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A Review on Some Fuzzy Integral Inequalities

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Abstract. In this paper, we introduce fuzzy measure and fuzzy integral concepts and express some of the fuzzy integral properties. The main purpose of this article is to reviewing of some important mathematical inequalities that have many applications in modeling mathematical problems. Firstly, we prove the related Gauss-Winkler type inequality for fuzzy integrals. Indeed, we prove fuzzy version provided by D. H. Hong. Another the famous mathematical inequality is Minkowski's inequality. It is an important inequality from both mathematical and application points of view. Here, we state a Minkowski type inequality for fuzzy integrals. The established results are based on the classical Minkowski's inequality for integrals. In the continue, we showed that by an example the classical Prékopa-Leindler type inequality is not valid for the Sugeno integral. We proved one version of the Prékopa-Leindler type inequality by adding concave fuzzy measure and quasi-concave fuzzy measure assumptions for the Sugeno integral with different proofs. Also, we obtained a derivation version of the Prékopa-Leindler inequality and illustrated all of the main results by examples. Finally, we investigate the Thunsdorff's inequality for Sugeno integral. By an example, we show that the classical form of this inequality does not hold for the Sugeno integral. Then, by reviewing the initial conditions, we prove two main theorems for this inequality.By checking the special case of the aforementioned Thunsdorff's inequality, we prove Frank-Pick type inequality for the Sugeno integral and illustrate it by an example.

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1. Introduction

Fuzzy meaures and fuzzy integrals, which were originally introduced by Sugeno in 1974 [62], are important analytical methods of measuring uncertain information. Fuzzy integrals or Sugeno integrals have very interesting properties from a mathematical point of view which have been studied by many authors, including Pap [48], Ralescu and Adams [50], Wang and Klir [63] among others. Ralesco and Adams [50] studied several equivalent definitions of fuzzy integrals, while Pap [48] and Wang and Klir [63], provided an overview of fuzzy measure theory. The fuzzy integral for monotone functions was presented in [51]. In fact, fuzzy measures and fuzzy integrals are versatile operators which can be used in different areas. They have a broad use in information fusion, electronic auctions, decision making, and etc. Chen et al. [8] employed fuzzy integral and fuzzy measure to establish a public attitude analysis model. Inequalities play an important role in mathematical analysis [44, 61] as well as in other fields, e.g., those used to estimate Chebyshev's functional, i.e., the difference between the integral of the product and the product of integrals [3], and to evaluate bounds for expectations of order and record statistics [4, 39]. For instance, integral inequalities play a major role in the development of a time scales calculus. The study of inequalities for Sugeno integrals was initiated by Román-Flores et al. [51, 59] and then followed by the authors [12, 22, 29, 33, 48, 56]. $\ddot{O}$zkan et al. [47] obtained H$\ddot{o}$lderâ s inequality, Minkowskiâ s inequality and Jensen's inequality on time scales. Some famous inequalities have been generalized to fuzzy integral. Román-Flores and Chalco-Cano [52] analyzed an interesting type of geometric inequalities for fuzzy integral with some applications to convex geometry. Román-Flores et al. [53, 54] studied a Jensen type inequality and a convolution type inequality for fuzzy integrals. Also, they have investigated a Chebyshev type inequality and a Stolarsky type inequality for fuzzy integrals [37, 55]. In [37], a fuzzy Chebyshev inequality for a special case was obtained which has been generalized by Ouyang et al. [46]. Furthermore, Chybyshev type inequalities for fuzzy integral were proposed in a rather general form by Mesiar and Ouyang [44]. In [56], RomÂ´an-Flores et al. proved a Hardy type inequality for fuzzy integrals. Recently, B. Daraby and et. al. proved related Gauss-Winkler type inequality for fuzzy and pseudo-integrals (see [33]). A Minkowski type inequality for fuzzy and Pseudo-integrals are studied by him and coauthors (see [**?**]). Also Daraby et al. proved a version of Thunsdorff's and Frank-Pick's inequality for the Sugeno integral in [34].

The classical Gauss-Winkler inequality provides the following inequality [6]:

$$\left(\int_0^\infty x^2 f(x)d\mu\right)^2 \le \frac{5}{9}\left(\int_0^\infty f(x)d\mu\right)\left(\int_0^\infty x^4 f(x)d\mu\right), \tag{1.1}$$

where $f : [0,\infty) \to [0,\infty)$ is decreasing. We suppose that all involved integrals exist.

Example 1.1. Let $f(x) = \frac{1}{1+x^6}$. We have

$$\int_0^\infty x^2 f(x)dx = \int_0^\infty \frac{x^2}{1+x^6}dx = \frac{\pi}{6},$$
$$\int_0^\infty f(x)dx = \int_0^\infty \frac{1}{1+x^6}dx = \frac{\pi}{3},$$
$$\int_0^\infty x^4 f(x)dx = \int_0^\infty \frac{x^4}{1+x^6}dx = \frac{\pi}{3}.$$

Finally

$$\left(\frac{\pi}{6}\right)^2 \le \frac{5}{9}\left(\frac{\pi}{3}\right)\left(\frac{\pi}{3}\right) \quad \Rightarrow \quad 0.0277 \le 0.0617.$$

In [42], Dug Hun Hong replaced the condition of decreasing with non-decreasing and reduced the domain of the integrals into the interval $[0, 1]$. Then he showed that the classical Gauss-Winkler inequality is not valid for the Sugeno integrals. In the continue, he only obtained the lowest optimal value for the related Gauss-winkler type inequality for fuzzy version and showed, by an example that, the obtained optimum value in the following Theorems are valid. D. H. Hong showed with an example that the bound is obtained, is optimal. Let's take a look at this example.

Example 1.2 ([42])**.** Let $f(x) = \begin{cases} 0 & x \in [0, 0.4255] \\ 3.1731 & x \in (0.4255, 1] \end{cases}$. A simple calculation shows that

$$\begin{aligned}\left(⨍_0^1 x^2 f(x)d\mu\right)^2 &= \left(⨍_0^1 3.1731x^2 d\mu\right)^2 \\ &= (0.5745)^2 \\ &= 0.3300,\end{aligned}$$

and

$$\begin{aligned}⨍_0^1 f(x)d\mu &= ⨍_0^{0.4255} 0d\mu \vee ⨍_{0.4255}^1 3.1731d\mu \\ &= 0.5745,\end{aligned}$$

and

$$\begin{aligned} \fint_0^1 x^4 f(x) d\mu &= \fint_0^1 3.1731 x^4 d\mu \\ &= 0.4030. \end{aligned}$$

Consequently,

$$\begin{aligned} 0.3300 &= \left(\fint_0^1 x^2 f(x) d\mu \right)^2 \\ &\approx 1.4255 \left(\fint_0^1 f(x) d\mu \right) \left(\fint_0^1 x^4 f(x) d\mu \right) \\ &= 0.3300. \end{aligned}$$

Therfore, the constant 1.4255 is optimal.

The classical Minkowski's inequality was published by Minkowski [45] in his famous book "Geometrie der Zahlen". A proof of Minkowski's inequality as well as several extensions, related results, and interesting geometrical interpretations can be found in [57, 58]. An extension of Minkowski's inequality, which is based on Hölder's inequality, is given in [63]. Applications of Minkowski's inequality have been studied by many authors. For example Özkan et al. [47] applied Minkowski's inequality, Hölder's inequality and Jensen's inequality on time scales. Lu et al. [43] used Minkowski's inequality for fast full search in motion estimation. The classical Minkowski's inequality [45] is as follows:

$$\left(\int_a^b (f(x) + g(x))^s dx \right)^{\frac{1}{s}} \leq \left(\int_a^b f(x)^s dx \right)^{\frac{1}{s}} + \left(\int_a^b g(x)^s dx \right)^{\frac{1}{s}} \tag{1.2}$$

where $1 \leq s < \infty$ and $f, g : [0,1] \to [0, \infty)$ are two non-negative functions.

Note we recall the following inequalities which are the fuzzy versions of Minkowski's inequality at two cases and appears in [1].

Theorem 1.3. *Let $f, g : [0,1] \to [0, \infty)$ be two real-valued functions and let μ be the Lebesgue measure on $\mathbb{R}$. If f, g are both continuous and strictly decreasing functions, then the inequality*

$$\left(\fint_0^1 (f+g)^s d\mu \right)^{\frac{1}{s}} \leq \left(\fint_0^1 f^s d\mu \right)^{\frac{1}{s}} + \left(\fint_0^1 g^s d\mu \right)^{\frac{1}{s}}$$

holds for all $1 \leq s < \infty$.

