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The Generalized Inequalities via Means and Positive Linear Mappings

Leila Nasiri[1] and Mehdi Shams[2*]

Abstract. In this paper, we establish further improvements of the Young inequality and its reverse. Then, we assert operator versions corresponding them. Moreover, an application including positive linear mappings is given. For example, if $A, B \in \mathbb{B}(\mathscr{H})$ are two invertible positive operators such that $0 < m \le A \le m^{'} < M^{'} \le B \le M$ or $0 < m \le B \le m^{'} < M^{'} \le A \le M$ for some positive real numbers $m, m^{'}, M, M^{'}$ and let Φ be a positive unital linear map, then for every $0 \le \nu < \mu \le 1$

$$\begin{aligned}\Phi^2\Bigg(&A\nabla_\nu B + rMm\left(A^{-1} + A^{-1}\sharp_\mu B^{-1} - 2\left(A^{-1}\sharp_{\frac{\mu}{2}} B^{-1}\right)\right)\\ &+\left(\frac{\nu}{\mu}\right)Mm\Big(A^{-1}\nabla_\mu B^{-1} - A^{-1}\sharp_\mu B^{-1}\Big)\Bigg)\\ &\le \left(\frac{K(h)}{K\left(\sqrt{h'^{\mu}}, 2\right)^{r'}}\right)^2 \Phi^2(A\sharp_\nu B),\end{aligned}$$

where $r = \min\{\nu, 1-\nu\}$, $K(h) = \frac{(1+h)^2}{4h}$, $h = \frac{M}{m}$, $h^{'} = \frac{M^{'}}{m^{'}}$ and $r^{'} = \min\{2r, 1-2r\}$. The results of this paper generalize the results of recent years.

1. Introduction

Suppose that a, b are two positive real numbers and $0 \le \nu \le 1$, the classical inequality

$$a^{1-\nu} b^{\nu} \le (1-\nu)a + \nu b, \tag{1.1}$$

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is known as Young inequality. The inequality (1.1) is also called the weighted arithmetic-geometric mean inequality when:

$$a\sharp_\nu b \le a\nabla_\nu b,$$

where $a\nabla_\nu b = (1-\nu)a + \nu b$ and $a\sharp_\nu b = a^{1-\nu}b^\nu$ [8]. For $a, b > 0$ and $0 \le \nu \le 1$, the authors, see [12], presented the following result:

$$K(h,2)^r a\sharp_\nu b \le a\nabla_\nu b, \tag{1.2}$$

where $r = \min\{\nu, 1-\nu\}$, $h = \frac{b}{a}$ and $K(h,2) = \frac{(1+h)^2}{4h}$ is the Kantorovich constant. After a short time, the authors in [7] showed a reversed version of (1.2) as follows:

$$a\nabla_\nu b \le K(h,2)^R a\sharp_\nu b, \tag{1.3}$$

where $R = \max\{\nu, 1-\nu\}$. Wu and Zhao [10] refined both inequalities (1.2) and (1.3) and obtained the following double inequality:

$$\begin{aligned} K\left(\sqrt{h},2\right)^{r'} a\sharp_\nu b + r\left(\sqrt{a}-\sqrt{b}\right)^2 &\le a\nabla_\nu b \\ &\le R\left(\sqrt{a}-\sqrt{b}\right)^2 + K\left(\sqrt{h},2\right)^{R'} a\sharp_\nu b, \end{aligned} \tag{1.4}$$

where $r = \min\{\nu, 1-\nu\}$, $R = \max\{\nu, 1-\nu\}$, $r' = \min\{2r, 1-2r\}$, $R' = \max\{2r, 1-2r\}$ and $h = \frac{b}{a}$.

Our aim, in this note, is to refine the double inequality (1.4) and obtain corresponding inequalities for positive operators. Moreover, as application of our work, we obtain some inequalities via positive linear maps which are improvements of the derived results in [9] and [11].

2. Main Results

2.1. **Inequalities for Positive Numbers.**

Theorem 2.1. *Let $a, b > 0$ and $0 \le \nu < \mu \le 1$. Then*

$$\begin{aligned} r\left(\sqrt{a}-\sqrt{a\sharp_\mu b}\right)^2 + K\left(\sqrt{h^\mu},2\right)^{r'} a\sharp_\nu b \\ \le a\nabla_\nu b - \left(\frac{\nu}{\mu}\right)(a\nabla_\mu b - a\sharp_\mu b) \\ \le R\left(\sqrt{a}-\sqrt{a\sharp_\mu b}\right)^2 + K\left(\sqrt{h^\mu},2\right)^{R'} a\sharp_\nu b, \end{aligned} \tag{2.1}$$

where $r=\min\left\{\frac{\nu}{\mu},1-\frac{\nu}{\mu}\right\}$, $R=\max\left\{\frac{\nu}{\mu},1-\frac{\nu}{\mu}\right\}$, $r^{'}=\min\{2r,1-2r\}$, $R^{'}=\max\{2r,1-2r\}$ *and* $h=\frac{b}{a}$. *Moreover,*

$$
\begin{aligned}
(2.2)\qquad & r\left(\sqrt{a}-\sqrt{a\sharp_{\nu}b}\right)^{2}+K\left(\sqrt{h^{1-\nu}},2\right)^{r^{'}}a\sharp_{\mu}b\\
&\leq a\nabla_{\mu}b-\left(\frac{1-\mu}{1-\nu}\right)(a\nabla_{\nu}b-a\sharp_{\nu}b)\\
&\leq R\left(\sqrt{a}-\sqrt{a\sharp_{\nu}b}\right)^{2}+K\left(\sqrt{h^{1-\nu}},2\right)^{R^{'}}a\sharp_{\mu}b,
\end{aligned}
$$

where $r=\min\left\{\frac{1-\mu}{1-\nu},1-\frac{1-\mu}{1-\nu}\right\}$, $R=\max\left\{\frac{1-\mu}{1-\nu},1-\frac{1-\mu}{1-\nu}\right\}$, $h=\frac{b}{a}$, $r^{'}=\min\{2r,1-2r\}$ *and* $R^{'}=\max\{2r,1-2r\}$.

