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Abstract. Some generalizations of Besselian, Hilbertian systems and frames in nonseparable Banach spaces with respect to some nonseparable Banach space K of systems of scalars are considered in this work. The concepts of uncountable K-Bessel, K-Hilbert systems, K-frames and K^*-Riesz bases in nonseparable Banach spaces are introduced. Criteria of uncountable K-Besselianness, K-Hilbertianness for systems, K-frames and unconditional K^*-Riesz basicity are found, and the relationship between them is studied. Unlike before, these new facts about Besselian and Hilbertian systems in Hilbert and Banach spaces are proved without using a conjugate system and, in some cases, a completeness of a system. Examples of K-Besselian systems which are not minimal are given. It is proved that every K-Hilbertian systems is minimal. The case where K is an space of systems of coefficients of uncountable unconditional basis of some space is also considered.

1. Introduction

The concepts of Besselian and Hilbertian sequences in the Hilbert space $L_2(a,b)$ were introduced by N.K. Bari [3] for minimal sequence $\{f_n\}_{n\in N}$ from $L_2(a,b)$ with a conjugate system $\{g_n\}_{n\in N}$ as follows:

Definition 1.1 ([3])**.** A sequence $\{f_n\}_{n\in N}$ is called Besselian if $\{(f, g_n)\}_{n\in N} \in l_2$ for every $f \in L_2(a,b)$, and it is called Hilbertian if for every $\lambda \in l_2$ there exists $f \in L_2(a,b)$ such that $\{(f, g_n)\}_{n\in N} = \lambda$.

The following Besselianness and Hilbertianness criteria for a pair of complete biorthogonal sequences in $L_2(a,b)$ are true.

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Theorem 1.2 ([3])**.** *A sequence* $\{f_n\}_{n\in N}$ *is Bessellian in* $L_2(a,b)$ *if and only if there exists the linear bounded operator* T *in* $L_2(a,b)$ *such that* $T(f_n)=\varphi_n, n\in N$, *where* $\{\varphi_n\}_{n\in N}$ *is an orthonormal basis for* $L_2(a,b)$.

Theorem 1.3 ([3])**.** *A sequence* $\{f_n\}_{n\in N}$ *is Hilbertian in* $L_2(a,b)$ *if and only if there exists the linear bounded operator* S *in* $L_2(a,b)$ *such that* $S(\varphi_n)=f_n, n\in N$, *where* $\{\varphi_n\}_{n\in N}$ *is an orthonormal basis for* $L_2(a,b)$.

It was also established that the complete minimal sequence which is both Besselian and Hilbertian forms a Riesz basis, i.e. image of an orthonormal basis for a bounded invertible operator. Later these concepts have been extended to abstract Hilbert spaces Riesz bases are equivalent to bounded unconditional bases.

Theorem 1.4. *Let* H *be a separable Hilbert space. Then a sequence* $\{\varphi_n\}_{n\in N}$ *forms a Riesz basis for* H *if and only if* $\{\varphi_n\}_{n\in N}$ *is complete in* H *and there exist the constants* $A>0$ *and* $B>0$ *such that*

$$A\sum_k |c_k|^2 \le \left\|\sum_k c_k\varphi_k\right\|_H^2 \le B\sum_k |c_k|^2,$$

for every finite set of numbers $\{c_k\}$.

More facts about the theory of Riesz bases in separable Hilbert spaces can be found in [22, 37].

Riesz bases are special cases of frames in Hilbert spaces introduced in 1952 by R.J. Duffin and A.C. Schaeffer in the study of nonharmonic Fourier series [19] for a perturbed exponential system.

Definition 1.5 ([19])**.** A sequence $\{\varphi_n\}_{n\in N}$ of elements of the Hilbert space H is a frame in H if there exist the constants $A>0$ and $B>0$ such that

$$A\|f\|_H^2 \le \sum_{n=1}^{\infty} |(f,\varphi_n)|^2 \le B\|f\|_H^2,$$

for every $f\in H$, where $\|\cdot\|_H$ is the norm on H generated by the inner product $(\cdot,\cdot)$.

The advantage of frames is that every element of H can be decomposed with respect to the frame elements. The interest to the frames grew as they found wide applications in various fields of mathematics. A lot of researches have been dedicated to frames (see [12, 18, 22, 30], etc.). For more details on the theory of frames we refer the readers to [15, 23, 32]. Frames in Banach spaces have been studied in [1, 2, 6, 13–17, 23–25, 27, 32]. The concepts of atomic decomposition and Banach frame have been introduced for the first time in [20, 21] (see also [1, 2, 4, 5, 12–15]). One of the important methods for establishing frames is a perturbation method. A lot of results have been obtained in this direction in the context of Paley-Wiener theorems [see [11, 13, 15, 16, 29].

Banach generalizations of the results obtained in [3] have been presented in [8, 10, 27, 31, 35, 36]. In [8], the concepts of K-Besselian and K-Hilbertian sequences have been given for a minimal sequence, and the concept of K-basis has been introduced which generalized the known ones, where K is some Banach space of numerical sequences. K-Besselian and K-Hilbertian sequences and frames with respect to some Banach space of vector-valued sequences were studied in [24, 27]. Uncountable Besselianness and Hilbertianness for a pair of complete biorthogonal system have been studied in [28]. The uncountable generalizations of frames were first considered in [9], where the concepts of uncountable Bessel system and uncountable frame were introduced and their properties were studied.

Note that uncountable Bessel and Hilbert systems, Riesz bases, and frames in nonseparable Banach spaces have not yet been studied. This work is dedicated to the study of these problems. We introduce the corresponding concepts in Banach space for a system from the conjugate space. The concepts of the uncountable K-Bessel and K-Hilbert systems and unconditional K-frames and K^*-Riesz bases in nonseparable Banach spaces are introduced. The characterization of such systems were established and the relationship between them are established. We obtain many of results of [3, 8] about Besselian and Hilbertian systems in Hilbert and Banach spaces without minimality condition for a system.

2. Preliminaries

Throughout this work, X and Y will be nonseparable Banach spaces with the norms $\|\cdot\|_X$ and $\|\cdot\|_Y$, respectively. An space conjugate to X will be denoted by X^*. By $L(X,Y)$ we denote an space of linearly bounded operators from X to Y. If $X = Y$, then $L(X,X)$ will be denoted by $L(X)$. The kernel and the image of the operator $T \in L(X,Y)$ are denoted by $\ker T$ and ImT, respectively. $spanM$ will be a linear span of the set $M \subset X$, $\overline{M}$ denotes a closure of the set M in X, $\delta_{\alpha\beta}$ is the Kronecker symbol.

Let I be an uncountable set of indices, I^a be a set of at most countable subsets of I, I_0 be a set of finite subsets of I, and system $\{x_\alpha^*\}_{\alpha\in I} \subset X^*$ be some system in X such that $I(x) = \{\alpha \in I : x_\alpha^*(x) \neq 0\} \in I^a$ for every $x \in X$.

Let K be some nonseparable Banach space of systems $\lambda = \{\lambda_\alpha\}_{\alpha\in I}$ of scalars such that $I(\lambda) \in I^a$.

The space K is called a CB-space if the system $\{\delta_\alpha\}_{\alpha\in I} \subset K$, $\delta_\alpha = \{\delta_{\alpha\beta}\}_{\beta\in I}$, forms an uncountable unconditional basis for K, i.e. for every

$\lambda = \{\lambda_\alpha\}_{\alpha\in I} \in K$ the relation

$$\lambda = \sum_{\alpha\in I} \lambda_\alpha \delta_\alpha = \sum_{i=1}^{\infty} \lambda_{\alpha_i} \delta_{\alpha_i}$$

holds, where $\{\lambda_{\alpha_i}\}_{i\in N}$ is a sequence of arbitrary permutations of non-zero elements $\lambda = \{\lambda_\alpha\}_{\alpha\in I}$. Let $\{\delta^*_\alpha\}_{\alpha\in I} \subset K^*$ be a system biorthogonal to $\{\delta_\alpha\}_{\alpha\in I}$.