Theorem 1.4. *Let $f, g : [0,1] \to [0, \infty)$ be two real-valued functions and let μ be the Lebesgue measure on $\mathbb{R}$. If f, g are both continuous and strictly increasing functions, then the inequality*

$$\left(\fint_0^1 (f+g)^s d\mu\right)^{\frac{1}{s}} \le \left(\fint_0^1 f^s d\mu\right)^{\frac{1}{s}} + \left(\fint_0^1 g^s d\mu\right)^{\frac{1}{s}} \tag{1.3}$$

holds for all $1 \le s < \infty$.

The following theorem is pseudo version of Minkowski's inequality and appears in [2].

Theorem 1.5. *Let* $f, g : X \to [0,\infty)$ *be two measurable functions and* $s \in [1,\infty)$. *If an additive generator* $g : [a,b] \to [0,1]$ *of the pseudo-addition* $\oplus$ *and the pseudo-multiplication* $\odot$ *are increasing, then for any* $\sigma - \oplus$*-measure* m *it holds:*

$$\left(\int_X^{\oplus} (f+g)^s d\mu\right)^{\frac{1}{s}} \le \left(\int_X^{\oplus} f^s d\mu\right)^{\frac{1}{s}} + \left(\int_X^{\oplus} g^s d\mu\right)^{\frac{1}{s}}. \tag{1.4}$$

The following theorem shows the new classical version of Minkowski's inequality and appears in [5].

Theorem 1.6. *Let* f *and* g *be positive functions satisfying* $0 < m \le \dfrac{f(x)}{g(x)} \le M, \quad \forall x \in [a,b]$, *we have*

$$\left(\int_a^b f^s(x)dx\right)^{\frac{1}{s}} + \left(\int_a^b g^s(x)dx\right)^{\frac{1}{s}} \le c\left(\int_a^b (f(x)+g(x))^s dx\right)^{\frac{1}{s}}, \tag{1.5}$$

where $1 \le s < \infty$ *and* $c = \dfrac{M(m+1)+(M+1)}{(m+1)(M+1)}$.

In [40], Gardner has presented Prékopa-Leindler inequality as follows.

Theorem 1.7 (Gardner [40])**.** *(Prékopa-Leindler inequality in* $\mathbb{R}^n$*). Let* $0 < \lambda < 1$ *and let* f, g, h *be non-negative integrable functions on* $\mathbb{R}^n$ *satisfying*

$$h\left((1-\lambda)x + \lambda y\right) \ge f(x)^{1-\lambda} g(y)^{\lambda}, \tag{1.6}$$

for all $x, y \in \mathbb{R}^n$. *Then*

$$\int_{\mathbb{R}^n} h(x)dx \ge \left(\int_{\mathbb{R}^n} f(x)dx\right)^{1-\lambda}\left(\int_{\mathbb{R}^n} g(x)dx\right)^{\lambda}.$$

In [9], an equivalent and shorter way of stating this result is presented, showing that for every $f, g : \mathbb{R}^n \to [0,\infty)$ and $\lambda \in (0,1)$ one has

$$\int_{\mathbb{R}^n} \sup_{z=(1-\lambda)x+\lambda y} \left(f(x)^{1-\lambda} g(y)^{\lambda}\right) dz \ge \left(\int_{\mathbb{R}^n} f\right)^{1-\lambda}\left(\int_{\mathbb{R}^n} g\right)^{\lambda}. \tag{1.7}$$

The Prékopa-Leindler inequality is a reverse form of Hölder's inequality and thus some conditions such as (1.6) are therefore necessary for it to hold. Extensions of the Prékopa-Leindler inequality can be used to obtain concavity properties of probability measures generated by densities of well-known distributions. By using the Prékopa-Leindler inequality, one can prove a modified logarithmic Sobolev inequality adapted for all measures on $\mathbb{R}^n$ with a strictly convex and super-linear potential [41]. Also, the Prékopa-Leindler inequality implies that if f is log concave and C is an open convex subset of its support, then the measure μ generated by f and C is also log concave.

All these inequalities for Sugeno integral were motivated by the related inequalities known for Riemann or Lebesgue integral. One of inequalities known for Riemann integral but it was not studied in the framework of Sugeno integral, namely Thunsdorff's inequality see [6], is given as follows:

If f is non-negative and concave on $[a, b]$ and if $0 < r < s$, then

$$\left(\frac{1+s}{b-a}\int_a^b f^s dx\right)^{1/s} \leq \left(\frac{1+r}{b-a}\int_a^b f^r dx\right)^{1/r}. \tag{1.8}$$

The case $r = 1$, $s = 2$ is the Frank-Pick inequality.

2. Preliminaries

In this section, some definitions and basic properties of the Sugeno integrals which will be used in the next sections are presented. We denote by $\mathbb{R}$ the set of all real numbers.

Let X be a non-empty set and Σ be a $\sigma-$algebra of subsets of X. Throughout this paper, all considered subsets are supposed to be in Σ.

Definition 2.1 (Gardner [40]). Let X and Y be sets in $\mathbb{R}^n$. We define their vector or Minkowski sum by $X + Y = \{x + y : x \in X, y \in Y\}$.

If $r \in \mathbb{R}$, let $rX = \{rx : \ x \in X\}$.

- If $r > 0$, then rX is the dilatation of X with factor r.
- If $r < 0$, it is the reflection of dilatation with factor $-r$ in the origin.
- If $r = 0$, $rX = \{0\}$, where $0 = (0, \ldots, 0)$.

If $0 < \lambda < 1$, the set $(1 - \lambda)X + \lambda Y$ is called convex combination of X and Y.

Definition 2.2 (Ralescu and Adams [50]). A set function $\mu : \Sigma \to [0, +\infty]$ is called a fuzzy measure if the following properties are satisfied:

(FM1) $\mu(\emptyset) = 0$,
(FM2) $A \subseteq B \Rightarrow \mu(A) \leq \mu(B)$,
(FM3) $A_1 \subseteq A_2 \subseteq \ldots \Rightarrow \lim \mu(A_i) = \mu\left(\bigcup_{i=1}^{\infty} A_i\right)$,

(FM4) $A_1 \supseteq A_2 \supseteq \dots$ and $\mu(A_1) < \infty \Rightarrow \lim \mu(A_i) = \mu\left(\bigcap_{i=1}^{\infty} A_i\right)$.

When μ is a fuzzy measure, the triple (X, Σ, μ) is called a fuzzy measure space.

Definition 2.3 (Prékopa [49]). A fuzzy measure μ on $(\mathbb{R}^n, \Sigma)$ is called concave if

$$\mu\left((1-\lambda)X + \lambda Y\right) \geq (1-\lambda)\mu(X) + \lambda\mu(Y), \tag{2.1}$$

for any $\lambda \in [0,1]$ and for any two convex subset X, Y in Σ such that $(1-\lambda)X + \lambda Y \in \Sigma$.

Under the same above conditions:

A fuzzy measure μ is called quasi-concave if

$$\mu\left((1-\lambda)X + \lambda Y\right) \geq \min\left\{\mu(X), \mu(Y)\right\}.$$

A fuzzy measure μ is said to be log-concave if

$$\mu\left((1-\lambda)X + \lambda Y\right) \geq \mu(X)^{1-\lambda}\mu(Y)^{\lambda}.$$

A fuzzy measure μ is called homogenous quasi-concave if for all $s, t > 0$,

$$\mu(sX + tY)^{1/n} \geq s\mu(X)^{1/n} + t\mu(Y)^{1/n}.$$

Definition 2.4 (Gardner [40]). A function f on $\mathbb{R}^n$ is concave on a convex set C if $f\left((1-\lambda)x + \lambda y\right) \geq (1-\lambda)f(x) + \lambda f(y)$, for all $x, y \in C$ and $0 < \lambda < 1$, and a function f is convex if $-f$ is concave. A non-negative function f is log-concave if $\log f$ is concave. Since the latter condition is equivalent to $f\left((1-\lambda)x + \lambda y\right) \geq f(x)^{1-\lambda}f(y)^{\lambda}$, the arithmetic-geometric mean inequality implies that each concave non-negative function is log-concave.