Proof. The definition of the weighted arithmetic mean implies that

$$
\begin{aligned}
(2.3)\qquad a\nabla_{\nu}b-\left(\frac{\nu}{\mu}\right)(a\nabla_{\mu}b-a\sharp_{\mu}b)&=\left(1-\frac{\nu}{\mu}\right)a+\frac{\nu}{\mu}(a\sharp_{\mu}b)\\
&=a\nabla_{\frac{\nu}{\mu}}(a\sharp_{\mu}b).
\end{aligned}
$$

Making use of the double inequality (1.4) for the inequality (2.3), we have

$$
\begin{aligned}
r\left(\sqrt{a}-\sqrt{a\sharp_{\mu}b}\right)^{2}+K\left(\sqrt{h^{\mu}},2\right)^{r^{'}}a^{1-\nu}b^{\nu}&\leq a\nabla_{\frac{\nu}{\mu}}(a\sharp_{\mu}b)\\
&\leq R\left(\sqrt{a}-\sqrt{a\sharp_{\mu}b}\right)^{2}\\
&\quad+K\left(\sqrt{h^{\mu}},2\right)^{R^{'}}a^{1-\nu}b^{\nu}.
\end{aligned}
$$

This proves the inequality (2.1) as desired. To prove the inequality (2.2), it is enough to apply the inequality (2.1) for $1-\mu<1-\nu$ and change a and b. ☐

Remark 2.2. In special case, when $\nu=\mu=\frac{1}{2}$, the obtained inequalities in 2.1 become to equalities.

In following, we obtain two inequalities that are the direct consequences of Theorem 2.1.

Corollary 2.3. *Let* $a,b>0$ *and* $0\leq\nu<\mu\leq1$. *Then*

$$
\begin{aligned}
&\frac{r}{2}\left[\left(\sqrt{a}-\sqrt{a\sharp_{\mu}b}\right)^{2}+\left(\sqrt{b}-\sqrt{a\sharp_{\mu}b}\right)^{2}\right]+K\left(\sqrt{h^{\mu}},2\right)^{r^{'}}H_{\nu}(a,b)\\
&\quad\leq\left(1-\frac{\nu}{\mu}\right)a\nabla b+\left(\frac{\nu}{\mu}\right)H_{\mu}(a,b)\\
&\quad\leq\frac{R}{2}\left[\left(\sqrt{a}-\sqrt{a\sharp_{\mu}b}\right)^{2}+\left(\sqrt{b}-\sqrt{a\sharp_{\mu}b}\right)^{2}\right]+K\left(\sqrt{h^{\mu}},2\right)^{R^{'}}H_{\nu}(a,b),
\end{aligned}
$$

where $r=\min\{\frac{\nu}{\mu},1-\frac{\nu}{\mu}\}$, $R=\max\{\frac{\nu}{\mu},1-\frac{\nu}{\mu}\}$, $r^{'}=\min\{2r,1-2r\}$, $R^{'}=\max\{2r,1-2r\}$ *and* $h=\frac{b}{a}$. *Moreover,*

$$\begin{aligned}\frac{r}{2}&\left[\left(\sqrt{a}-\sqrt{a\sharp_\nu b}\right)^2+\left(\sqrt{b}-\sqrt{a\sharp_\nu b}\right)^2\right]+K\left(\sqrt{h^{1-\nu}},2\right)^{r^{'}}H_\mu(a,b)\\ &\leq\left(1-\frac{1-\mu}{1-\nu}\right)a\nabla b+\left(\frac{1-\mu}{1-\nu}\right)H_\nu(a,b)\\ &\leq\frac{R}{2}\left[\left(\sqrt{a}-\sqrt{a\sharp_\nu b}\right)^2+\left(\sqrt{b}-\sqrt{a\sharp_\nu b}\right)^2\right]\\ &\quad+K\left(\sqrt{h^{1-\nu}},2\right)^{R^{'}}H_\mu(a,b),\end{aligned}$$

where $r=\min\left\{\frac{1-\mu}{1-\nu},1-\frac{1-\mu}{1-\nu}\right\}$, $R=\max\left\{\frac{1-\mu}{1-\nu},1-\frac{1-\mu}{1-\nu}\right\}$, $h=\frac{b}{a}$, $r^{'}=\min\{2r,1-2r\}$ *and* $R^{'}=\max\{2r,1-2r\}$.

2.2. **Inequalities via Positive Operators.** Let $\mathbb{B}(\mathscr{H})$ be the C^*- algebra of all bounded linear operators on a complex Hilbert space $\mathscr{H}$ whose identity is denoted by I. A Hermitian operator $A\in\mathbb{B}(\mathscr{H})$ is called positive (strictly positive, resp.) and we write $A\geq 0(A>0$, resp.) if $\langle Ax,x\rangle\geq 0$ ($\langle Ax,x\rangle>0$ resp.) for all $x\in\mathscr{H}$. If $A,B\in\mathbb{B}(\mathscr{H})$ are two Hermitian operators, the order relation $A\geq B(A>B$, resp.) means, as usual, that $A-B\geq 0(A-B>0$, resp.). The Gelfand map $f(t)\mapsto f(A)$ is an isometrical $*$-isomorphism between the C^*-algebra $C(\mathrm{Sp}(A))$ of continuous functions on the spectrum $\mathrm{Sp}(A)$ of a self-adjoint operator A and the C^*-algebra generated by A and I. If $f,g\in C(\mathrm{Sp}(A))$, then $f(t)\geq g(t)$ $(t\in\mathrm{Sp}(A))$ implies that $f(A)\geq g(A)$. If $A,B\in\mathbb{B}(\mathscr{H})$ be strictly positive and $0\leq\nu\leq 1$, we use following notations to define the $\nu-$weighted arithmetic mean and geometric mean of A and B, respectively [8]:

$$\begin{aligned}A\nabla_\nu B&=(1-\nu)A+\nu B,\\ A\sharp_\nu B&=A^{\frac{1}{2}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^\nu A^{\frac{1}{2}}.\end{aligned}$$