Let K be a reflexive CB-space with an uncountable unconditional basis $\{\delta_\alpha\}_{\alpha\in I}$. It is easily shown that the conjugated space K^* is isometrically isomorphic to the CB-space of systems of scalars $\mu = \{\mu_\alpha\}_{\alpha\in I}$ such that $I(\mu) \in I^a$ with the norm

$$\left\|\{\mu_\alpha\}_{\alpha\in I}\right\| = \sup_{\|\lambda\|=1} \left|\sum_{\alpha\in I} \lambda_\alpha \mu_\alpha\right|.$$

The system $\{\delta^*_\alpha\}_{\alpha\in I}$ forms an uncountable unconditional basis for K^* (see [28]).

Example 2.1. Let $l_p(I)$, $1 \le p < +\infty$, be a set of systems of scalars $\lambda = \{\lambda_\alpha\}_{\alpha\in I}$ such that $I(\lambda) \in I^a$ and $\sum_{\alpha\in I(\lambda)} |\lambda_\alpha|^p < +\infty$. $l_p(I)$ is nonseparable CB-space with the norm

$$\|\lambda\| = \left(\sum_{\alpha\in I(\lambda)} |\lambda_\alpha|^p\right)^{\frac{1}{p}}, \qquad \lambda = \{\lambda_\alpha\}_{\alpha\in I} \in l_p(I).$$

3. Main Results

The next definition is a generalization of the corresponding concept of Bessel sequence given in [3, 8].

Definition 3.1. The system $\{x^*_\alpha\}_{\alpha\in I}$ is called K-Besselian in X, if for every $x \in X$ the condition $\{x^*_\alpha(x)\}_{\alpha\in I} \in K$ holds.

We have the following criteria of uncountable K-Besselianness of a system.

Theorem 3.2. *Let K be a CB-space. In order for system $\{x^*_\alpha\}_{\alpha\in I}$ to be K-Besselian in X, it is necessary and sufficient that there exists the operator $S \in L(X, K)$ such that $S^*(\delta^*_\alpha) = x^*_\alpha$ for every $\alpha \in I$.*

Proof. **Necessity.** Let the system $\{x^*_\alpha\}_{\alpha\in I}$ be K-Besselian in X. Then $\forall x \in X$ we have $\{x^*_\alpha(x)\}_{\alpha\in I} \in K$, the series $\sum_{\alpha\in I} x^*_\alpha(x)\delta_\alpha$ converges unconditionally to $\{x^*_\alpha(x)\}_{\alpha\in I}$ for every $x \in X$. Consider the linear operator $S : X \to K$ defined by the formula

$$S(x) = \sum_{\alpha\in I} x^*_\alpha(x)\delta_\alpha, \quad \forall x \in X.$$

Let's show that the operator S is bounded. For each $\omega \in I^a$ define the linear operator $S_\omega : X \to K$ by the equality $S_\omega(x) = \sum\limits_{\alpha\in\omega} x^*_\alpha(x)\delta_\alpha$, $\forall x \in X$. As is well known [8], the operator S_ω is bounded. For each $x \in X$ from

$$\begin{aligned} S_\omega(x) &= \sum_{\alpha\in\omega} x^*_\alpha(x)\delta_\alpha \\ &= \frac{1}{2}\left(\sum_{\alpha\in I(x)} x^*_\alpha(x)\delta_\alpha + \sum_{\alpha\in I(x)} \varepsilon_\alpha x^*_\alpha(x)\delta_\alpha\right), \end{aligned}$$

we obtain

$$\|S_\omega(x)\|_K \le \frac{3}{2}\left\|\{x^*_\alpha(x)\}_{\alpha\in I(x)}\right\|_K,$$

where

$$\varepsilon_\alpha = \begin{cases} 1, & \alpha \in \omega \\ -1, & \alpha \notin \omega. \end{cases}$$

Then, by Banach-Steinhaus theorem, the system $\{S_\omega\}_{\omega\in I_a}$ is bounded, i.e. $\exists M > 0$ such that $\|S_\omega\| \le M$. Consequently,

$$\|S(x)\|_K \le \sup_{\omega\in I^a} \|S_\omega(x)\|_K \le M\,\|x\|_X, \quad \forall x \in X.$$

On the other hand, for each $x \in X$ the equality

$$\delta^*_\alpha(S(x)) = x^*_\alpha(x)$$

holds, i.e. $S^*\delta^*_\alpha = x^*_\alpha$ for every $\alpha \in I$.

Sufficiency. Let there exist the operator $S \in L(X, K)$ such that $S^*\delta^*_\alpha = x^*_\alpha$, for every $\alpha \in I$. We take $x \in X$. Assume that $S(x) = \{\lambda_\alpha\}_{\alpha\in I}$. Then for each $x \in X$ we have

$$\begin{aligned} x^*_\alpha(x) &= S^*(\delta^*_\alpha)(x) \\ &= \delta^*_\alpha(S(x)) \\ &= \lambda_\alpha. \end{aligned}$$

Thus, $\forall x \in X$, $\{x^*_\alpha(x)\}_{\alpha\in I} \in K$, i.e. $\{x^*_\alpha\}_{\alpha\in I}$ is a K-Besselian system in X. □

In the case where $X = H$ is a separable Hilbert space and $K = l_2$ and $\{x^*_\alpha\}_{\alpha\in N}$ is complete minimal sequence in H, from the proved theorem we obtain Theorem 1.2.

Corollary 3.3. *Let K be a CB-space, and the system $\{x^*_\alpha\}_{\alpha\in I}$ be uncountably K-Besselian in X. Then there exists $B>0$ such that*

$$\left\|\{x^*_\alpha(x)\}_{\alpha\in I}\right\|_K \le B\left\|x\right\|_X, \quad \forall x\in X. \tag{3.1}$$

Proof. In fact, if the system $\{x^*_\alpha\}_{\alpha\in I}$ is K-Besselian in X, then by virtue of Theorem 3.2, the operator $S: X\to K$, defined by the formula $S(x) = \{x^*_\alpha(x)\}_{\alpha\in I}$, is bounded, i.e. there exists $B>0$ such that $\|S\|\le B$. Hence, $\forall x\in X$ we have

$$\begin{aligned}\left\|\{x^*_\alpha(x)\}_{\alpha\in I}\right\|_K &= \left\|S(x)\right\|_K \\ &\le B\left\|x\right\|_X.\end{aligned}$$

□

Theorem 3.4. *Let K be a reflexive CB-space with an uncountable unconditional basis $\{\delta_\alpha\}_{\alpha\in I}$, and X be reflexive. Then, in order for the system $\{x^*_\alpha\}_{\alpha\in I}$ to be K-Bessel in X, it is necessary and sufficient that there exists $B>0$ such that for every $\mu\in K^*$, $I(\mu)\in I_0$ the relation*

$$\left\|\sum_{\alpha\in I}\mu_\alpha x^*_\alpha\right\| \le B\left\|\{\mu_\alpha\}_{\alpha\in I}\right\|_{K^*} \tag{3.2}$$

holds.