If f is a non-negative real-valued function on X, we will denote

$$\begin{aligned} F_\alpha &= L_\alpha f \\ &= \{x \in X \mid f(x) \geq \alpha\} \\ &= \{f \geq \alpha\}, \end{aligned}$$

the α-level of f, for $\alpha > 0$. $L_0 f = \overline{\{x \in X \mid f(x) > 0\}} = \operatorname{supp}(f)$ is the support of f. We know that:

$$\alpha \leq \beta \quad \Rightarrow \quad \{f \geq \beta\} \subseteq \{f \geq \alpha\}.$$

If μ is a fuzzy measure on (X, Σ), we define the following:

$$\begin{aligned} \mathfrak{F}^\sigma(X) &= \{f : X \to [0,\infty) | f \text{ is } \mu - \text{measurable}\}, \\ \mathfrak{F}^\sigma_c(X) &= \{f \in \mathfrak{F}^\sigma(X) | L_\alpha f \text{ is convex } \forall \alpha \in [0,\infty)\} \end{aligned}$$

Definition 2.5 (Wang and Klir [63]). Let μ be a fuzzy measure on (X, Σ). If $f \in \mathfrak{F}^{\mu}(X)$ and $A \in \Sigma$, then the Sugeno integral of f on A, with respect to the fuzzy measure μ, is defined as

$$\fint_A f d\mu = \bigvee_{\alpha \geq 0} (\alpha \wedge \mu(A \cap F_\alpha)).$$

Where $\vee$ and $\wedge$ denotes the operations *sup* and *inf* on $[0, \infty]$, respectively. In particular, if $A = X$ then

$$\begin{aligned} \fint_X f d\mu &= \fint f d\mu \\ &= \bigvee_{\alpha \geq 0} (\alpha \wedge \mu(F_\alpha)). \end{aligned}$$

In [63], Wang and Klir proved the following theorem.

Theorem 2.6 (Wang and Klir [63]). *If μ is a fuzzy measure and $f \in \mathfrak{F}^{\sigma}(X)$ is an measurable function, then*

$$\fint_X f d\mu = \fint_0^\infty \mu(\{f \geq \alpha\}) \, d\alpha.$$

The integral in the right-side of the last equation is the Sugeno integral of $F(\alpha) = \mu(L_\alpha f)$ with respect to the Lebesgue measure.

The following properties of the Sugeno integral can be found in [63].

Proposition 2.7 (Wang and Klir [63]). *Let (X, Σ, μ) be a fuzzy measure space, with $A, B \in \sum$ and $f, g \in \mathfrak{F}^{\sigma}(X)$. We have*

(1) $\fint_A f d\mu \leq \mu(A)$.
(2) $\fint_A k d\mu = k \wedge \mu(A)$, *for k non-negative constant.*
(3) *If $A \subset B$, then* $\fint_A f d\mu \leq \fint_B f d\mu$.
(4) $\fint_{A \cup B} f d\mu \geq \fint_A f d\mu \vee \fint_B f d\mu$.
(5) *If $\mu(A) < \infty$, then* $\fint_A f d\mu \geq \alpha \Leftrightarrow \mu(A \cap \{f \geq \alpha\}) \geq \alpha$.
(6) $\fint_A f d\mu \leq \alpha \Leftrightarrow \mu(A \cap F_{\alpha^+}) \leq \alpha \Leftarrow \mu(A \cap \{f \geq \alpha\}) \leq \alpha$.
(7) $\fint_A f d\mu > \alpha \Leftrightarrow \mu(A \cap F_{\alpha^+}) > \alpha \Rightarrow \mu(A \cap \{f \geq \alpha\}) \leq \alpha$.

Remark 2.8. Consider the survival function F associated to f on A, that is to say,

$$F(\alpha) = \mu(A \cap \{f \geq \alpha\}).$$

Then

$$F(\alpha) = \alpha \quad \Rightarrow \quad \fint_A f d\mu = \alpha.$$

Thus, from a numerical (or computational) point of view, the Sugeno integral can be calculated by solving the equation $F(\alpha) = \alpha$ (if the solution exists).

Now, we recall some integral inequalities at the following which are used in the next section.

Theorem 2.9 (Fuzzy Chebyshev's inequality [7])**.** *Suppose that f, g are two real-valued functions from $[0,1]$ to $[0,\infty)$ and that μ is the Lebesgue measure. If f and g are non-decreasing functions, then the inequality*

$$\tag{2.2} ⨍_0^1 f \cdot g d\mu \leq \left(⨍_0^1 f d\mu \right) \left(⨍_0^1 g d\mu \right),$$

holds.

Lemma 2.10 ([22])**.** *Let (X, Σ, μ) be a fuzzy measure space, let $A \in \Sigma$ and let $f : X \to \mathbb{R}$ be a measurable function such that $⨍_A f d\mu \leq 1$. Then, for any $s \geq 1$, we have*

$$⨍_A f^s d\mu \geq \left(⨍_A f d\mu \right)^s .$$

Lemma 2.11. *Let (X, Σ, μ) be a fuzzy measure space, let $A \in \Sigma$ and let $f : X \to \mathbb{R}$ be a measurable function such that $⨍_A f d\mu \leq 1$. Then*

$$⨍_X f d\mu = ⨍_X \min(1, f) d\mu.$$

3. Related Gauss-Winkler Type Inequalities for Fuzzy Integrals

In this section, as main results, we prove related Gauss-Winkler type inequalities for fuzzy integrals.

Theorem 3.1 (Fuzzy Gauss-Winkler inequality: non-decreasing case)**.** *Let $f : [0,1] \to [0,\infty)$ be a non-decreasing function and μ be the Lebesgue measure on $\mathbb{R}$. Then the inequality*

$$\tag{3.1} \left(⨍_0^1 x^2 f(x) d\mu \right)^2 \leq 1.4255 \left(⨍_0^1 f(x) d\mu \right) \left(⨍_0^1 x^4 f(x) d\mu \right),$$

holds.

Proof. Since 1.4255 is greater than 1, it can be written:

$$1.4255 \left(⨍_0^1 f(x) d\mu \right) \left(⨍_0^1 x^4 f(x) d\mu \right) \geq \left(⨍_0^1 f(x) d\mu \right) \left(⨍_0^1 x^4 f(x) d\mu \right).$$

Now, from the Chebyshev inequality, we have:

$$\begin{aligned} &1.4255 \left(⨍_0^1 f(x) d\mu \right) \left(⨍_0^1 x^4 f(x) d\mu \right) \\ &\quad \geq \left(⨍_0^1 f(x) d\mu \right) \left(⨍_0^1 x^4 d\mu \right) \left(⨍_0^1 f(x) d\mu \right) \end{aligned}$$

$$= \left(⨍_0^1 f(x)d\mu\right)^2 \left(⨍_0^1 (x^2)^2 d\mu\right).$$

Now by using Lemma 2.10, we get:

$$1.4255 \left(⨍_0^1 f(x)d\mu\right)\left(⨍_0^1 x^4 f(x)d\mu\right) \geq \left(⨍_0^1 f(x)d\mu\right)^2 \left(⨍_0^1 x^2 d\mu\right)^2$$
$$= \left(\left(⨍_0^1 f(x)d\mu\right)\left(⨍_0^1 x^2 d\mu\right)\right)^2.$$

Applying Chebyshev inequality, it follows that

$$1.4255 \left(⨍_0^1 f(x)d\mu\right)\left(⨍_0^1 x^4 f(x)d\mu\right) \geq \left(⨍_0^1 f(x)\cdot x^2 d\mu\right)^2.$$

Thereby, the theorem is proved. □

Theorem 3.2 (Fuzzy Gauss-Winkler inequality: non-increasing case)**.** *Let $f:[0,1]\to[0,\infty)$ be a non-increasing function and μ be the Lebesgue measure on $\mathbb{R}$. Then the inequality*

$$\left(⨍_0^1 (1-x)^2 f(x)d\mu\right)^2 \leq 1.4255 \left(⨍_0^1 f(x)d\mu\right)\left(⨍_0^1 (1-x)^4 f(x)d\mu\right), \tag{3.2}$$

holds.

Proof. The proof is similar to the proof of the Theorem 3.1. □

In the following, we present an example to illustrate the validity of Theorem 3.1.

Example 3.3. Let $f(x) = \begin{cases} x & 0 \leq x \leq \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} < x \leq 1 \end{cases}$ and μ be the Lebesgue measure on $\mathbb{R}$. A straightforward calculus shows that $⨍_0^1 f(x)d\mu = 0.5$, $⨍_0^1 x^2 f(x)d\mu = 0.267$ and $⨍_0^1 x^4 f(x)d\mu = 0.202$. Consequently

$$\begin{aligned} 0.071 &= \left(⨍_0^1 x^2 f(x)d\mu\right)^2 \\ &\leq 1.4255\left(⨍_0^1 f(x)d\mu\right)\left(⨍_0^1 x^4 f(x)d\mu\right) \\ &= 1.4255(0.5)(0.202) \\ &= 0.144. \end{aligned}$$

4. Minkowski's Inequality for Fuzzy Integrals

In this section, by an example we show that the Theorem 1.3 is not valid for the Sugeno integral.