Now, we are ready to state operator versions corresponding to the inequalities (2.1) and (2.2) and Corollary 2.3 via the Gelfand map.

Theorem 2.4. *Suppose that* $A,B\in\mathbb{B}(\mathscr{H})$ *are two invertible positive operators such that* $0<m\leq A\leq m^{'}<M^{'}\leq B\leq M$ *or* $0<m\leq B\leq m^{'}<M^{'}\leq A\leq M$ *for some positive real numbers* $m,m^{'},M,M^{'}$. *Then for every* $0\leq\nu<\mu\leq 1$*:*

$$r\left(A+A\sharp_\mu B-2\left(A\sharp_{\frac{\mu}{2}}B\right)\right)+K\left(\sqrt{h^{'\mu}},2\right)^{r^{'}}A\sharp_\nu B \tag{2.4}$$

$$\leq A\nabla_\nu B - \left(\frac{\nu}{\mu}\right)(A\nabla_\mu B - A\sharp_\mu B)$$
$$\leq R\left(A + A\sharp_\mu B - 2\left(A\sharp_{\frac{\mu}{2}} B\right)\right) + K\left(\sqrt{h^\mu}, 2\right)^{R'} A\sharp_\nu B,$$

where $r = \min\left\{\frac{\nu}{\mu}, 1 - \frac{\nu}{\mu}\right\}$, $R = \max\left\{\frac{\nu}{\mu}, 1 - \frac{\nu}{\mu}\right\}$, $r' = \min\{2r, 1-2r\}$, $R' = \max\{2r, 1-2r\}$, $h = \frac{M}{m}$ *and* $h = \frac{M'}{m'}$. *Also*

$$r\left(A + A\sharp_\nu B - 2\left(A\sharp_{\frac{\nu}{2}} B\right)\right) + K\left(\sqrt{h'^{1-\nu}}, 2\right)^{r'} A\sharp_\mu B \tag{2.5}$$
$$\leq A\nabla_\mu B - \left(\frac{1-\mu}{1-\nu}\right)(A\nabla_\nu B - A\sharp_\nu B)$$
$$\leq R\left(A + A\sharp_\nu B - 2\left(A\sharp_{\frac{\nu}{2}} B\right)\right) + K\left(\sqrt{h^{1-\nu}}, 2\right)^{R'} A\sharp_\mu B,$$

where $r = \min\left\{\frac{1-\mu}{1-\nu}, 1 - \frac{1-\mu}{1-\nu}\right\}$, $R = \max\left\{\frac{1-\mu}{1-\nu}, 1 - \frac{1-\mu}{1-\nu}\right\}$, $r' = \min\{2r, 1-2r\}$, $R' = \max\{2r, 1-2r\}$, $h = \frac{M}{m}$ *and* $h = \frac{M'}{m'}$.

Proof. By taking $a = 1$ and $b = x$ in double inequalities (2.1), we have

$$r\left(1 + x^\mu - 2x^{\frac{\mu}{2}}\right) + K\left(\sqrt{x^\mu}, 2\right)^{r'} x^\nu$$
$$\leq (1-\nu) + \nu x - \left(\frac{\nu}{\mu}\right)((1-\mu) + \mu x - x^\mu)$$
$$\leq R\left(1 + x^\mu - 2x^{\frac{\mu}{2}}\right) + K\left(\sqrt{x^\mu}, 2\right)^{R'} x^\nu.$$

It follows from $0 < m \leq A \leq m' < M' \leq B \leq M$ that

$$1 < h'I = \frac{M'}{m'}I \leq A^{-\frac{1}{2}}BA^{-\frac{1}{2}} \leq \frac{M}{m}I = hI.$$

By putting $X = A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$, we find that $\mathrm{Sp}(X) \subseteq [h', h] \subset (1, +\infty)$. Now, by applying the Gelfand map for the above inequalities and by the increasing property of the function $K(t)$ on the interval $(1, +\infty)$; see [4], we get

$$r\left(I + X^\mu - 2X^{\frac{\mu}{2}}\right) + K\left(\sqrt{h'^\mu}, 2\right)^{r'} X^\nu \tag{2.6}$$
$$\leq (1-\nu)I + \nu X - \left(\frac{\nu}{\mu}\right)((1-\mu)I + \mu X - X^\mu)$$
$$\leq R\left(I + X^\mu - 2X^{\frac{\mu}{2}}\right) + K\left(\sqrt{h^\mu}, 2\right)^{R'} X^\nu.$$

Finally, multiplying the both sides of double inequalities (2.6) by $A^{\frac{1}{2}}$ in a similar way, we obtain the desired result. The inequalities in (2.4) can be proved under the condition $0 < m \leq B \leq m^{'} < M^{'} \leq A \leq M$. The proof of (2.5) is completely similar. So, we leave it to the reader. □

It is obvious that for $\nu = \mu = \frac{1}{2}$, the inequalities become to equalities.