Proof. **Necessity.** Let the system $\{x^*_\alpha\}_{\alpha\in I}$ be K-Bessel in X. Take an arbitrary $\mu\in K^*$, $I(\mu)\in I_0$. Taking (3.1) into account, we obtain

$$\begin{aligned}\left\|\sum_{\alpha\in I}\mu_\alpha x^*_\alpha\right\|_{X^*} &= \sup_{\|x\|=1}\left|\sum_{\alpha\in I}\mu_\alpha x^*_\alpha(x)\right| \\ &= \sup_{\|x\|=1}\left|\left(\{\mu_\alpha\}_{\alpha\in I}, \{x^*_\alpha(x)\}_{\alpha\in I}\right)\right| \\ &\le B\left\|\{\mu_\alpha\}_{\alpha\in I}\right\|_{K^*}.\end{aligned}$$

Sufficiency. Assume that (3.2) holds for every $\mu\in K^*$, $I(\mu)\in I_0$. Consider arbitrary $\mu = \{\mu_\alpha\}_{\alpha\in I}\in K^*$. Since K is reflexive for $\mu = \{\mu_\alpha\}_{\alpha\in I}$, the following expansion $\mu = \sum\limits_{\alpha\in I}\mu_\alpha\delta^*_\alpha = \sum\limits_{i=1}^{\infty}\mu_{\alpha_i}\delta^*_{\alpha_i}$ holds, where $\{\alpha_i\}_{i\in N}$ is enumerate the elements of the set $I(\mu)$. Consequently, $\forall\varepsilon>0$ $\exists i_0$ $\forall i\ge i_0$ $\forall m\in N$

$$\left\|\sum_{k=i+1}^{i+m}\mu_{\alpha_k}\delta^*_{\alpha_k}\right\|_{K^*} < \varepsilon.$$

Then for each $x\in X$: $\|x\|_X = 1$ using (3.2) we get

$$\left|\sum_{k=i+1}^{i+m}\mu_{\alpha_k}x^*_{\alpha_k}(x)\right| \le \left\|\sum_{k=i+1}^{i+m}\mu_{\alpha_k}x^*_{\alpha_k}\right\|_{X^*}$$

$$\le B \left\| \sum_{k=i+1}^{i+m} \mu_{\alpha_k} \delta^*_{\alpha_k} \right\|_{X^*}$$
$$< B\varepsilon,$$

i.e. the series $\sum_{i=1}^{+\infty} \mu_{\alpha_i} x^*_{\alpha_i}(x)$ is convergent. Thus, for every $x \in X$ the series $\sum_{\alpha\in I} \mu_\alpha x^*_\alpha(x)$ is unconditionally convergent, and so $\{x^*_\alpha(x)\}_{\alpha\in I} \in K$, i.e. the system $\{x^*_\alpha\}_{\alpha\in I}$ is K-Besselian in X. □

The following definition generalizes the concept of Hilbert system in Hilbert and Banach spaces (see [3, 8]).

Definition 3.5. The system $\{x^*_\alpha\}_{\alpha\in I}$is called K-Hilbertian in X, if for every $\lambda = \{\lambda_\alpha\}_{\alpha\in I} \in K$ there exists $x \in X$ such that $\lambda = \{x^*_\alpha(x)\}_{\alpha\in I}$.

The following theorem is true.

Theorem 3.6. *Let K be a CB-space. In order for the system $\{x^*_\alpha\}_{\alpha\in I}$ to be K-Hilbertian in X, it is sufficient and, if $\{x^*_\alpha\}_{\alpha\in I}$ is complete in X^*, necessary that there exists the operator $R \in L(K, X)$ such that $R^*(x^*_\alpha) = \delta^*_\alpha$ for every $\alpha \in I$.*

Proof. **Necessity.** Let $\{x^*_\alpha\}_{\alpha\in I}$ be complete in X^* and K-Hilbertian in X. Then for every $\lambda \in K$ there exists $x \in X$ such that $\{x^*_\alpha(x)\}_{\alpha\in I} = \lambda$. Note that, due to the completeness of $\{x^*_\alpha\}_{\alpha\in I}$, such an element is unique. Consequently, the linear operator $R : K \to X$ is defined by the equality $R(\lambda) = x$, where $\{x^*_\alpha(x)\}_{\alpha\in I} = \lambda$. Let's show the closedness of this operator. Assume that the sequence $\left\{\lambda^{(n)}\right\}_{n\in N} \subset K$ converges to λ, and the sequence $\left\{R(\lambda^{(n)})\right\}_{n\in N} \subset X$ converges to x. Let $\lambda^{(n)} = \left\{\lambda^{(n)}_\alpha\right\}_{\alpha\in I}$, $\lambda = \{\lambda_\alpha\}_{\alpha\in I}$ and $R(\lambda^{(n)}) = x_n$. Take $\alpha \in I$. We have

$$\left|\lambda_\alpha - \lambda^{(n)}_\alpha\right| = \left|\delta^*_\alpha(\lambda - \lambda^{(n)})\right|$$
$$\le \|\delta^*_\alpha\|_{K^*} \left\|\lambda - \lambda^{(n)}\right\|_K .$$

Hence, from $\lim_{n\to\infty} \lambda^{(n)} = \lambda$ it follows that $\lim_{n\to\infty} \lambda^{(n)}_\alpha = \lambda_\alpha$. From $\lim_{n\to\infty} x_n = x$, due to the continuity of x^*_α, we obtain $\lim_{n\to\infty} x^*_\alpha(x_n) = x^*_\alpha(x)$. As $\lambda^{(n)}_\alpha = x^*_\alpha(x_n)$, we have $\lambda_\alpha = x^*_\alpha(x)$, i.e. $R(\lambda) = x$. So R is a closed linear operator. By closed graph theorem, the operator R is bounded. Further, for every $\lambda \in K$ we have

$$\delta^*_\alpha(\lambda) = x^*_\alpha(R(\lambda))$$
$$= R^*(x^*_\alpha)(\lambda),$$

i.e. $R^*(x_\alpha^*) = \delta_\alpha^*$ for every $\alpha \in I$.

Sufficiency. Let there exist the operator $R \in L(K, X)$ such that $R^*(x_\alpha^*) = \delta_\alpha^*$ for every $\alpha \in I$. Take any $\lambda \in K$. Denote $R(\lambda) = x$. Then for every $\alpha \in I$ we have

$$\begin{aligned} x_\alpha^*(x) &= x_\alpha^*(R(\lambda)) \\ &= R^*(x_\alpha^*)(\lambda) \\ &= \delta_\alpha^*(\lambda) \\ &= \lambda_\alpha. \end{aligned}$$

Thus, $\{x_\alpha^*(x)\}_{\alpha\in I} = \lambda$, i.e. $\{x_\alpha^*\}_{\alpha\in I}$ is K-Hilbertian in X. □

In the case where $X = H$ is separable Hilbert space and $K = l_2$, and $\{x_\alpha^*\}$ is complete minimal sequence in H, from the proved theorem we obtain Theorem 1.3.

K-Hilbertianness criterion for a system implies the following statements.

Corollary 3.7. *Let K be a CB-space, and the system $\{x_\alpha^*\}_{\alpha\in I}$ be K-Hilbertian in X and complete in X^*. Then there exists $A > 0$ such that for each $x \in X$ satisfying the condition $\{x_\alpha^*(x)\}_{\alpha\in I} \in K$ the inequality*

$$A\,\|x\|_X \le \left\|\{x_\alpha^*(x)\}_{\alpha\in I}\right\|_K \tag{3.3}$$

holds.