Example 4.1. Let $f(x) = x + 1$, $g(x) = 2x + 1$ and $s = 1$. We have $0 < \frac{2}{3} \leq \frac{f(x)}{g(x)} \leq 1$ and

$$\text{(i)} \qquad \fint_0^1 f(x)d\mu = \bigvee_{\alpha\in[0,1]} [\alpha \wedge \mu(\{x+1 \geq \alpha\})] = \bigvee_{\alpha\in[0,1]} [\alpha \wedge (\alpha - 1)] = 1,$$

$$\text{(ii)} \qquad \fint_0^1 g(x)d\mu = \bigvee_{\alpha\in[0,1]} [\alpha \wedge \mu(\{2x+1 \geq \alpha\})] = \bigvee_{\alpha\in[0,1]} \left[\alpha \wedge \left(\frac{\alpha - 1}{2}\right)\right] = 1,$$

$$\text{(iii)} \qquad \fint_0^1 (f(x)+g(x))d\mu = \bigvee_{\alpha\in[0,2]} [\alpha \wedge \mu(\{3x+2 \geq \alpha\})] = \bigvee_{\alpha\in[0,2]} \left[\alpha \wedge \left(\frac{\alpha - 2}{3}\right)\right] = \frac{5}{4}.$$

Consequently,

$$2 = \fint_0^1 f(x)d\mu + \fint_0^1 g(x)d\mu \nleq c \fint_0^1 (f(x)+g(x))d\mu = \frac{11}{10} \times \frac{5}{4} = \frac{11}{8}.$$

Hence the Inequality (1.2) is not valid for fuzzy integrals.

In the following theorem we show a Minkowski type inequality derived from (1.5) for the Sugeno integral.

Theorem 4.2 (Fuzzy Minkowski's inequality, decreasing case)**.** *Let $f, g : [0,1] \to [0,\infty)$ be two real-valued and non-negative functions and let μ be the Lebesgue measure on $\mathbb{R}$. Let f, g be both continuous and strictly decreasing functions. If functions satisfying*

$$0 < m \le \frac{f(x)}{g(x)} \le M, \quad \forall x \in [0,1]$$

then the inequality

(4.1)

$$\left(⨍_0^1 f^s(x)d\mu\right)^{\frac{1}{s}} + \left(⨍_0^1 g^s(x)d\mu\right)^{\frac{1}{s}} \le 2c\left(⨍_0^1 (f(x)+g(x))^s d\mu\right)^{\frac{1}{s}},$$

holds, where $1 \le s < \infty$ and $c = \dfrac{M(m+1)+(M+1)}{(m+1)(M+1)}$.

Proof. Since $\frac{f(x)}{g(x)} \le M, f \le M(f(x)+g(x)) - Mf(x)$. Therefore

$$(M+1)^s f(x)^s \le M^s (f(x)+g(x))^s$$

and so,

$$f(x)^s \le \frac{M^s}{(M+1)^s}(f(x)+g(x))^s.$$

Now we have

$$\left(⨍_0^1 f(x)^s d\mu\right)^{\frac{1}{s}} \le \left(⨍_0^1 \left(\frac{M}{M+1}\right)^s (f(x)+g(x))^s d\mu\right)^{\frac{1}{s}}. \tag{4.2}$$

By Lemma 2.8 (i) we have

$$\begin{aligned} ⨍_0^1 \frac{M}{M+1} dx &< ⨍_0^1 1dx \\ &= 1. \end{aligned} \tag{4.3}$$

So by (3.2) and (3.3) we can write

$$\left(⨍_0^1 f(x)^s d\mu\right)^{\frac{1}{s}} \le \left(⨍_0^1 (f(x)+g(x))^s d\mu\right)^{\frac{1}{s}}. \tag{4.4}$$

On the other hand, since $mg(x) \le f(x)$, Hence

$$g \le \frac{1}{m}(f(x)+g(x)) - \frac{1}{m}g(x).$$

Therefore,

$$\left(\frac{1}{m}+1\right)^s g(x)^s \le \left(\frac{1}{m}\right)^s (f(x)+g(x))^s$$

and so, by Lemma 2.8 (i) we have

$$\left(⨍_0^1 g(x)^s d\mu\right)^{\frac{1}{s}} \leq \left(⨍_0^1 \left(\frac{1}{m+1}\right)^s (f(x)+g(x))^s d\mu\right)^{\frac{1}{s}}. \tag{4.5}$$

Since $\dfrac{1}{m+1} < 1$, then

$$\begin{aligned} ⨍_0^1 \frac{1}{m+1} dx &< ⨍_0^1 1dx \\ &= 1. \end{aligned} \tag{4.6}$$

The inequalities (4.5) and (4.6) follows that

$$\left(⨍_0^1 g(x)^s d\mu\right)^{\frac{1}{s}} \leq \left(⨍_0^1 (f(x)+g(x))^s d\mu\right)^{\frac{1}{s}}. \tag{4.7}$$

Now, by adding the Inequalities (3.4) and (3.7) we have

$$\begin{aligned} \left(⨍_0^1 f(x)^s d\mu\right)^{\frac{1}{s}} + \left(⨍_0^1 g(x)^s d\mu\right)^{\frac{1}{s}} &\leq 2\left(⨍_0^1 (f(x)+g(x))^s d\mu\right)^{\frac{1}{s}} \\ &\leq 2c\left(⨍_0^1 (f(x)+g(x))^s d\mu\right)^{\frac{1}{s}}. \end{aligned}$$

The proof is now complete. □

Example 4.3. Let $f, g : [0,1] \to [0,\infty)$ be two real-valued functions defined as $f(x) = 1-x$, $g(x) = 1-x^2$ and μ be the Lebesgue measure on $\mathbb{R}$. Let $s = 1$. A straightforward calculus shows that $0 < \frac{1}{2} \leq \frac{f}{g} \leq 1$ and

(i)
$$\begin{aligned} ⨍_0^1 f(x) d\mu &= \bigvee_{\alpha\in[0,1]} [\alpha \wedge \mu(\{1-x \geq \alpha\})] \\ &= \bigvee_{\alpha\in[0,1]} [\alpha \wedge (1-\alpha)] \\ &= \frac{1}{2} \\ &= 0.5, \end{aligned}$$

(ii)
$$\begin{aligned} ⨍_0^1 g(x) d\mu &= \bigvee_{\alpha\in[0,1]} [\alpha \wedge \mu(\{1-x^2 \geq \alpha\})] \\ &= \bigvee_{\alpha\in[0,1]} [\alpha \wedge \sqrt{1-\alpha}] \end{aligned}$$

$$= 0.618,$$

(iii)
$$\begin{aligned}
\int_0^1 (f+g)d\mu &= \bigvee_{\alpha\in[0,2]} \left[\alpha \wedge \mu\left(\{-x^2 - x + 2 \geq \alpha\}\right)\right] \\
&= \bigvee_{\alpha\in[0,2]} \left[\alpha \wedge \left(-\frac{1}{2} + \frac{1}{2}\sqrt{(9-4\alpha)}\right)\right] \\
&= 0.732.
\end{aligned}$$

Therefore

$$\begin{aligned}
1.118 &= 0.5 + 0.618 \\
&= \left(\int_0^1 f d\mu\right) + \left(\int_0^1 g d\mu\right) \\
&\leq 2c\left(\int_0^1 (f+g)d\mu\right) \\
&= 2c \times 0.732 \\
&= 1.464c.
\end{aligned}$$

Theorem 4.4. *(Fuzzy Minkowski's inequality, increasing case). Let $f, g : [0,1] \to [0,\infty)$ be two real-valued and non-negative functions and let μ be the Lebesgue measure on $\mathbb{R}$. Let f, g be both continuous and strictly increasing functions and satisfying*

$$0 < m \leq \frac{f(x)}{g(x)} \leq M, \quad \forall x \in [0,1]$$

then the inequality

$$\left(\int_0^1 f(x)^s d\mu\right)^{\frac{1}{s}} + \left(\int_0^1 g(x)^s d\mu\right)^{\frac{1}{s}} \leq 2c\left(\int_0^1 (f(x)+g(x))^s d\mu\right)^{\frac{1}{s}},$$

holds, where $1 \leq s < \infty, n \geq 2$ and $c = \dfrac{M(m+1)+(M+1)}{(m+1)(M+1)}$.

Proof. The proof is similar Theorem 4.2. □

5. Prékopa-Leindler's Inequality for the Sugeno Integral

In this section, we prove Prékopa-Leindler's type inequality for Sugeno integral.