Corollary 2.5. *Suppose that $A, B \in \mathbb{B}(\mathscr{H})$ are two invertible positive operators such that $0 < m \leq A \leq m^{'} < M^{'} \leq B \leq M$ or $0 < m \leq B \leq m^{'} < M^{'} \leq A \leq M$ for some positive real numbers $m, m^{'}, M, M^{'}$. Then for every $0 \leq \nu < \mu \leq 1$, we have*

$$\begin{aligned}
&\frac{r}{2}\left[A + B + 2\left(A\sharp_{\mu}B\right) - 2\left(A\sharp_{\frac{\mu}{2}}B\right) - 2\left(A\sharp_{\frac{1+\mu}{2}}B\right)\right] \\
&\quad + K\left(\sqrt{h^{'\mu}}, 2\right)^{r^{'}} H_{\nu}(A, B) \\
&\leq \left(1 - \frac{\nu}{\mu}\right) A\nabla B + \left(\frac{\nu}{\mu}\right) H_{\mu}(A, B) \\
&\leq \frac{R}{2}\left[A + B + 2\left(A\sharp_{\mu}B\right) - 2\left(A\sharp_{\frac{\mu}{2}}B\right) - 2\left(A\sharp_{\frac{1+\mu}{2}}B\right)\right] \\
&\quad + K\left(\sqrt{h^{\mu}}, 2\right)^{R^{'}} H_{\nu}(A, B),
\end{aligned}$$

$$\begin{aligned}
&\frac{r}{2}\left[A + B + 2\left(A\sharp_{\nu}B\right) - 2\left(A\sharp_{\frac{\nu}{2}}B\right) - 2\left(A\sharp_{\frac{1+\nu}{2}}B\right)\right] \\
&\quad + K\left(\sqrt{h^{'1-\nu}}, 2\right)^{r^{'}} H_{\mu}(A, B) \\
&\leq \left(1 - \frac{1-\mu}{1-\nu}\right) A\nabla B + \left(\frac{1-\mu}{1-\nu}\right) H_{\nu}(A, B) \\
&\leq \frac{R}{2}\left[A + B + 2\left(A\sharp_{\nu}B\right) - 2\left(A\sharp_{\frac{\nu}{2}}B\right) - 2\left(A\sharp_{\frac{1+\nu}{2}}B\right)\right] \\
&\quad + K\left(\sqrt{h^{1-\nu}}, 2\right)^{R^{'}} H_{\mu}(A, B).
\end{aligned}$$

3. Operator Inequalities via Positive Linear Maps

In [4], the authors proved an operator version of the Young inequality as follows:

$$A\sharp_{\nu}B \leq A\nabla_{\nu}B.$$

For $\nu = \frac{1}{2}$, it follows that

$$A\sharp B \leq A\nabla B.$$

Lin [6] reversed the inequality above utilizing the Kantorovich constant in following form:

$$\Phi(A\nabla B) \le K(h)\Phi(A\sharp B), \tag{3.1}$$

where $A, B \in \mathbb{B}(\mathscr{H})$ satisfy $m \le A, B \le M$. In recent years, generalizations, refinements and improvements of (3.1) have been presented. For example, M. Bakherad in [1] established the next result which both generalize and improve the inequality (3.1).

Theorem 3.1 ([1]). *Let $0 < m \le A, B \le M$. Then for every positive unital linear map Φ, $0 \le \nu \le 1$ and for every $p > 0$:*

$$\Phi^p\left(A\nabla_\nu B + 2rMm\left(A^{-1}\nabla B^{-1} - A^{-1}\sharp B^{-1}\right)\right) \le \alpha^p \Phi^p(A\sharp_\nu B),$$

$$\Phi^p\left(A\nabla_\nu B + 2rMm\left(A^{-1}\nabla B^{-1} - A^{-1}\sharp B^{-1}\right)\right) \le \alpha^p (\Phi(A)\sharp_\nu \Phi(B))^p,$$

where $r = \min\{\nu, 1-\nu\}$ and $\alpha = \max\left\{\frac{(M+m)^2}{4Mm}, \frac{(M+m)^2}{4^{\frac{2}{p}}Mm}\right\}$.

After short time, the authors of [11] gave another refinement of the inequality (3.1) and obtained the following inequalities:

Theorem 3.2 ([11]). *Let $0 < m \le A \le m' < M' \le B \le M$. Then for every positive unital linear map Φ, $0 \le \nu \le 1$ and for every $p \ge 2$:*

$$\Phi^p(A\nabla_\nu B) \le \left(\frac{K(h)}{4^{\frac{2}{p}-1}K^r(h')}\right)^p \Phi^p(A\sharp_\nu B),$$

$$\Phi^p(A\nabla_\nu B) \le \left(\frac{K(h)}{4^{\frac{2}{p}-1}K^r(h')}\right)^p (\Phi(A)\sharp_\nu \Phi(B))^p,$$

where $r = \min\{\nu, 1-\nu\}$, $h = \frac{M}{m}$, $h' = \frac{M'}{m'}$ and $K(h) = \frac{(1+h)^2}{4h}$.

The following Lemmas are essential to give the results of this section.

Lemma 3.3 ([2]). (*Choi's inequality*) *Let $A \in \mathbb{B}(\mathscr{H})$ be positive and Φ be a positive unital linear map. Then*

$$\Phi(A)^{-1} \le \Phi\left(A^{-1}\right). \tag{3.2}$$

Lemma 3.4 ([1, 3, 5]). *Let $A, B \in \mathbb{B}(\mathscr{H})$ be positive and $\alpha > 0$. Then*

(i) $\|\mathrm{AB}\| \le \frac{1}{4}\|A+B\|^2$.
(ii) $\|A^\alpha + B^\alpha\| \le \|(A+B)^\alpha\|$.
(iii) $A \le \alpha B$ *if and only if* $\left\|A^{\frac{1}{2}}B^{-\frac{1}{2}}\right\| \le \alpha^{\frac{1}{2}}$.