Proof. In fact, if the system $\{x_\alpha^*\}_{\alpha\in I}$ is K-Hilbertian in X and complete in X^*, then by Theorem 3.6 there exists the linear bounded operator $R : K \to X$ such that for every $\lambda \in K$ from $R(\lambda) = x$ we get $\lambda = \{x_\alpha^*(x)\}_{\alpha\in I}$. Let $A > 0$ be such that $\|R\| \le \frac{1}{A}$. Then for each $x \in X$ such that $\{x_\alpha^*(x)\}_{\alpha\in I} \in K$ we have

$$\begin{aligned} A\,\|x\|_X &= A\left\|R\left(\{x_\alpha^*(x)\}_{\alpha\in I}\right)\right\|_X \\ &\le \left\|\{x_\alpha^*(x)\}_{\alpha\in I}\right\|_K. \end{aligned}$$

□

Corollary 3.8. *Let K be a CB-space, and the system $\{x_\alpha^*\}_{\alpha\in I}$ be K-Hilbertian in X and complete in X^*. Then the system $\{x_\alpha^*\}_{\alpha\in I}$ is minimal.*

Proof. Let $\{x_\alpha^*\}_{\alpha\in I}$ be K-Hilbertian in X and complete in X^*. Assume the contrary. Then there exists $\alpha_0 \in I$ such that $x_{\alpha_0}^* \in \overline{span}\,\{x_\alpha^*\}_{\alpha\neq\alpha_0}$. From here it follows that for each $\varepsilon > 0$ there exists $x^* \in span\,\{x_\alpha^*\}_{\alpha\neq\alpha_0}$ such that

$$\left\|x_{\alpha_0}^* - x^*\right\|_{X^*} < \varepsilon.$$

By Theorem 3.6 there exists the linear bounded operator $R : K \to X$ such that $R^*(x^*_\alpha) = \delta^*_\alpha$ for every $\alpha \in I$. Assume that $\varepsilon = \|R(\delta_{\alpha_0})\|^{-1}$. It is clear that $R^*(x^*)(\delta_{\alpha_0}) = 0$. Consequently,

$$\begin{aligned}
1 &= \delta^*_{\alpha_0}(\delta_{\alpha_0}) \\
&= (R^*(x^*_{\alpha_0}) - R^*(x^*))(\delta_{\alpha_0}) \\
&= (x^*_{\alpha_0} - x^*)(R(\delta_{\alpha_0})) \\
&\le \left\|x^*_{\alpha_0} - x^*\right\|_{X^*} \|R(\delta_{\alpha_0})\|_X \\
&< \varepsilon \|R(\delta_{\alpha_0})\| \\
&= 1.
\end{aligned}$$

The obtained contradiction proves that $x^*_{\alpha_0} \notin \overline{span}\,\{x^*_\alpha\}_{\alpha \neq \alpha_0}$, i.e. the system $\{x^*_\alpha\}_{\alpha\in I}$ is minimal. Note that the system $\{x_\alpha\}_{\alpha\in I}$, where $x_\alpha = R(\delta_\alpha)$, is conjugate to the system $\{x^*_\alpha\}_{\alpha\in I}$. Indeed, for every $\alpha, \beta \in I$ we have

$$\begin{aligned}
\delta_{\alpha\beta} &= \delta^*_\alpha(\delta_\beta) \\
&= R^*(x^*_\alpha)(\delta_\beta) \\
&= x^*_\alpha(R(\delta_\beta)) \\
&= x^*_\alpha(x_\beta).
\end{aligned}$$

□

Also, the following theorem is valid.

Theorem 3.9. *Let K be a reflexive CB-space with an uncountable unconditional basis $\{\delta_\alpha\}_{\alpha\in I}$, and X be reflexive. In order for the system $\{x^*_\alpha\}_{\alpha\in I}$ to be K-Hilbertian in X, it is sufficient and, if the system $\{x^*_\alpha\}_{\alpha\in I}$ is complete in X^*, necessary that there exists $A > 0$ such that for each $\mu \in K^*$, $I(\mu) \in I_0$ the condition*

$$A \left\|\{\mu_\alpha\}_{\alpha\in I}\right\|_{K^*} \le \left\|\sum_{\alpha\in I} \mu_\alpha x^*_\alpha\right\|_{X^*} \tag{3.4}$$

holds.

Proof. **Necessity.** Let $\{x^*_\alpha\}_{\alpha\in I}$ be complete in X^* and K-Hilbertian in X. Let's take an arbitrary $\mu \in K^*$, $I(\mu) \in I_0$. By reflexivity of K there exists $\lambda = \{\lambda_\alpha\}_{\alpha\in I} \in K$ such that $\|\lambda\|_K = 1$ and $\mu(\lambda) = \|\mu\|_{K^*}$. Since $\{x^*_\alpha\}_{\alpha\in I}$ is K-Hilbertian in X, there exists $x \in X$: $x^*_\alpha(x) = \lambda_\alpha$. According to Corollary 3.7, the inequality (3.3) is true. Then

$$\begin{aligned}
\left\|\{\mu_\alpha\}_{\alpha\in I}\right\|_{K^*} &= \mu(\lambda) \\
&= \sum_{\alpha\in J_0} \mu_\alpha x^*_\alpha(x)
\end{aligned}$$

$$\leq \|x\|_X \left\| \sum_{\alpha \in I} \mu_\alpha x_\alpha^* \right\|_{X^*}$$
$$\leq \frac{1}{A} \left\| \sum_{\alpha \in I} \mu_\alpha x_\alpha^* \right\|_{X^*}.$$

Sufficiency. Let there exist $A > 0$ such that the condition (3.4) is satisfied. We take an arbitrary $\lambda = \{\lambda_\alpha\}_{\alpha \in I} \in K$. Let's define on *span* $\{x_\alpha^*\}_{\alpha \in I}$ a linear functional φ by the formula:

$$\varphi\left(\sum_{\alpha \in J} \mu_\alpha x_\alpha^*\right) = \sum_{\alpha \in J} \mu_\alpha \lambda_\alpha.$$

Obviously, a fact that a functional is well defined follows from inequality (3.4). Using (3.4), we get

$$\left|\varphi\left(\sum_{\alpha \in J} \mu_\alpha x_\alpha^*\right)\right| = \left|\sum_{\alpha \in J} \mu_\alpha \lambda_\alpha\right|$$
$$\leq \left\|\{\mu_\alpha\}_{\alpha \in J}\right\|_{K^*} \|\lambda\|_K$$
$$\leq \frac{1}{A} \|\lambda\|_K \left\| \sum_{\alpha \in J} \mu_\alpha x_\alpha^* \right\|_{X^*},$$

i.e. the functional φ is bounded on *span* $\{x_\alpha^*\}_{\alpha \in I}$ and $\|\varphi\| \leq \frac{1}{A} \|\lambda\|_K$. By the Hahn-Banach theorem, we continue φ to the linear continuous functional on the whole X^* preserving the norm. We denote this functional also by φ. Thus $\varphi \in X^{**}$ and

$$|\varphi(f)| \leq \frac{1}{A} \|\lambda\|_K \|f\|_{X^*}, \quad f \in X^*.$$

By the reflexivity of X there exists $x \in X$ such that $\varphi(f) = f(x)$, $f \in X^*$ and $\|\varphi\| = \|x\|_X$. It is clear that

$$x_\alpha^*(x) = \varphi(x_\alpha^*)$$
$$= \lambda_\alpha.$$

Consequently, the system $\{x_\alpha^*\}_{\alpha \in I}$ is K-Hilbertian in X. □

The following theorem establishes relation between the K-Besselian and K-Hilbertian systems.