Unfortunately, the classical Prékopa-Leindler's inequality is not valid for the Sugeno integral as it is shown in the following example.

Example 5.1. Let $f, g : [0,1] \to [0,\infty)$ be defined by $f(x) = x^2$, $g(x) = x$ and $m = \frac{\mu}{2}$, where μ is the Lebesgue measure. From (1.7), by choosing $\lambda = \frac{1}{2}$, we have

$$\begin{aligned} h(z) &= \sup_{z=(1-\lambda)x+\lambda y} \left(f(x)^{1-\lambda} g(y)^{\lambda} \right) \\ &= z\sqrt{z}. \end{aligned}$$

A straightforward calculus show that

$$\begin{aligned} \text{(i)} \qquad \int_0^1 h dm &= \sup_{\alpha\in[0,1]} \left(\alpha \wedge \mu \left\{ z\sqrt{z} \geq \alpha \right\} \right) \\ &= \sup_{\alpha\in[0,1]} \left(\alpha \wedge \frac{\left(1 - \sqrt[3]{\alpha^2}\right)}{2} \right) \\ &= 0.2839, \end{aligned}$$

$$\begin{aligned} \text{(ii)} \qquad \int_0^1 f dm &= \sup_{\alpha\in[0,1]} \left(\alpha \wedge \mu \left\{ x^2 \geq \alpha \right\} \right) \\ &= \sup_{\alpha\in[0,1]} \left(\alpha \wedge \frac{(1-\alpha)^2}{2} \right) \\ &= 0.2679, \end{aligned}$$

$$\begin{aligned} \text{(iii)} \qquad \int_0^1 g dm &= \sup_{\alpha\in[0,1]} \left(\alpha \wedge \mu \left\{ x \geq \alpha \right\} \right) \\ &= \sup_{\alpha\in[0,1]} \left(\alpha \wedge \frac{(1-\alpha)}{2} \right) \\ &= 0.3333. \end{aligned}$$

Thus

$$\begin{aligned} 0.2839 &= \int_0^1 h dm \\ &\not\geq \left(\int_0^1 f dm \right)^{\frac{1}{2}} \left(\int_0^1 g dm \right)^{\frac{1}{2}} \\ &= 0.2987, \end{aligned}$$

Therefore, Inequality (1.7) does not holds for Sugeno integral.

The following theorem shows a version of the Prékopa-Leindler's inequality for Sugeno integral in relation to a concave fuzzy measure.

Theorem 5.2 (Fuzzy Prékopa-Leindler's inequality)**.** *Let* $f, g, h \in \mathcal{F}_c^{\sigma}([0,1])$ *be such that for all* $x, y \in [0,1]$ *and some* $\lambda \in (0,1)$*, the inequality* (1.6) *is satisfied. If* μ *is a concave fuzzy measure, then the inequality*

$$\fint_0^1 h d\mu \geq \fint_0^1 f d\mu \wedge \fint_0^1 g d\mu, \tag{5.1}$$

holds.

Proof. If $\alpha \in [0, \infty)$, $f(x) \geq \alpha$ and $g(y) \geq \alpha$. Then,

$$h((1-\lambda)x + \lambda y) \geq \alpha.$$

It is easy to check that $(1-\lambda)L_\alpha f + \lambda L_\alpha g \subseteq L_\alpha h$. From monotonicity of μ, we have

$$\mu\left((1-\lambda)L_\alpha f + \lambda L_\alpha g\right) \leq \mu(L_\alpha h). \tag{5.2}$$

Due to Theorem 2.6 and Proposition 2.7 (3), we get

$$\begin{aligned} \fint_0^1 h d\mu &= \fint_0^\infty \mu(L_\alpha h) d\alpha \\ &\geq \fint_0^1 \mu(L_\alpha h) d\alpha. \end{aligned} \tag{5.3}$$

Now from (5.9) and (2.1), we can obtain that

$$\begin{aligned} \fint_0^1 h d\mu &\geq \fint_0^1 \mu\left((1-\lambda)L_\alpha f + \lambda L_\alpha g\right) d\alpha \\ &\geq \fint_0^1 \left((1-\lambda)\mu(L_\alpha f) + \lambda \mu(L_\alpha g)\right) d\alpha. \end{aligned}$$

Using Proposition 2.7 (4),

$$\begin{aligned} \fint_0^1 h d\mu &\geq \fint_0^1 \left((1-\lambda)\mu(L_\alpha f) + \lambda \mu(L_\alpha g)\right) d\alpha \\ &\geq \fint_{[0,p]} \left((1-\lambda)\mu(L_\alpha f) + \lambda \mu(L_\alpha g)\right) d\alpha \\ &\quad \bigvee \fint_{(p,1]} \left((1-\lambda)\mu(L_\alpha f) + \lambda \mu(L_\alpha g)\right) d\alpha. \end{aligned}$$

Without the loss of generality, we suppose that p and q are chosen in a such way that

$$\fint_0^1 f d\mu = p, \qquad \fint_0^1 g d\mu = q, \qquad q \geq p. \tag{5.4}$$

From Proposition 2.7 (7), we have

$$\mu(\{f \geq p\}) \geq p, \qquad \mu(\{g \geq q\}) \geq q.$$

From monotonicity of μ, for all $\alpha \in [0, p]$ we have

$$\mu(\{f \geq \alpha\}) \geq \mu(\{f \geq p\}) \geq p, \tag{5.5}$$
$$\mu(\{g \geq \alpha\}) \geq \mu(\{g \geq q\}) \geq q.$$

By using Proposition 2.7 (1) and (5.12),

(5.6)
$$\begin{aligned}\fint_0^1 hd\mu &\geq \fint_{[0,p]} ((1-\lambda)p + \lambda q)\, d\alpha \bigvee \fint_{(p,1]} ((1-\lambda)\mu(L_\alpha f) + \lambda\mu(L_\alpha g))\, d\alpha \\ &= p \vee \fint_{(p,1]} ((1-\lambda)\mu(L_\alpha f) + \lambda\mu(L_\alpha g))\, d\alpha \\ &= p.\end{aligned}$$

Now, from (5.11) that we supposed $q \geq p$, it is easy to see that

$$p \wedge q = p. \tag{5.7}$$

From (5.13) and (5.14), we obtain that

$$\begin{aligned}\fint_0^1 hd\mu &\geq p \wedge q \\ &= \fint_0^1 fd\mu \wedge \fint_0^1 gd\mu.\end{aligned}$$

Now, proof is complete. □

Example 5.3. Let f and g be measurable functions on $[0, 1]$ given by $f(x) = 1 - x$ and $g(x) = \dfrac{1}{1+x}$ and let μ be the Lebesgue measure on Borel subsets of $[0, 1]$. By choosing $\lambda = \dfrac{1}{2}$,

$$\begin{aligned}h(z) &= \sup_{z=(1-\lambda)x+\lambda y} \left(f(x)^{1-\lambda} g(y)^\lambda\right) \\ &= \sqrt{\frac{1-z}{1+z}}.\end{aligned}$$

Then, a straightforward calculus shows that

$$\begin{aligned}\text{(i)} \qquad \fint_0^1 f(x)d\mu &= \bigvee_{\alpha\in[0,1]} [\alpha \wedge \mu(\{1 - x \geq \alpha\})] \\ &= \bigvee_{\alpha\in[0,1]} [\alpha \wedge (1-\alpha)] \\ &= 0.5,\end{aligned}$$

(ii)
$$\fint_0^1 g(x)d\mu = \bigvee_{\alpha\in[0,1]} \left[\alpha \wedge \mu\left(\left\{\frac{1}{x+1} \geq \alpha\right\}\right)\right]$$
$$= \bigvee_{\alpha\in[0,1]} \left[\alpha \wedge \left(\frac{1-\alpha}{\alpha}\right)\right]$$
$$= 0.618,$$

(iii)
$$\fint_0^1 h(z)d\mu = \bigvee_{\alpha\in[0,1]} \left[\alpha \wedge \mu\left(\left\{\sqrt{\frac{1-z}{1+z}} \geq \alpha\right\}\right)\right]$$
$$= \bigvee_{\alpha\in[0,1]} \left[\alpha \wedge \left(\frac{1-\alpha^2}{1+\alpha^2}\right)\right]$$
$$= 0.544.$$

Therefore

$$\begin{aligned} 0.544 &= \fint_0^1 h d\mu \\ &\geq \fint_0^1 f d\mu \wedge \fint_0^1 g d\mu \\ &= (0.5) \wedge (0.618) \\ &= 0.5. \end{aligned}$$

It follows that the Inequality (5.8) is valid.