Theorem 3.5. *Suppose that $A, B \in \mathbb{B}(\mathscr{H})$ are two invertible positive operators such that $0 < m \le A \le m' < M' \le B \le M$*

or $0 < m \leq B \leq m' < M' \leq A \leq M$ *for some positive real numbers* m, m', M, M' *and let* Φ *be a positive unital linear map. Then for every* $0 \leq \nu < \mu \leq 1$*:*

$$\begin{aligned}\Phi^2\Bigg(&A\nabla_\nu B + rMm\left(A^{-1} + A^{-1}\sharp_\mu B^{-1} - 2\left(A^{-1}\sharp_{\frac{\mu}{2}} B^{-1}\right)\right)\\ &+ \left(\frac{\nu}{\mu}\right) Mm\Big(A^{-1}\nabla_\mu B^{-1} - A^{-1}\sharp_\mu B^{-1}\Big)\Bigg)\\ &\leq \left(\frac{K(h)}{K\left(\sqrt{h'^{\mu}}, 2\right)^{r'}}\right)^2 \Phi^2(A\sharp_\nu B),\end{aligned} \tag{3.3}$$

and

$$\begin{aligned}\Phi^2\Bigg(&A\nabla_\nu B + rMm\left(A^{-1} + A^{-1}\sharp_\mu B^{-1} - 2\left(A^{-1}\sharp_{\frac{\mu}{2}} B^{-1}\right)\right)\\ &+ \left(\frac{\nu}{\mu}\right) Mm\Big(A^{-1}\nabla_\mu B^{-1} - A^{-1}\sharp_\mu B^{-1}\Big)\Bigg)\\ &\leq \left(\frac{K(h)}{K\left(\sqrt{h'^{\mu}}, 2\right)^{r'}}\right)^2 (\Phi(A)\sharp_\nu \Phi(B))^2,\end{aligned} \tag{3.4}$$

where $r = \min\{\nu, 1-\nu\}$, $K(h) = \frac{(1+h)^2}{4h}$, $h = \frac{M}{m}$, $h' = \frac{M'}{m'}$ *and* $r' = \min\{2r, 1-2r\}$.

Proof. By applying the left-hand side of the inequality (2.4) for A^{-1} and B^{-1} instead of A and B, respectively, and by using linear property of Φ, it follows that

$$\begin{aligned}\Phi\Bigg(&r\left(A^{-1} + A^{-1}\sharp_\mu B^{-1} - 2\left(A^{-1}\sharp_{\frac{\mu}{2}} B^{-1}\right)\right)\\ &+ \left(\frac{\nu}{\mu}\right)\left(A^{-1}\nabla_\mu B^{-1} - A^{-1}\sharp_\mu B^{-1}\right)\\ &+ K\left(\sqrt{h'^{\mu}}, 2\right)^{r'}\left(A^{-1}\sharp_\nu B^{-1}\right)\Bigg)\\ \leq\ &\Phi\left(A^{-1}\nabla_\nu B^{-1}\right).\end{aligned} \tag{3.5}$$

From $0 < m \leq A, B \leq M$ and by the fact that Φ is linear, one can get

$$\Phi(A) + Mm\Phi(A^{-1}) \leq Mm,$$
$$\Phi(B) + Mm\Phi(B^{-1}) \leq Mm.$$

By multiplying the above inequalities by $(1-\nu)$ and ν, respectively, summing up the derived results and using the linearity of Φ, respectively, we get

$$\Phi\left(A\nabla_\nu B\right)+Mm\Phi\Big(A^{-1}\nabla_\nu B^{-1}\Big)\le M+m. \tag{3.6}$$

Also we have

$$\begin{aligned}
&\Big\|\Phi\Big(A\nabla_\nu B+rMm\left(A^{-1}+A^{-1}\sharp_\mu B^{-1}-2\left(A^{-1}\sharp_{\frac{\mu}{2}}B^{-1}\right)\right)\\
&\quad+\left(\frac{\nu}{\mu}\right)Mm\Big(A^{-1}\nabla_\mu B^{-1}-A^{-1}\sharp_\mu B^{-1}\Big)\Big)MmK\Big(\sqrt{h'^{\mu}},2\Big)^{r'}\Phi^{-1}(A\sharp_\nu B)\Big\|\\
&\le\frac{1}{4}\Big\|\Phi\Big(A\nabla_\nu B+rMm\left(A^{-1}+A^{-1}\sharp_\mu B^{-1}-2\left(A^{-1}\sharp_{\frac{\mu}{2}}B^{-1}\right)\right)\\
&\quad+\left(\frac{\nu}{\mu}\right)Mm\Big(A^{-1}\nabla_\mu B^{-1}-A^{-1}\sharp_\mu B^{-1}\Big)\Big)\\
&\quad+MmK\Big(\sqrt{h'^{\mu}},2\Big)^{r'}\Phi^{-1}(A\sharp_\nu B)\Big\|^2 \text{ (by Lemma 3.4(i))}\\
&\le\frac{1}{4}\Big\|\Phi\Big(A\nabla_\nu B+rMm\left(A^{-1}+A^{-1}\sharp_\mu B^{-1}-2\left(A^{-1}\sharp_{\frac{\mu}{2}}B^{-1}\right)\right)\\
&\quad+\left(\frac{\nu}{\mu}\right)Mm\left(A^{-1}\nabla_\mu B^{-1}-A^{-1}\sharp_\mu B^{-1}\right)\Big)\\
&\quad+MmK\Big(\sqrt{h'^{\mu}},2\Big)^{r'}\Phi(A^{-1}\sharp_\nu B^{-1})\Big\|^2 \text{ (by (3.2))}\\
&=\frac{1}{4}\Big\|\Phi(A\nabla_\nu B)+Mm\Big(\Phi\Big(r\left(A^{-1}+A^{-1}\sharp_\mu B^{-1}-2A^{-1}\sharp_{\frac{\mu}{2}}B^{-1}\right)\\
&\quad+\left(\frac{\nu}{\mu}\right)\Big(A^{-1}\nabla_\mu B^{-1}-A^{-1}\sharp_\mu B^{-1}\Big)\\
&\quad+K\Big(\sqrt{h'^{\mu}},2\Big)^{r'}\left(A^{-1}\sharp_\nu B^{-1}\right)\Big)\Big)\Big\|^2 \text{ (since } f \text{ is linear)}\\
&\le\frac{1}{4}\Big\|\Phi\left(A\nabla_\nu B\right)+Mm\Phi\Big(A^{-1}\nabla_\nu B^{-1}\Big)\Big\|^2 \text{ (by (3.5))}\\
&\le\frac{(M+m)^2}{4} \text{ (by (3.6))}.
\end{aligned}$$