Theorem 3.10. *Let X be a reflexive space and K be a reflexive CB-space. In order for the system $\{x_\alpha^*\}_{\alpha \in I}$ to be K-Hilbertian in X, it is sufficient and, if $\{x_\alpha^*\}_{\alpha \in I}$ complete in X^*, necessary that the following conditions hold:*

1) *$\{x_\alpha^*\}_{\alpha \in I}$ is minimal in X^*;*

2) $\{x_\alpha\}_{\alpha\in I}$ *is* K^**-Besselian in* X^**, where* $\{x_\alpha\}_{\alpha\in I}$ *is a system conjugate to* $\{x^*_\alpha\}_{\alpha\in I}$.

Proof. **Necessity.** Let the system $\{x^*_\alpha\}_{\alpha\in I}$ be complete in X^* and K-Hilbertian in X. By Corollary 3.8, $\{x^*_\alpha\}_{\alpha\in I}$ is minimal in X^*. Therefore, $\{x^*_\alpha\}_{\alpha\in I}$ has a conjugate system $\{x_\alpha\}_{\alpha\in I}$. According to Theorem 3.6 there exists the operator $R \in L(K, X)$ such that $R^*(x^*_\alpha) = \delta^*_\alpha$ for every $\alpha \in I$. As the space K is reflexive, the system $\{\delta^*_\alpha\}_{\alpha\in I}$ forms an uncountable unconditional basis for K^* (see [25], Corollary 2.4). On the other hand, $x_\alpha = R(\delta_\alpha)$, $\forall\alpha \in I$, is valid. Then it follows from Theorem 3.2 that the system $\{x_\alpha\}_{\alpha\in I}$ is K^*-Besselian in X^*.

Sufficiency. Let the conditions (1) and (2) hold. By Theorem 3.2, there exists the operator $S \in L(K, X)$ such that $x_\alpha = S(\delta_\alpha)$. For each $\lambda \in K$ the series $\sum_{\alpha\in I} \lambda_\alpha x_\alpha$ converges unconditionally. In fact, applying the operator S to both sides of the equality $\lambda = \sum_{\alpha\in I} \lambda_\alpha \delta_\alpha$, we obtain

$$S(\lambda) = \sum_{\alpha\in I} \lambda_\alpha S(\delta_\alpha) = \sum_{\alpha\in I} \lambda_\alpha x_\alpha.$$

Assume that $x = \sum_{\alpha\in I} \lambda_\alpha x_\alpha$. Then $x^*_\alpha(x) = \lambda_\alpha$. Thus, $\{x^*_\alpha\}_{\alpha\in I}$ is K-Hilbertian in X. □

Now let's consider the case where K is an space of system of coefficients of an uncountable unconditional basis of some nonseparable Banach space.

Let Y be a nonseparable Banach space with an uncountable unconditional basis $\varphi = \{\varphi_\alpha\}_{\alpha\in I}$. Denote by K_φ an space of system of coefficients of the basis $\varphi = \{\varphi_\alpha\}_{\alpha\in I}$, i.e. K_φ consists of all systems of scalars $\lambda = \{\lambda_\alpha\}_{\alpha\in I}$, such that $I(\lambda) \in I^a$ and the series $\sum_{\alpha\in I} \lambda_\alpha \varphi_\alpha$ is convergent unconditionally. K_φ is a nonseparable Banach space (see [28]) with the norm

$$\|\lambda\|_{K_\varphi} = \sup_{J\in I_0} \left\| \sum_{\alpha\in J} \lambda_\alpha \varphi_\alpha \right\|_X, \quad \lambda = \{\lambda_\alpha\}_{\alpha\in I} \in K.$$

Then the operator $T : K_\varphi \to Y$ defined by $T(\lambda) = y$ is an isomorphism, where $y = \sum_{\alpha\in I} \lambda_\alpha \varphi_\alpha$. Let $\{\varphi^*_\alpha\}_{\alpha\in I}$ be a system conjugate to the uncountable unconditional basis $\{\varphi_\alpha\}_{\alpha\in I}$.

The following theorem is true.

Theorem 3.11. *In order for the system $\{x^*_\alpha\}_{\alpha\in I}$ to be K_φ-Besselian in X, it is necessary and sufficient that there exists the operator $S \in L(X, Y)$ such that $S^*(\varphi^*_\alpha) = x^*_\alpha$ for every $\alpha \in I$.*

Proof. **Necessity.** Let $\{x^*_\alpha\}_{\alpha\in I}$ be K_φ-Besselian in X. Consider the operator $S : X \to Y$ defined by the formula

$$S(x) = \sum_{\alpha\in I} x^*_\alpha(x)\varphi_\alpha, \quad \forall x \in X.$$

It is clear that the canonical system $\{\delta_\alpha\}_{\alpha\in I}$ forms an uncountable unconditional basis for in K_φ. Take an arbitrary $\lambda \in K_\varphi$. Then by Theorem 3.2, there exists the operator $S_0 \in L(X, K_\varphi)$ such that $S_0(x) = \{x^*_\alpha(x)\}_{\alpha\in I}$, $x \in X$. We have

$$\begin{aligned} TS_0(x) &= \sum_{\alpha\in I} x^*_\alpha(x)\varphi_\alpha \\ &= S(x). \end{aligned}$$

From here it directly follows that $S = TS_0$. As a result, we obtain $S \in L(X, Y)$. Further, for each $x \in X$ we have

$$\begin{aligned} x^*_\alpha(x) &= \varphi^*_\alpha(S(x)) \\ &= S^*(\varphi^*_\alpha)(x), \quad \forall \alpha \in I, \end{aligned}$$

i.e. $S^*(\varphi^*_\alpha) = x^*_\alpha$, $\forall \alpha \in I$.

Sufficiency. Let there exist the operator $S \in L(X, Y)$ such that $S^*(\varphi^*_\alpha) = x^*_\alpha$ for every $\alpha \in I$. Take an arbitrary $x \in X$. Denote $S(x) = \lambda$. Assume that $\lambda = \sum\limits_{\alpha\in I} \lambda_\alpha\varphi_\alpha$. Hence, taking into account the relation $S^*(\varphi^*_\alpha) = x^*_\alpha$, $\forall \alpha \in I$, we have

$$\begin{aligned} x^*_\alpha(x) &= S^*(\varphi^*_\alpha)(x) \\ &= \varphi^*_\alpha(S(x)) \\ &= \lambda_\alpha. \end{aligned}$$

Thus, $\{x^*_\alpha\}_{\alpha\in I}$ is K_φ-Besselian in X. □

Theorem 3.12. *Let the system $\varphi = \{\varphi_\alpha\}_{\alpha\in I}$ form an uncountable unconditional basis for X with the space of system of coefficients K_φ. In order for the system $\{x^*_\alpha\}_{\alpha\in I}$ to be K_φ-Besselian in X, it is necessary and sufficient that the linear operator $A : K_\varphi \to K_\varphi$ defined by the formula $A(\lambda) = \left\{\sum\limits_{\alpha\in I} x^*_\beta(\varphi_\alpha)\lambda_\alpha\right\}_{\beta\in I}$, $\lambda = \{\lambda_\alpha\}_{\alpha\in I} \in K_\varphi$, is bounded in K_φ.*

Proof. **Necessity.** Let the system $\{x^*_\alpha\}_{\alpha\in I}$ be K_φ-Besselian in X. Then Theorem 3.11 implies the boundedness of the linear operator $S : X \to X$ defined by the formula $S(x) = \sum_{\alpha\in I} x^*_\alpha(x)\varphi_\alpha$, $\forall x \in X$, and also $S^*\varphi^*_\alpha = x^*_\alpha$ for every $\alpha \in I$. Consider the operator $T^{-1}ST$, where T is an isomorphism of the spaces K_φ and X. For each $x \in X$ we have $T^{-1}(x) = \{\varphi^*_\alpha(x)\}_{\alpha\in I}$. For every $\lambda \in K$ we have

$$\begin{aligned}
T^{-1}ST(\lambda) &= \{\varphi^*_\alpha(S(T(\lambda)))\}_{\alpha\in I} \\
&= \{S^*(\varphi^*_\alpha)(T(\lambda))\}_{\alpha\in I} \\
&= \left\{x^*_\beta(T(\lambda))\right\}_{\beta\in I} \\
&= \left\{x^*_\beta(\sum_{\alpha\in I} x^*_\alpha(x)\varphi_\alpha)\right\}_{\beta\in I} \\
&= \left\{\sum_{\alpha\in I} x^*_\beta(\varphi_\alpha)\lambda_\alpha\right\}_{\beta\in I} \\
&= A(\lambda).
\end{aligned}$$

Hence, we get $T^{-1}ST = A$ and $A \in L(K_\varphi)$.