In the next theorem, we show that if we replace concave fuzzy measure with quasi-concave measure, Fuzzy Prékopa-Leindler's inequality is still valid.

Theorem 5.4 (Fuzzy Prékopa-Leindler's inequality)**.** *Let* $f, g, h \in \mathcal{F}_c^\sigma([0,1])$ *be such that for all* $x, y \in [0,1]$ *and some* $\lambda \in (0,1)$*, the inequality* (1.6) *is satisfied. If* μ *is a quasi-concave fuzzy measure, then the inequality*

$$\fint_0^1 h d\mu \geq \fint_0^1 f d\mu \wedge \fint_0^1 g d\mu, \tag{5.8}$$

holds.

Proof. If $\alpha \in [0,\infty)$, $f(x) \geq \alpha$ and $g(y) \geq \alpha$. Then,

$$h((1-\lambda)x + \lambda y) \geq \alpha.$$

It is easy to check that $(1-\lambda)L_\alpha f + \lambda L_\alpha g \subseteq L_\alpha h$. From monotonicity of μ, we have

$$\mu\left((1-\lambda)L_\alpha f + \lambda L_\alpha g\right) \leq \mu(L_\alpha h). \tag{5.9}$$

Due to Theorem 2.6 and Proposition 2.7 (3), we get

$$\begin{aligned} ⨍_0^1 h d\mu &= ⨍_0^\infty \mu(L_\alpha h) d\alpha \\ &\geq ⨍_0^1 \mu(L_\alpha h) d\alpha. \end{aligned} \tag{5.10}$$

Now from (5.9) and (2.1), we can obtain that

$$\begin{aligned} ⨍_0^1 h d\mu &\geq ⨍_0^1 \mu\left((1-\lambda)L_\alpha f + \lambda L_\alpha g\right) d\alpha \\ &\geq ⨍_0^1 \left((1-\lambda)\mu(L_\alpha f) + \lambda\mu(L_\alpha g)\right) d\alpha. \end{aligned}$$

Using Proposition 2.7 (4),

$$\begin{aligned} ⨍_0^1 h d\mu &\geq ⨍_0^1 \left((1-\lambda)\mu(L_\alpha f) + \lambda\mu(L_\alpha g)\right) d\alpha \\ &\geq ⨍_{[0,p]} \left((1-\lambda)\mu(L_\alpha f) + \lambda\mu(L_\alpha g)\right) d\alpha \\ &\quad \bigvee ⨍_{(p,1]} \left((1-\lambda)\mu(L_\alpha f) + \lambda\mu(L_\alpha g)\right) d\alpha. \end{aligned}$$

Without the loss of generality, we suppose that p and q are chosen in a such way that

$$⨍_0^1 f d\mu = p, \qquad ⨍_0^1 g d\mu = q, \qquad q \geq p. \tag{5.11}$$

From Proposition 2.7 (7), we have

$$\mu(\{f \geq p\}) \geq p, \qquad \mu(\{g \geq q\}) \geq q.$$

From monotonicity of μ, for all $\alpha \in [0,p]$ we have

$$\begin{aligned} \mu(\{f \geq \alpha\}) &\geq \mu(\{f \geq p\}) \geq p, \\ \mu(\{g \geq \alpha\}) &\geq \mu(\{g \geq q\}) \geq q. \end{aligned} \tag{5.12}$$

By using Proposition 2.7 (1) and (5.12),

(5.13)

$$⨍_0^1 h d\mu \geq ⨍_{[0,p]} \left((1-\lambda)p + \lambda q\right) d\alpha \bigvee ⨍_{(p,1]} \left((1-\lambda)\mu(L_\alpha f) + \lambda\mu(L_\alpha g)\right) d\alpha$$

$$= p \vee \fint_{(p,1]} ((1-\lambda)\mu(L_\alpha f) + \lambda\mu(L_\alpha g))\, d\alpha$$
$$= p.$$

Now, from (5.11) that we supposed $q \geq p$, it is easy to see that

$$p \wedge q = p. \tag{5.14}$$

From (5.13) and (5.14), we obtain that

$$\begin{aligned} \fint_0^1 h d\mu &\geq p \wedge q \\ &= \fint_0^1 f d\mu \wedge \fint_0^1 g d\mu. \end{aligned}$$

Now, proof is complete. □

In the following, we present an example to illustrate the validity of Theorem 5.4.

Example 5.5. Let f and g be measurable functions on $[0,1]$ given by $f(x) = 1 - x$ and $g(x) = \dfrac{1}{1+x}$ and let μ be the Lebesgue measure on Borel subsets of $[0,1]$. By choosing $\lambda = \dfrac{1}{2}$,

$$\begin{aligned} h(z) &= \sup_{z=(1-\lambda)x+\lambda y} \left(f(x)^{1-\lambda} g(y)^{\lambda}\right) \\ &= \sqrt{\frac{1-z}{1+z}}. \end{aligned}$$

Then, a straightforward calculus shows that

$$\text{(i)} \qquad \begin{aligned} \fint_0^1 f(x) d\mu &= \bigvee_{\alpha\in[0,1]} [\alpha \wedge \mu(\{1 - x \geq \alpha\})] \\ &= \bigvee_{\alpha\in[0,1]} [\alpha \wedge (1-\alpha)] \\ &= 0.5, \end{aligned}$$

$$\text{(ii)} \qquad \begin{aligned} \fint_0^1 g(x) d\mu &= \bigvee_{\alpha\in[0,1]} \left[\alpha \wedge \mu\left(\left\{\frac{1}{x+1} \geq \alpha\right\}\right)\right] \\ &= \bigvee_{\alpha\in[0,1]} \left[\alpha \wedge \left(\frac{1-\alpha}{\alpha}\right)\right] \\ &= 0.618, \end{aligned}$$

$$\text{(iii)} \qquad \fint_0^1 h(z)d\mu = \bigvee_{\alpha\in[0,1]} \left[\alpha \wedge \mu\left(\left\{\sqrt{\frac{1-z}{1+z}} \geq \alpha\right\}\right)\right]$$
$$= \bigvee_{\alpha\in[0,1]} \left[\alpha \wedge \left(\frac{1-\alpha^2}{1+\alpha^2}\right)\right]$$
$$= 0.544.$$

Therefore

$$\begin{aligned} 0.544 &= \fint_0^1 hd\mu \\ &\geq \fint_0^1 fd\mu \wedge \fint_0^1 gd\mu \\ &= (0.5) \wedge (0.618) \\ &= 0.5. \end{aligned}$$

It follows that the Inequality (5.8) is valid.

The following theorem is a derived from the Prékopa-Leindler inequality. This theorem holds for any quasi-concave fuzzy measure μ homogeneous of degree n, i.e., $\mu(sX) = s^n \cdot \mu(X)$ for any $s > 0, X, sX \in \Sigma$.

Theorem 5.6. *Let $\lambda \in (0,1)$ and X and Y be nonempty bounded measurable sets in $\mathbb{R}^n$ such that $(1-\lambda)X + \lambda Y$ is also measurable. If $\mu(X)\mu(Y) < \infty$ and μ is a homogenous (of degree n) quasi-concave fuzzy measure, then*

$$\mu((1-\lambda)X + \lambda Y)^{1/n} \geq (1-\lambda)\mu(X)^{1/n} + \lambda\mu(Y)^{1/n}. \tag{5.15}$$

Proof. Let $\lambda' = \dfrac{\mu(Y)^{1/n}}{\mu(X)^{1/n} + \mu(Y)^{1/n}}$, $X' = \mu(X)^{-1/n}X$ and $Y' = \mu(Y)^{-1/n}Y$. Then

$$1 - \lambda' = \frac{\mu(X)^{1/n}}{\mu(X)^{1/n} + \mu(Y)^{1/n}}, \qquad \mu(X') = \mu(Y') = 1.$$

Due quasi-concavity of μ applied to X', Y' and λ',

$$\begin{aligned} \mu((1-\lambda')X' + \lambda'Y') &\geq \mu(X') \wedge \mu(Y') \\ &= 1, \end{aligned} \tag{5.16}$$