This proves inequality (3.3) as desired. In a similar way, one can proves inequality (3.4). □

Making use of Theorem 3.5 and Lemma 3.4 (ii), we obtain the following result:

Corollary 3.6. *Under assumptions of Theorem 3.5, for every positive unital linear map* Φ, $0 \leq \nu < \mu \leq 1$ *and for every* $p > 0$ *we have:*

$$\begin{aligned}\Phi^p\Bigg(& A\nabla_\nu B + rMm\left(A^{-1} + A^{-1}\sharp_\mu B^{-1} - 2\left(A^{-1}\sharp_{\frac{\mu}{2}} B^{-1}\right)\right) \\ &+ \left(\frac{\nu}{\mu}\right) Mm\Big(A^{-1}\nabla_\mu B^{-1} - A^{-1}\sharp_\mu B^{-1}\Big)\Bigg) \\ &\leq \left(\frac{K(h)}{K\left(\sqrt{h'^{\mu}}, 2\right)^{r'}}\right)^p \Phi^p(A\sharp_\nu B),\end{aligned} \tag{3.7}$$

and

$$\begin{aligned}\Phi^p\Bigg(& A\nabla_\nu B + rMm\left(A^{-1} + A^{-1}\sharp_\mu B^{-1} - 2\left(A^{-1}\sharp_{\frac{\mu}{2}} B^{-1}\right)\right) \\ &+ \left(\frac{\nu}{\mu}\right) Mm\Big(A^{-1}\nabla_\mu B^{-1} - A^{-1}\sharp_\mu B^{-1}\Big)\Bigg) \\ &\leq \left(\frac{K(h)}{K\left(\sqrt{h'^{\mu}}, 2\right)^{r'}}\right)^p (\Phi(A)\sharp_\nu \Phi(B))^p,\end{aligned} \tag{3.8}$$

where $r = \min\{\nu, 1-\nu\}$, $K(h) = \frac{(1+h)^2}{4h}$, $h = \frac{M}{m}$, $h' = \frac{M'}{m'}$ *and* $r' = \min\{2r, 1-2r\}$.

Proof. If we consider $0 < p \leq 2$, then by applying Theorem 3.5 and the Lowner-Heinz inequality, the result follows. If $p \geq 2$, then making use of Theorem 3.5 and Lemma 3.4 (ii), we get the desired result. □

Remark 3.7. It is obvious that

$$\begin{aligned}\Phi^p(A\nabla_\nu B) \leq\ & \Phi^p(A\nabla_\nu B) \\ &+ M^p m^p \Phi^p\Bigg(r\left(A^{-1} + A^{-1}\sharp_\mu B^{-1} - 2\left(A^{-1}\sharp_{\frac{\mu}{2}} B^{-1}\right)\right) \\ &+ \left(\frac{\nu}{\mu}\right)\left(A^{-1}\nabla_\mu B^{-1} - A^{-1}\sharp_\mu B^{-1}\right)\Bigg).\end{aligned}$$

Therefore,

$$\begin{aligned}&\|\Phi^p(A\nabla_\nu B)\| \\ &\leq \Bigg\|\Phi^p(A\nabla_\nu B) + M^p m^p \Phi^p\Bigg(r\left(A^{-1} + A^{-1}\sharp_\mu B^{-1} - 2\left(A^{-1}\sharp_{\frac{\mu}{2}} B^{-1}\right)\right) \\ &\quad + \left(\frac{\nu}{\mu}\right)\left(A^{-1}\nabla_\mu B^{-1} - A^{-1}\sharp_\mu B^{-1}\right)\Bigg)\Bigg\|\end{aligned}$$

$$\leq \left\| \Phi^p \Big(A\nabla_\nu B + Mmr \left(A^{-1} + A^{-1}\sharp_\mu B^{-1} - 2\left(A^{-1}\sharp_{\frac{\mu}{2}} B^{-1} \right) \right) + \left(\frac{\nu}{\mu}\right) Mm \Big(A^{-1}\nabla_\mu B^{-1} - A^{-1}\sharp_\mu B^{-1} \Big) \Big) \right\|.$$

Therefore, the left-hand side of the inequalities (3.7) and (3.8) are the norm refinements of the inequalities given in [1] and [11]. On the other hand, since the Kantorovich constant $K(h)$ is an increasing function on the interval $(1, +\infty)$ and also $K(h) \geq 1$ for every $h > 0$, so the right-hand side of inequalities (3.7) and (3.8) refine the right-hand side of inequalities obtained in [1], respectively.