Sufficiency. Let the linear operator $A : K_\varphi \to K_\varphi$ be such that $A(\lambda) = \left\{\sum_{\alpha\in I} x^*_\beta(\varphi_\alpha)\lambda_\alpha\right\}_{\beta\in I}$, $\lambda = \{\lambda_\alpha\}_{\alpha\in I} \in K_\varphi$, is bounded. Assume that $S = TAT^{-1}$. Then $S \in L(X)$. For each $x \in X$ we obtain

$$\begin{aligned}
S(x) &= TA(T^{-1}x) \\
&= TA\left(\{\varphi^*_\alpha(x)\}_{\alpha\in I}\right) \\
&= T\left(\left\{\sum_{\alpha\in I} x^*_\beta(\varphi_\alpha)\varphi^*_\alpha(x)\right\}_{\beta\in I}\right) \\
&= T\left(\left\{x^*_\beta\left(\sum_{\alpha\in I} \varphi^*_\alpha(x)\varphi_\alpha\right)\right\}_{\beta\in I}\right) \\
&= T\left(\left\{x^*_\beta(x)\right\}_{\beta\in I}\right) \\
&= \sum_{\alpha\in I} x^*_\alpha(x)\varphi_\alpha.
\end{aligned}$$

Thus, $S(x) = \sum_{\alpha\in I} x^*_\alpha(x)\varphi_\alpha$, $\forall x \in X$, and, consequently, $S^*(\varphi^*_\alpha) = x^*_\alpha$ for every $\alpha \in I$. Applying Theorem 3.11 finishes the proof. □

The next theorem provides the K_φ-Hilbertianness criterion for a system.

Theorem 3.13. *In order for the system $\{x_\alpha^*\}_{\alpha\in I}$ to be K_φ-Hilbertian in X, it is sufficient and, if $\{x_\alpha^*\}_{\alpha\in I}$ is complete in X^*, necessary that there exists the operator $R \in L(Y,X)$ such that $R^*(x_\alpha^*) = \varphi_\alpha^*$ for every $\alpha \in I$.*

Proof. **Necessity.** Let $\{x_\alpha^*\}_{\alpha\in I}$ be K_φ-Hilbertian in X. Take an arbitrary $\lambda \in K_\varphi$. Then by Theorem 3.6 there exists the operator $R_0 \in L(K_\varphi, X)$ such that $R_0(\lambda) = x$, where $x_\alpha^*(x) = \lambda_\alpha$. Assume that $R = R_0T^{-1}$. Then $R \in L(Y,X)$ and for each $y \in Y$ we obtain

$$\begin{aligned} R^*(x_\alpha^*)(y) &= x_\alpha^*(R(y)) \\ &= x_\alpha^*(R_0T^{-1}(y)) \\ &= x_\alpha^*(R_0(\{\varphi_\beta^*(y)\}_{\beta\in I})) \\ &= \varphi_\alpha^*(y), \quad \forall \alpha \in I. \end{aligned}$$

Therefore, $R^*(x_\alpha^*) = \varphi_\alpha^*$ for every $\alpha \in I$.

Sufficiency. Let there exist the operator $R \in L(Y,X)$ such that $R^*(x_\alpha^*) = \varphi_\alpha^*$ for every $\alpha \in I$. Take an arbitrary $\lambda \in K$. Let $x = RT(\lambda)$. For every $\alpha \in I$ we have

$$\begin{aligned} x_\alpha^*(x) &= x_\alpha^*(RT(\lambda)) \\ &= R^*(x_\alpha^*)(T(\lambda)) \\ &= \varphi_\alpha^*(T(\lambda)) \\ &= \lambda_\alpha, \end{aligned}$$

i.e. $\{x_\alpha^*\}_{\alpha\in I}$ is K_φ-Hilbertian in X. □

This theorem, combined with Theorems 3.10 and 3.12, implies

Theorem 3.14. *Let X be a reflexive space and $\varphi = \{\varphi_\alpha\}_{\alpha\in I}$ be an uncountable unconditional basis for X with the space of systems of coefficients K_φ. In order for the system $\{x_\alpha^*\}_{\alpha\in I}$ to be K_φ-Hilbertian in X, it is sufficient and, if $\{x_\alpha^*\}_{\alpha\in I}$ is complete in X^*, necessary that the following conditions hold:*

1) *$\{x_\alpha^*\}_{\alpha\in I}$ is minimal in X^*;*
2) *the linear operator $A : K_\varphi^* \to K_\varphi^*$ defined by the formula $A(\mu) = \left\{\sum_{\alpha\in I} \varphi_\alpha^*(x_\beta)\mu_\alpha\right\}_{\beta\in I}$ is bounded, where $\{x_\alpha\}_{\alpha\in I}$ is a system conjugate to $\{x_\alpha^*\}_{\alpha\in I}$.*

The concept below is a generalization of an uncountable frame in nonseparable Hilbert space [9].

Definition 3.15. The system $\{x^*_\alpha\}_{\alpha\in I}$ is called an uncountable K-frame in X, if for every $x \in X$, $\{x^*_\alpha(x)\}_{\alpha\in I} \in K$, there exist absolute constants A and B such that

$$A\,\|x\|_X \le \left\|\{x^*_\alpha(x)\}_{\alpha\in I}\right\|_K \le B\,\|x\|_X\,. \tag{3.5}$$

The numbers A and B are called the lower and upper bound of uncountable K-frame, respectively.

The following theorem is a criterion for uncountable K-frame.

Theorem 3.16. *Let K be a reflexive CB-space with the uncountable unconditional basis $\{\delta_\alpha\}_{\alpha\in I}$. In order for the system $\{x^*_\alpha\}_{\alpha\in I}$ to be a K-frame in X, it is necessary and sufficient that the operator $T : K^* \to X^*$ defined by the formula*

$$T(\mu) = \sum_{\alpha\in I} \mu_\alpha x^*_\alpha, \quad \forall \mu \in K^*, \tag{3.6}$$

is a bounded, surjective operator.

Proof. **Necessity.** Let $\{x^*_\alpha\}_{\alpha\in I}$ form the uncountable K-frame for X with the bounds A and B. Then the operator $U : X \to K$ defined by the formula $U(x) = \{x^*_\alpha(x)\}_{\alpha\in I}$ satisfies the condition $A\,\|x\|_X \le \|U(x)\|_K \le B\,\|x\|_X$, $\forall x \in X$. Therefore, $ImU^* = K^*$ is fulfilled, i.e. the operator U^* is surjective. By Theorem 3.4, it follows that the series $\sum\limits_{\alpha\in I} \mu_\alpha x^*_\alpha$ converge unconditionally for every $\mu \in K^*$, and the operator T defined by the formula (3.6) is bounded. For each $\mu \in K^*$ and $\forall x \in X$ we have

$$\begin{aligned} T(\mu)(x) &= \sum_{\alpha\in I} \mu_\alpha x^*_\alpha(x) \\ &= \mu(U(x)) \\ &= U^*(\mu)(x), \end{aligned} \tag{3.7}$$

i.e. $T = U^*$ and therefore the operator T is surjective.