But

$$\mu((1-\lambda')X' + \lambda'Y') = \mu\left(\frac{(X+Y)}{\mu(X)^{1/n} + \mu(Y)^{1/n}}\right)$$

$$= \frac{\mu(X+Y)}{\left(\mu(X)^{1/n} + \mu(Y)^{1/n}\right)^n},$$

that from (5.16) it follows that

$$\frac{\mu(X+Y)}{\left(\mu(X)^{1/n} + \mu(Y)^{1/n}\right)^n} \geq 1.$$

Now, we have

$$\mu(X+Y)^{1/n} \geq \mu(X)^{1/n} + \mu(Y)^{1/n}. \tag{5.17}$$

By replacing X by $(1-\lambda)X$ and Y by λY in (5.17), proof is complete. □

Example 5.7. Let $X = Y = \underbrace{[0,m] \times \cdots \times [0,m]}_{n-\text{times}}$, $m \in \mathbb{N}$ and $\mu = \sqrt[n]{|X|}$ where $|X|$ is the length of X. With these assumptions, we obtain that $\mu(X) = \mu(Y) = m$. Because of that μ is quasi-concave measure, we have

$$\begin{aligned} \mu\left((1-\lambda)X + \lambda Y\right) &\geq \min\{m, m\} \\ &= m. \end{aligned}$$

Now,

$$\begin{aligned} \mu\left((1-\lambda)X + \lambda Y\right)^{1/n} &\geq m^{1/n} \\ &= \sqrt[n]{m} \\ &= (1-\lambda)\sqrt[n]{m} + \lambda\sqrt[n]{m} \\ &= (1-\lambda)\mu(X)^{1/n} + \lambda\mu(Y)^{1/n}. \end{aligned}$$

Finally, we have

$$\mu\left((1-\lambda)X + \lambda Y\right)^{1/n} \geq (1-\lambda)\mu(X)^{1/n} + \lambda\mu(Y)^{1/n}.$$

6. Frank-Pick's Inequality for Sugeno Integral

In the following, we state and prove a fuzzy version of Frank-Pick's inequality if $f : [0,1] \to [0,\infty]$ and $r, s \in (0,\infty)$.

Theorem 6.1. *Let (X, Σ, μ) be a fuzzy measure space with normed fuzzy measure μ and $f \in \mathfrak{F}^\sigma(X)$. Then*

$$\left(1 + \left(\frac{1}{2}\right) \cdot ⨍_X f^2 d\mu\right)^{\frac{1}{2}} \leq 1 + ⨍_X f d\mu \tag{6.1}$$

holds.

Proof. Based on Proposition 2.7 (1), $⨍_X f d\mu \leq \mu(X) = 1$. Then, due to Lemma 2.11.,

$$⨍_X f d\mu = ⨍_X \min(1, f) d\mu.$$

Similarly one can show

$$\begin{aligned}\fint_X f^2 d\mu &= \fint_X \left(\min\left(1, f^2\right) d\mu\right) \\ &= \fint_X \left(\min\left(1, f\right)\right)^2 d\mu\end{aligned}$$

and

$$\begin{aligned}\fint_X f d\mu &= \fint_X \left(\min\left(1, f\right) d\mu\right) \\ &= \fint_X \left(\min\left(1, f\right)\right) d\mu.\end{aligned}$$

It is evident that $(\min(1,f))^2 \leq (\min(1,f))$, what ensures

$$\begin{aligned}\fint_X f^2 d\mu &= \fint_X \left(\min\left(1, f\right)\right)^2 d\mu \\ &\leq \fint_X \left(\min\left(1, f\right)\right) d\mu \\ &= \fint_X f d\mu.\end{aligned}$$

Consequently,

$$\left(\frac{1}{2}\right) \cdot \fint_X f^2 d\mu \leq \fint_X f d\mu \tag{6.2}$$

Adding 1 to both parts of the last inequality one obtain

$$\begin{aligned}1 &\leq 1 + \left(\frac{1}{2}\right) \cdot \fint_X f^2 d\mu \\ &\leq 1 + \fint_X f d\mu\end{aligned}$$

and therefore

$$\left(1 + \left(\frac{1}{2}\right) \cdot \fint_X f^2 d\mu\right)^{\frac{1}{2}} \leq \left(1 + \fint_X f d\mu\right),$$

proving the validity of (6.1). □

Now, by an example, we illustrate validity of theorem.

Example 6.2. Let $f : [0,1] \to [0,1]$ be defined as $f(x) = \dfrac{x+1}{2}$. A simple calculation shows that

$$\begin{aligned}\left(1 + \left(\frac{1}{2}\right) \cdot \fint_X f^2 d\mu\right)^{\frac{1}{2}} &= \left(1 + \left(\frac{1}{2}\right) \cdot \fint_X \left(\frac{x+1}{2}\right)^2 d\mu\right)^{\frac{1}{2}} \\ &= 1.1267,\end{aligned}$$

and similarly,

$$1+\fint_X f d\mu = 1+\fint_X \frac{x+1}{2} d\mu$$
$$= 1.\overline{6}.$$

Finally the relations (6.1) and (6.2) are valid.

7. Thunsdorff's Inequality for Sugeno Integral

In this section, we prove Thunsdorff's inequalities for Sugeno integral.

Firstly, by an example, we show that (1.8) is not valid for Sugeno integral.

Example 7.1. Let $f(x)=\sqrt{x}$ and let μ be the Lebesque measure.

(i) Suppose that f is defined from $[0,1]$ to $[0,1]$, $r=\frac{1}{3}$ and $s=\frac{1}{2}$. Simple calculations show that

$$\left(\frac{1+r}{b-a}\fint_0^1 f^r(x)d\mu\right)^{1/r} = \left(\left(1+\frac{1}{3}\right)\fint_0^1 \left(\sqrt{x}\right)^{1/3} d\mu\right)^3$$
$$= 1.1166,$$

and

$$\left(\frac{1+s}{b-a}\fint_0^1 f^s(x)d\mu\right)^{1/s} = \left(\left(1+\frac{1}{2}\right)\fint_0^1 \left(\sqrt{x}\right)^{1/2} d\mu\right)^2$$
$$= 1.1810.$$

Therefore,

$$1.1166 = \left(\frac{1+r}{b-a}\fint_0^1 f^r(x)d\mu\right)^{1/r}$$
$$\ngeq \left(\frac{1+s}{b-a}\fint_0^1 f^s(x)d\mu\right)^{1/s}$$
$$= 1.1810.$$

(ii) If f defined from $[1,3]$ to $[1,3]$, $r=1$ and $s=2$. We have

$$\left(\frac{1+r}{b-a}\fint_a^b f^r(x)d\mu\right)^1 = \frac{2}{2}\fint_1^3 \left(\sqrt{x}\right)^1 d\mu$$
$$= 1.3028.$$

As the same way

$$\left(\frac{1+s}{b-a}\fint_a^b f^s(x)d\mu\right)^{1/s} = \left(\frac{3}{2}\fint_1^3 \left(\sqrt{x}\right)^2 d\mu\right)^{1/2}$$
$$= 1.5.$$

It follows that

$$1.3028 \not\geq 1.5.$$

In other words, (1.8) is not valid for Sugeno integral.

Remark 7.2. Thunsdorff's inequality (1.8) can be rewritten in the following form:

$$\left((1+s)\cdot\int_{[a,b]} f^s d\lambda_{[a,b]}\right)^{\frac{1}{s}} \leq \left((1+r)\cdot\int_{[a,b]} f^r d\lambda_{[a,b]}\right)^{\frac{1}{r}},$$

where $\lambda_{[a,b]}$ is the normed Lebesgue measure on Borel subsets of $[a,b[$, i.e.,

$$\begin{aligned}\lambda_{[a,b]}(E) &= \frac{\lambda(E)}{\lambda([a,b])} \\ &= \frac{\lambda(E)}{(b-a)}.\end{aligned}$$

Also, defining a real function $\phi_{[a,b]}, f :]0,\infty[\to [0,\infty[$ by

$$\phi_{[a,b]}, f(r) = \left((1+r).\int_{[a,b]} f^r d\lambda_{[a,b]}\right)^{\frac{1}{r}},$$

the Thunsdorff inequality is equivalent to the decreasingness of $\phi_{[a,b]}, f$.

In the sequel, we present and prove a fuzzy version of (1.8) if $f : [0,1] \to [0,\infty]$ and $r, s \in (0,\infty)$.