Recently, the authors of [11] proved the following inequalities for $p \geq 4$:

$$\Phi^p(A\nabla_\nu B) \leq \left(\frac{K(h)\left(M^2+m^2\right)}{4^{\frac{2}{p}} Mm K^r(h')} \right)^p \Phi^p(A\sharp_\nu B); \tag{3.9}$$

$$\Phi^p(A\nabla_\nu B) \leq \left(\frac{K(h)\left(M^2+m^2\right)}{4^{\frac{2}{p}} Mm K^r(h')} \right)^p (\Phi(A)\sharp_\nu \Phi(B))^p. \tag{3.10}$$

The following theorem refine inequalities (3.9) and (3.10), respectively.

Theorem 3.8. *Under assumptions of Theorem 3.5, for every positive unital linear map* Φ, $0 \leq \nu < \mu \leq 1$ *and for every* $p \geq 4$ *we have:*

$$\begin{aligned} &\Phi^p \Big(A\nabla_\nu B + rMm \left(A^{-1} + A^{-1}\sharp_\mu B^{-1} - 2\left(A^{-1}\sharp_{\frac{\mu}{2}} B^{-1} \right) \right) \\ &\quad + \left(\frac{\nu}{\mu}\right) Mm \Big(A^{-1}\nabla_\mu B^{-1} - A^{-1}\sharp_\mu B^{-1} \Big) \Big) \\ &\leq \left(\frac{K(h)\left(M^2+m^2\right)}{4^{\frac{2}{p}} Mm K\left(\sqrt{h'^{\mu}}, 2\right)^{r'}} \right)^p \Phi^p(A\sharp_\nu B), \end{aligned} \tag{3.11}$$

and

$$\begin{aligned} &\Phi^p \Big(A\nabla_\nu B + rMm \left(A^{-1} + A^{-1}\sharp_\mu B^{-1} - 2\left(A^{-1}\sharp_{\frac{\mu}{2}} B^{-1} \right) \right) \\ &\quad + \left(\frac{\nu}{\mu}\right) Mm \Big(A^{-1}\nabla_\mu B^{-1} - A^{-1}\sharp_\mu B^{-1} \Big) \Big) \\ &\leq \left(\frac{K(h)\left(M^2+m^2\right)}{4^{\frac{2}{p}} Mm K\left(\sqrt{h'^{\mu}}, 2\right)^{r'}} \right)^p (\Phi(A)\sharp_\nu \Phi(B))^p, \end{aligned} \tag{3.12}$$

where $r=\min\{\nu,1-\nu\}$, $K(h)=\frac{(1+h)^2}{4h}$, $h=\frac{M}{m}$, $h^{'}=\frac{M^{'}}{m^{'}}$ *and* $r^{'}=\min\{2r,1-2r\}$.

Proof. According to Theorem 3.5, we have

$$\begin{aligned}\Phi^2\Bigg(&A\nabla_\nu B+rMm\left(A^{-1}+A^{-1}\sharp_\mu B^{-1}-2\left(A^{-1}\sharp_{\frac{\mu}{2}}B^{-1}\right)\right)\\&+\left(\frac{\nu}{\mu}\right)Mm\Big(A^{-1}\nabla_\mu B^{-1}-A^{-1}\sharp_\mu B^{-1}\Big)\Bigg)\\&\le\left(\frac{K(h)}{K\left(\sqrt{h^{'\mu}},2\right)^{r^{'}}}\right)^2\Phi^2(A\sharp_\nu B).\end{aligned}\tag{3.13}$$

Making use of monotonicity of the geometric mean and the linearity of Φ, $0<m\le\Phi\left(A\sharp_\nu B\right)\le M$. On the other hand, for every $T\in\mathbb{B}(\mathscr{H})$ such that $0<m\le T\le M$, we have

$$M^2m^2T^{-2}+T^2\le M^2+m^2.$$

Now, by applying the latter inequality for $\Phi\left(A\sharp_\nu B\right)$, we have

$$M^2m^2\Phi\left(A\sharp_\nu B\right)^{-2}+\Phi\left(A\sharp_\nu B\right)^2\le M^2+m^2.\tag{3.14}$$

By a simple computation, we obtain

$$\begin{aligned}M^{\frac{p}{2}}m^{\frac{p}{2}}\Bigg\|&\Phi^{\frac{p}{2}}\Bigg(A\nabla_\nu B+rMm\left(A^{-1}+A^{-1}\sharp_\mu B^{-1}-2\left(A^{-1}\sharp_{\frac{\mu}{2}}B^{-1}\right)\right)\\&+\left(\frac{\nu}{\mu}\right)Mm\Big(A^{-1}\nabla_\mu B^{-1}-A^{-1}\sharp_\mu B^{-1}\Big)\Bigg)\Phi^{-\frac{p}{2}}(A\sharp_\nu B)\Bigg\|\\\le\frac{1}{4}\Bigg\|&\frac{K\left(\sqrt{h^{'\mu}},2\right)^{\frac{r^{'}p}{4}}}{K^{\frac{p}{4}}(h)}\Phi^{\frac{p}{2}}\Bigg(A\nabla_\nu B+rMm\Big(A^{-1}+A^{-1}\sharp_\mu B^{-1}\\&-2\left(A^{-1}\sharp_{\frac{\mu}{2}}B^{-1}\right)\Big)+\left(\frac{\nu}{\mu}\right)Mm\Big(A^{-1}\nabla_\mu B^{-1}-A^{-1}\sharp_\mu B^{-1}\Big)\Bigg)\\&+\left(\frac{M^2m^2K(h)}{K\left(\sqrt{h^{'\mu}},2\right)^{\frac{r^{'}p}{4}}}\right)\Phi^{-\frac{p}{2}}(A\sharp_\nu B)\Bigg\|^2\text{ (by Lemma 3.4 (i))}\\\le\frac{1}{4}\Bigg\|&\Bigg(\frac{K\left(\sqrt{h^{'\mu}},2\right)^{r^{'}}}{K(h)}\Phi^2\Bigg(A\nabla_\nu B+rMm\Big(A^{-1}+A^{-1}\sharp_\mu B^{-1}\end{aligned}$$