Sufficiency. Let the operator $T : K^* \to X^*$ defined by the formula (3.6) be bounded and surjective. By Theorem 3.4, the system $\{x^*_\alpha\}_{\alpha\in I}$ is K-Besselian in X, i.e. right-hand side of the inequality (3.5) holds. As $ImT = K^*$, from equality (3.7) it follows that the left-hand side of the inequality (3.5) holds. Thus, $\{x^*_\alpha\}_{\alpha\in I}$ forms uncountable K-frame in X. □

The definition below is the generalization of the concept of K-basis in separable Banach space.

Definition 3.17. An uncountable unconditional basis $\varphi = \{\varphi_\alpha\}_{\alpha\in I}$ for X is called an uncountable unconditional K-basis, if K is an space of system of coefficients of this basis.

The next theorem provides a K-basisity of the minimal system, the conjugate of which is complete K-Besselian and K-Hilbertian.

Theorem 3.18. *Let K be a CB-space, and the system $\{x_\alpha\}_{\alpha\in I}$ from X be minimal in X with a conjugate system $\{x_\alpha^*\}_{\alpha\in I}$. In order for the system $\{x_\alpha\}_{\alpha\in I}$ to be uncountable unconditional K-basis for X, it is necessary and, if $\{x_\alpha^*\}_{\alpha\in I}$ is complete, sufficient that $\{x_\alpha^*\}_{\alpha\in I}$ is both K-Besselian and K-Hilbertian system.*

Proof. **Necessity.** Let $\{x_\alpha\}_{\alpha\in I}$ be uncountable unconditional K-basis for X. Take an arbitrary $x \in X$. The following decomposition holds:

$$x = \sum_{\alpha\in I} x_\alpha^*(x)x_\alpha.$$

As $\{x_\alpha^*(x)\}_{\alpha\in I} \in K$, the system $\{x_\alpha^*\}_{\alpha\in I}$ is K-Besselian in X. Further, let $\lambda \in K$. The series $\sum_{\alpha\in I} \lambda_\alpha x_\alpha$ is convergent unconditionally. Assume that $x = \sum_{\alpha\in I} \lambda_\alpha x_\alpha$. Obviously, $\{x_\alpha^*(x)\}_{\alpha\in I} = \lambda$, i.e.$\{x_\alpha^*\}_{\alpha\in I}$ is K-Hilbertian in X.

Sufficiency. Let the system $\{x_\alpha^*\}_{\alpha\in I}$ be complete in X^*, both K-Besselian and K-Hilbertian in X. Take an arbitrary $x \in X$. Then $\{x_\alpha^*(x)\}_{\alpha\in I} \in K$ and by Theorem 3.2 the linear operator $S : X \to K$ defined by

$$S(x) = \sum_{\alpha\in I} x_\alpha^*(x)\delta_\alpha$$

is bounded. The system $\{x_\alpha^*\}_{\alpha\in I}$ being K-Hilbertian in X and complete in X^* for every $\lambda \in K$ implies that there exists a unique $x \in X$ such that $\lambda = \{x_\alpha^*(x)\}_{\alpha\in I}$. By Theorem 3.6, the linear operator $R : K \to X$ defined by $R(\lambda) = x$ is bounded. We have

$$\begin{aligned} x &= R(\lambda) \\ &= RS(x) \\ &= \sum_{\alpha\in I} x_\alpha^*(x)R(\delta_\alpha) \\ &= \sum_{\alpha\in I} x_\alpha^*(x)x_\alpha. \end{aligned}$$

Thus, the system $\{x_\alpha\}_{\alpha\in I}$ forms an uncountable unconditional K-basis for X. $\square$

This theorem, combined with Corollaries 3.3 and 3.7, implies the following.

Corollary 3.19. *Let K be a CB-space. Let the system $\{x_\alpha\}_{\alpha\in I}$ form an uncountable unconditional K-basis for X and the conjugate system $\{x_\alpha^*\}_{\alpha\in I}$ be complete in X^*. Then there exist $A>0$ and $B>0$ such that*

$$A\left\|x\right\|_K \leq \left\|\{x_\alpha^*(x)\}_{\alpha\in I}\right\|_K \leq B\left\|x\right\|_X, \quad \forall x\in X,$$

i.e. $\{x_\alpha^*\}_{\alpha\in I}$ forms an uncountable K-frame in X.

The next definition is the generalization of the Riesz basis.

Definition 3.20. The system $\{x_\alpha^*\}_{\alpha\in I}$ is called uncountable K^*-Riesz basis for X^*, if $\{x_\alpha^*\}_{\alpha\in I}$ is complete in X^* and there exist the constants $A>0$ and $B>0$ such that

$$A\left\|\{\mu_\alpha\}_{\alpha\in I}\right\|_{K^*} \leq \left\|\sum_{\alpha\in I}\mu_\alpha x_\alpha^*\right\|_{X^*} \leq B\left\|\{\mu_\alpha\}_{\alpha\in I}\right\|_{K^*}, \tag{3.8}$$

for all $\{\mu_\alpha\}_{\alpha\in I}\in K^*$. The constants A and B are called the lower and the upper bound of uncountable K^*-Riesz basis $\{x_\alpha^*\}_{\alpha\in I}$, respectively.

It is easily shown that the system $\{x_\alpha^*\}_{\alpha\in I}$ forms an uncountable K^*-Riesz basis for X^* if and only if the operator $T: K^*\to X^*$ defined by the formula (3.6) is bounded and boundedly invertible.

The following theorem is true.

Theorem 3.21. *Let K be reflexive CB-space with uncountable unconditional basis $\{\delta_\alpha\}_{\alpha\in I}$ and X be reflexive. Then the following conditions are equivalent:*

(1) *$\{x_\alpha^*\}_{\alpha\in I}$ forms uncountable K^*-Riesz basis for X^* with the bounds A and B;*
(2) *$\{x_\alpha^*\}_{\alpha\in I}$ is uncountable K-frame in X with the bounds A and B, and is minimal in X^*;*
(3) *$\{x_\alpha^*\}_{\alpha\in I}$ is complete in X^*, both K-Besselian and K-Hilbertian in X.*

Proof. Let's show the validity of the equivalence of (1) and (2). First, suppose that (1) is satisfied. Consequently, the operator $T: K^*\to X^*$ defined by the formula (3.6) is boundedly invertible. Then by Theorem 3.16, from the inequalities (3.2) and (3.4) it follows that the system $\{x_\alpha^*\}_{\alpha\in I}$ is a K-frame in X with the bounds A and B. From the equality (3.7) it follows that the operator $U: X\to K$ defined by the formula $U(x)=\{x_\alpha^*(x)\}_{\alpha\in I}$, $x\in X$, is boundedly invertible and $U^*=T$. Assume that $U^{-1}(\delta_\alpha)=x_\alpha$, $\alpha\in I$. For each $\alpha,\beta\in I$ we have

$$x_\alpha^*(x_\beta)=x_\alpha^*(U^{-1}(\delta_\beta))$$

$$
\begin{aligned}
&= T^{-1}(x_\alpha^*)(\delta_\beta) \\
&= \delta_\alpha^*(\delta_\beta) \\
&= \delta_{\alpha\beta},
\end{aligned}
$$

i.e. systems $\{x_\alpha^*\}_{\alpha\in I}$ and $\{x_\alpha\}_{\alpha\in I}$ are biorthogonal. Consequently, the system $\{x_\alpha^*\}_{\alpha\in I}$ is minimal in X^*, and therefore (2) holds. Conversely, suppose that (2) is true. Then by Theorem 3.16 the operator T, defined by (3.6), is bounded and surjective. From the minimality of the system $\{x_\alpha^*\}_{\alpha\in I}$ in X^* it follows that $\ker T = \{0\}$. Therefore, the operator T is boundedly invertible. Thus, $\{x_\alpha^*\}_{\alpha\in I}$ forms an uncountable K^*-Riesz basis for X^* with the bounds A and B, i.e. (1) holds. From Theorems 3.4 and 3.9 it follows that the conditions (1) and (3) are equivalent. □

Assuming $X = H$ is a separable Hilbert space and $K = l_2$, we obtain Theorem 1.4.