Theorem 7.3. *Let (X,Σ,μ) be a fuzzy measure space with normed fuzzy measure μ, $f \in \mathfrak{F}^\sigma(X)$. Then for any positive real constants r, s with $r < s$, it holds*

$$\left(1+\left(\frac{1}{s}\right)\cdot ⨍_X f^s d\mu\right)^{\frac{1}{s}} \leq \left(1+\left(\frac{1}{r}\right)\cdot ⨍_X f^r d\mu\right)^{\frac{1}{r}} \tag{7.1}$$

Proof. Based on Proposition 2.7 (1), $⨍_X f d\mu \leq \mu(X) = 1$. Then, due to Lemma 2.11,

$$⨍_X f d\mu = ⨍_X \min(1,f) d\mu.$$

Similarly one can show

$$\begin{aligned}⨍_X f^s d\mu &= ⨍_X \left(\min\left(1, f^s\right) d\mu\right) \\ &= ⨍_X \left(\min\left(1, f\right)\right)^s d\mu\end{aligned}$$

and

$$\begin{aligned}\fint_X f^r d\mu &= \fint_X (\min(1, f^r)\, d\mu) \\ &= \fint_X (\min(1, f))^r\, d\mu.\end{aligned}$$

As far as $r < s$, it is evident that $(\min(1,f))^s \leq (\min(1,f))^r$, what ensures

$$\begin{aligned}\fint_X f^s d\mu &= \fint_X (\min(1, f))^s\, d\mu \\ &\leq \fint_X (\min(1, f))^r\, d\mu \\ &= \fint_X f^r d\mu.\end{aligned}$$

More, $0 < r < s$ implies $0 < \frac{1}{s} < \frac{1}{r}$, and, consequently,

$$\left(\frac{1}{s}\right) \cdot \fint_X f^s d\mu \leq \left(\frac{1}{r}\right) \cdot \fint_X f^r d\mu \tag{7.2}$$

Adding 1 to both parts of the last inequality one obtain

$$\begin{aligned}1 &\leq 1 + \left(\frac{1}{s}\right) \cdot \fint_X f^s d\mu \\ &\leq 1 + \left(\frac{1}{r}\right) \cdot \fint_X f^r d\mu\end{aligned}$$

and therefore

$$\left(1 + \left(\frac{1}{s}\right) \cdot \fint_X f^s d\mu\right)^{\frac{1}{s}} \leq \left(1 + \left(\frac{1}{r}\right) \cdot \fint_X f^r d\mu\right)^{\frac{1}{r}},$$

proving the validity of (7.1). □

Remark 7.4. (i) As a by-product, a new integral inequality (7.2) for Sugeno integral was obtained.
(ii) Define two real functions τ_X, f, μ and $\eta_X, f, \mu :]0, \infty[\to [0, \infty[$ by

$$\tau_X, f, \mu(r) = \left(1 + \left(\frac{1}{r}\right) \cdot \fint_X f^r d\mu\right)^{\frac{1}{r}}$$

and

$$\eta_X, f, \mu(r) = \left(\frac{1}{r}\right) \cdot \fint_X f^r d\mu.$$

Then our version of Thunsdorff's inequality (7.1) is equivalent to the decreasingness in r of the function τ_X, f, μ. Similarly, integral inequality

(7.2) is equivalent to the decreasingness of the function η_X, f, μ (for all $f \in \mathfrak{F}^\sigma(X)$) and any normed fuzzy measure μ on Σ.

Example 7.5. (i) Let $f : [0,1] \to [0,1]$ be defined as $f(x) = \frac{1}{x+1}$, $r = \frac{1}{3}$ and $s = \frac{1}{2}$. With simple calculations, we have

$$\begin{aligned}\left(1+\left(\frac{1}{s}\right)⨍_0^1 f^s(x)d\mu\right)^{1/s} &= \left(1+\left(\frac{1}{\frac{1}{2}}\right)⨍_0^1 \left(\frac{1}{x+1}\right)^{1/2} d\mu\right)^2 \\ &= 6.3001,\end{aligned}$$

and

$$\begin{aligned}\left(1+\left(\frac{1}{r}\right)⨍_0^1 f^r(x)d\mu\right)^{1/r} &= \left(1+\left(\frac{1}{\frac{1}{3}}\right)⨍_0^1 \left(\frac{1}{x+1}\right)^{1/3} d\mu\right)^3 \\ &= 41.318.\end{aligned}$$

Therefore, (7.1) (also (7.2)) is valid.

(ii) Let $f(x) = x^2$, $r = 1$ and $s = 2$. Then, we get

$$\begin{aligned}\left(1+\left(\frac{1}{s}\right)⨍_0^1 f^s(x)d\mu\right)^{1/s} &= \left(1+\left(\frac{1}{2}\right)⨍_0^1 (x^2)^2 d\mu\right)^{1/2} \\ &= 1.0666,\end{aligned}$$

and

$$\begin{aligned}\left(1+\left(\frac{1}{r}\right)⨍_0^1 f^r(x)d\mu\right)^{1/r} &= 1+⨍_0^1 x^2 d\mu \\ &= 1.382.\end{aligned}$$

The proof of Theorem 7.3 can be applied also in the case when the fuzzy measure μ is not normed, only then we have some constraint on the original Sugeno integral.

Corollary 7.6. *Let (X, Σ, μ) be a fuzzy measure space and $f \in \mathfrak{F}^\sigma(X)$ be such that $⨍_X f d\mu \le 1$. Then for any positive real constants $r, s, r < s$, it holds*

$$\left(1+\left(\frac{1}{s}\right)\cdot ⨍_X f^s d\mu\right)^{\frac{1}{s}} \le \left(1+\left(\frac{1}{r}\right)\cdot ⨍_X f^r d\mu\right)^{\frac{1}{r}}$$

and

$$\left(\frac{1}{s}\right)\cdot ⨍_X f^s d\mu \le \left(\frac{1}{r}\right)\cdot ⨍_X f^r d\mu.$$

Proof. The proof follows from Proposition 2.7 (5), using similar argumentation as in the proof of Theorem 7.3. □

Also the next result can be shown, considering Proposition 2.7 (5) and similar reasoning as in Theorem 7.3.

Corollary 7.7. *Let* (X, Σ, μ) *be a fuzzy measure space and* $f \in \mathfrak{F}^{\sigma}(X)$ *be such that* $⨍_X f d\mu \geq 1$. *Then*

$$r. ⨍_X f^r d\mu \leq s. ⨍_X f^s d\mu. \tag{7.3}$$

Note that if $⨍_X f d\mu = 1$, we can consider both Corollary 7.6 and Corollary 7.7, i.e., both inequalities (7.2) and (7.3) should be valid.

This is obviously true, as then $⨍_X f^r d\mu = 1$ for any $r \in (0, \infty)$, and (7.2) turns into $\frac{1}{s} \leq \frac{1}{r}$, while (7.3) turns into $r \leq s$, which are equivalent inequalities.

8. Conclusion

We have proved the related Gauss-Winkler type inequality for fuzzy integrals. More precisely, we show that

$$\left(⨍_0^1 x^2 f(x) d\mu\right)^2 \leq 1.4255 \left(⨍_0^1 f(x) d\mu\right)\left(⨍_0^1 x^4 f(x) d\mu\right),$$

holds where $f : [0,1] \to [0, \infty)$ is a non-decreasing function and μ is the Lebesgue measure on $\mathbb{R}$. In Section 4, the paper proposed a Minkowski type inequality for fuzzy integrals. We know that, the Prékopa-Leindler inequality is useful in the theory of log-concave distributions, as it can be used to show that log-concavity is preserved by marginalization and independent summation of log-concave distributed random variables. The Prékopa-Leindler inequality is the functional form of Brunn-Minkowski inequality. The Brunn-Minkowski inequality plays a central role in geometric convex analysis. It can be stated as follows: If X, Y are bounded measurable sets such that $(1-\lambda)X + \lambda Y$ is also measurable with $\lambda \in (0,1)$, then

$$\mu\left((1-\lambda)X + \lambda Y\right) \geq \mu(X)^{1-\lambda}\mu(Y)^{\lambda},$$

where μ is the Lebesgue measure on $\mathbb{R}^n$. The main result of this paper has shown a Prékopa-Leindler type inequality for the Sugeno integral obtained under relation to a concave/quasi-concave fuzzy measure. Finally in this paper, we prove the Thunsdorff's and Frank-Pick's inequalities for Sugeno integrals. By considering the initial conditions for the Thunsdorff's inequality, we proved this inequality. Indeed, we showed that:

If $f : [0,1] \to [0,\infty]$, $r, s \in (0,\infty)$ and $r < s$ then:

$$\left(1+\left(\frac{1}{s}\right)\cdot \fint_X f^s d\mu\right)^{\frac{1}{s}} \leq \left(1+\left(\frac{1}{r}\right)\cdot \fint_X f^r d\mu\right)^{\frac{1}{r}},$$

holds, and if f is defined as $f : [0,1] \to [0,1]$, then

$$\left(1+\left(\frac{1}{2}\right)\cdot \fint_X f^2 d\mu\right)^{\frac{1}{2}} \leq 1+\fint_X f d\mu,$$

holds. Also, by examples, we show the validity of theorems.

In the future works, we aim to discuss these inequalities for pseudo and Choquet integrals.

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