^2;x\right) \odot h''(x)\right)$$
$$\leq \sup_{r\in[0,1]} \max\left\{2\omega_1\left(h^r_-;\sqrt{\frac{p^{n-1}\alpha\,(b_n-\alpha)}{[n]_{p,q}}}\right), 2\omega_1\left(h^r_+;\sqrt{\frac{p^{n-1}\alpha\,(b_n-\alpha)}{[n]_{p,q}}}\right)\right\}$$
$$+ \sup_{r\in[0,1]} \max\left\{\frac{p^{n-1}\alpha\,(b_n-\alpha)}{2\,[n]_{p,q}}\left\|(h'')^r_-\right\|, \frac{p^{n-1}\alpha\,(b_n-\alpha)}{2\,[n]_{p,q}}\left\|(h'')^r_+\right\|\right\},$$

which implies

$$D\left(C^{\mathcal{F}}_{n,p,q}(h;x), h(x) \oplus \frac{1}{2} C^{p,q}_{n}\left((t-x)^2;x\right) \odot h''(x)\right)$$
$$\leq D\left(\omega^{\mathcal{F}}_1\left(h;\sqrt{\frac{p^{n-1}\alpha\,(b_n-\alpha)}{[n]_{p,q}}}\right),\tilde{o}\right) + \frac{p^{n-1}\alpha\,(b_n-\alpha)}{2\,[n]_{p,q}} D^*\left(h'',\tilde{0}\right),$$

for each $x \in [0,\alpha]$. In the last inequality, taking on supremum the right hand side, we get the desired result. □

6. Conclusions

Any real number $x_0 \in \mathbb{R}$ can be identified with $\chi_{\{x_0\}}$, which satisfies the properties (i)-(iv) of Definition 3.2. Then, it is clear that $\mathbb{R} \subset \mathbb{R}_{\mathcal{F}}$[19]. We define the fuzzy (p,q)-Bernstein Chlodowsky operators with the help of all functions h from $[0,\alpha]$ to $\mathbb{R}_{\mathcal{F}}$ in fuzzy function space. If we take $\mathbb{R}$ instead of $\mathbb{R}_{\mathcal{F}}$ in definition of h, then all results in this paper are reduced to real function space.

In this sense, this study is crucial in terms of extending the approximation properties of (p,q)-Bernstein-Chlodowsky operators from real function space to fuzzy function space.

Let (p_n) and (q_n) be any two sequences fulfilling (3.12). In the definition of fuzzy (p,q)-Bernstein-Chlodowsky operators, if we select the sequences (p_n) and (q_n) instead of p and q, respectively, then

$$\lim_{n\to\infty}\sqrt{\frac{p^{n-1}\alpha\,(b_n-\alpha)}{[n]_{p,q}}} = 0.$$

Therefore

$$\lim_{n\to\infty}\omega^{\mathcal{F}}_1\left(h;\sqrt{\frac{p^{n-1}\alpha\,(b_n-\alpha)}{[n]_{p,q}}}\right) = 0,$$

for each $h \in C_{\mathcal{F}}\,[0,1]$.

Thus, in Theorem 4.1 and 4.2, for each $h \in C_{\mathcal{F}}\,[0,1]$, we obtain

$$\lim_{n\to\infty} D^*\left(C^{\mathcal{F}}_{n,p_n,q_n}(h;.), h\right) = 0,$$

i.e. $C^{\mathcal{F}}_{n,p_n,q_n}(h)$ converges to h uniformly in fuzzy sense, and also in Theorem 5.1, for each $h \in C_{\mathcal{F}}[0,1]$, we have

$$\lim_{n\to\infty} D^*\left(C^{\mathcal{F}}_{n,p_n,q_n}(h;.), h \oplus \frac{1}{2} C^{p,q}_{n}(\sigma;.) \odot h''\right) = 0.$$

Consequently, Theorem 3.7 demonstrates that any sequence of the fuzzy (p,q)-Bernstein-Chlodowsky operators converges uniformly respect to the fuzzy metric D^* in fuzzy sense. Theorem 4.1 and 4.2 estimate the degree of the fuzzy approximation of the fuzzy (p,q)-Bernstein-Chlodowsky operators, and Theorem 5.1 implies that asymptotic expansions of the fuzzy (p,q)-Bernstein-Chlodowsky operators $C^{\mathcal{F}}_{n,p_n,q_n}(h)$ converge to h respect to the metric D^* for all $h \in C^2_{\mathcal{F}}[0,1]$ in fuzzy sense.

It can be also obtained fuzzy approximation properties of other fuzzy positive linear operators by the similar methods used in this paper. Therefore, this study will contribute to the readers in making new studies.

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