Example 3.22. Let $e_\alpha(t) = e^{i\alpha t}$, $t \in R$. Assume that $V = span\,\{e_\alpha\}_{\alpha\in R}$. Obviously, $\forall x \in V\ \exists M_x$: $|x(t)| \le M_x$. Therefore $\forall p \in [1, +\infty)$ there exists

$$\overline{\lim_{T\to\infty}} \frac{1}{2T} \int_{-T}^{T} |x(t)|^p \, dt.$$

Denote by $L_p^V(R)$, $1 \le p < +\infty$, the completion of the space V_p with the norm

$$\|x\|_p = \overline{\lim_{T\to\infty}} \left(\frac{1}{2T} \int_{-T}^{T} |x(t)|^p \, dt \right)^{\frac{1}{p}}.$$

The space $L_p^V(R)$ is nonseparable (see [28]). The system $\{e_\alpha\}_{\alpha\in R}$ is complete and orthonormal in the space $L_2^V(R)$ with respect to the iner product

$$(x, y)_V = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x(t)\overline{y(t)} dt.$$

It follows from Bessel's inequality that the set $I(x)$ is at most countable for each $x \in L_2^V(R)$. Therefore, Parseval's equality

$$\|x\|_2^2 = \sum_{\alpha\in R} |(x, e_\alpha)_V|^2$$

is true. In the space $L_2^V(R)$, consider the system

$$
\begin{aligned}
x_{\alpha_0} &= e_{\alpha_0}, x_\alpha \\
&= e_\alpha - e_{\alpha-\alpha_0}, \quad \alpha \ne \alpha_0.
\end{aligned}
$$

Let's show that the system $\{x_\alpha\}_{\alpha\in R}$ is $l_2(R)$-Besselian in $L_2^V(R)$, but not minimal in $L_2^V(R)$. Indeed, the set $\{\alpha : (x, x_\alpha)_V \neq 0\}$ is at most countable and by Cauchy-Schwarz inequality we have

$$\begin{aligned}\sum_{\alpha\in R} |(x, x_\alpha)_V|^2 &= |(x, x_{\alpha_0})_V|^2 + \sum_{\alpha\neq\alpha_0} |(x, x_\alpha)_V|^2 \\ &= |(x, e_{\alpha_0})_V|^2 + \sum_{\alpha\neq\alpha_0} |(x, e_\alpha)_V - (x, e_{\alpha-\alpha_0})_V|^2 \\ &\leq |(x, e_{\alpha_0})_V|^2 + 2\sum_{\alpha\neq\alpha_0} |(x, e_\alpha)_V|^2 + 2\sum_{\alpha\neq\alpha_0} |(x, e_{\alpha-\alpha_0})_V|^2 \\ &\leq 2\sum_{\alpha\in R} |(x, e_\alpha)_V|^2 + 2\sum_{\alpha\neq 0} |(x, e_\alpha)_V|^2 \\ &\leq 4\,\|x\|_2^2,\end{aligned}$$

i.e. the system $\{x_\alpha\}_{\alpha\in R}$ is $l_2(R)$-Besselian in $L_2^V(R)$. Let the system $\{x_\alpha\}_{\alpha\in R}$ have a conjugate system $\{\varphi_\alpha\}_{\alpha\in R}$. From $(\varphi_\beta, x_\alpha) = \delta_{\alpha\beta}$ we obtain

$$\begin{aligned}&(\varphi_{\alpha_0}, e_{\alpha_0}) = 1, \\ &(\varphi_{\alpha_0}, e_\alpha) - (\varphi_{\alpha_0}, e_{\alpha-\alpha_0}) = 0, \quad \alpha \neq \alpha_0.\end{aligned}$$

Hence we get $(\varphi_{\alpha_0}, e_{n\alpha_0}) = 1$, $\forall n \in N$, which contradicts the condition $\|\varphi_{\alpha_0}\|_2^2 = \sum\limits_{\alpha\in R} |(\varphi_{\alpha_0}, e_\alpha)_V|^2$.

Example 3.23. Define a functional in V_p by the following equality:

$$e_\alpha^*(x) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} x(t)e^{-i\alpha t}dt.$$

The linearity of e_α^* is obvious. We have

$$\begin{aligned}|e_\alpha^*(x)| &\leq \overline{\lim_{T\to\infty}} \frac{1}{2T}\int_{-T}^{T} |x(t)|\, dt \\ &\leq \overline{\lim_{T\to\infty}} \left(\frac{1}{2T}\int_{-T}^{T} |x(t)|^p\, dt\right)^{\frac{1}{p}},\end{aligned}$$

i.e. $e_\alpha^* \in (V_p)^*$. Extending e_α^* by continuity onto $L_p^V(R)$, we obtain $e_\alpha^* \in (L_p^V(R))^*$.

The system $\{e_\alpha^*\}_{\alpha\in R}$ is $l_p(R)$-Besselian in $L_p^V(R)$ for $p \geq 2$. Indeed, for each $x \in L_p^V(R)$ we have $x \in L_2^V(R)$. As $x \in L_2^V(R)$, it follows from Bessels's inequality that there are no more than a countable number of Fourier coefficients $e_\alpha^*(x) = (x, e_\alpha)_V$ which are different from zero and

$\{e^*_\alpha(x)\}_{\alpha\in R} \in l_2(R)$. Using $l_2(R) \subset l_p(R)$, we obtain $\{e^*_\alpha(x)\}_{\alpha\in R} \in l_p(R)$, i.e. for $p \geq 2$ the system $\{e^*_\alpha\}_{\alpha\in R}$ is $l_p(R)$-Besselian in $L^V_p(R)$.

The system $\{e^*_\alpha\}_{\alpha\in R}$ is $l_p(R)$-Hilbertian in $L^V_p(R)$ for $p \leq 2$. Indeed, take $\lambda = \{\lambda_\alpha\}_{\alpha\in R} \in l_p(R)$. Then from $l_p(R) \subset l_2(R)$ it follows that $\lambda \in l_2(R)$. As $\{e_\alpha\}_{\alpha\in R}$ is an orthonormal system in $L^V_2(R)$, there exists $x \in L^V_2(R)$ such that $e^*_\alpha(x) = (x, e_\alpha)_V = \lambda_\alpha$. From $L^V_2(R) \subset L^V_p(R)$ we obtain $x \in L^V_p(R)$. Hence $\{e_\alpha\}_{\alpha\in R}$ is $l_p(R)$-Hilbertian in $L^V_p(R)$ for $p \leq 2$.

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