$$- 2\left(A^{-1}\sharp_{\frac{\mu}{2}}B^{-1}\right)\Big) + \left(\frac{\nu}{\mu}\right)Mm\Big(A^{-1}\nabla_\mu B^{-1} - A^{-1}\sharp_\mu B^{-1}\Big)\Big)$$

$$+ \frac{M^2m^2K(h)}{K\left(\sqrt{h'^{\mu}},2\right)^{r'}}\Phi^{-2}(A\sharp_\nu B)\Bigg)^{\frac{p}{4}}\Bigg\|^2 \text{ (by Lemma 3.4 (ii))}$$

$$= \frac{1}{4}\Bigg\|\frac{K\left(\sqrt{h'^{\mu}},2\right)^{r'}}{K(h)}\Phi^2\Big(A\nabla_\nu B + rMm\Big(A^{-1} + A^{-1}\sharp_\mu B^{-1}$$

$$- 2\left(A^{-1}\sharp_{\frac{\mu}{2}}B^{-1}\right)\Big) + \left(\frac{\nu}{\mu}\right)Mm\Big(A^{-1}\nabla_\mu B^{-1} - A^{-1}\sharp_\mu B^{-1}\Big)\Big)$$

$$+ \frac{M^2m^2K(h)}{K\left(\sqrt{h'^{\mu}},2\right)^{r'}}\Phi^{-2}(A\sharp_\nu B)\Bigg\|^{\frac{p}{2}}$$

$$\leq \frac{1}{4}\Bigg\|\frac{K(h)}{K\left(\sqrt{h'^{\mu}},2\right)^{r'}}\left(\Phi^2\left(A\sharp_\nu B\right) + M^2m^2\Phi^{-2}\left(A\sharp_\nu B\right)\right)\Bigg\|^{\frac{p}{2}} \text{ (by (3.13))}$$

$$\leq \frac{1}{4}\left(\frac{K(h)(M^2+m^2)}{K\left(\sqrt{h'^{\mu}},2\right)^{r'}}\right)^{\frac{p}{2}} \text{ (by (3.14)).}$$

By Lemma 3.4(iii), the last inequality implies (3.11). Analogously, we can prove the inequality (3.12). □

The next Theorem generalize the results of this section.

Theorem 3.9. *Under assumptions of Theorem 3.5, for every positive unital linear map* Φ, $0 \leq \nu < \mu \leq 1$, $1 \leq \alpha \leq 2$ *and* $p \geq 2\alpha$ *we have:*

$$\Phi^p\Big(A\nabla_\nu B + rMm\left(A^{-1} + A^{-1}\sharp_\mu B^{-1} - 2\left(A^{-1}\sharp_{\frac{\mu}{2}}B^{-1}\right)\right)$$
$$+ \left(\frac{\nu}{\mu}\right)Mm\Big(A^{-1}\nabla_\mu B^{-1} - A^{-1}\sharp_\mu B^{-1}\Big)\Big)$$
$$\leq \frac{\left(K\left(\sqrt{h'^{\mu}},2\right)^{-\frac{r'\alpha}{2}}K^{\frac{\alpha}{2}}(h)(M^\alpha + m^\alpha)\right)^{\frac{2p}{\alpha}}}{16M^pm^p}\Phi^p(A\sharp_\nu B), \tag{3.15}$$

and
(3.16)

$$\Phi^p\Bigg(A\nabla_\nu B + rMm\left(A^{-1}+A^{-1}\sharp_\mu B^{-1}-2\left(A^{-1}\sharp_{\frac{\mu}{2}}B^{-1}\right)\right) + \left(\frac{\nu}{\mu}\right)Mm\Big(A^{-1}\nabla_\mu B^{-1}-A^{-1}\sharp_\mu B^{-1}\Big)\Bigg) \le \frac{\left(K\left(\sqrt{h'^{\mu}},2\right)^{-\frac{r'\alpha}{2}}K^{\frac{\alpha}{2}}(h)(M^\alpha+m^\alpha)\right)^{\frac{2p}{\alpha}}}{16M^pm^p}(\Phi(A)\sharp_\nu\Phi(B))^p,$$

where $r=\min\{\nu,1-\nu\}$, $K(h)=\frac{(1+h)^2}{4h}$, $h=\frac{M}{m}$, $h'=\frac{M'}{m'}$ *and* $r'=\min\{2r,1-2r\}$.

Proof. By the property of the arithmetic mean (see [4]) and the linearity of Φ, we have:

$$m=m\sharp_\nu m\le\Phi\left(A\sharp_\nu B\right)\le M\sharp_\nu M=M.$$

On the other hand, for every $T\in\mathbb{B}(\mathscr{H})$ such that $0<m\le T\le M$, we have $0<T^\alpha-m^\alpha$ and $0<T^{-\alpha}-M^{-\alpha}$, whence $0<(T^\alpha-m^\alpha)(T^{-\alpha}-M^{-\alpha})$ or equivalently

$$M^\alpha m^\alpha T^{-\alpha}+T^\alpha\le M^\alpha+m^\alpha.$$

Applying the latter inequality for $\Phi\left(A\sharp_\nu B\right)$, we obtain

$$M^\alpha m^\alpha\Phi\left(A\sharp_\nu B\right)^{-\alpha}+\Phi\left(A\sharp_\nu B\right)^\alpha\le M^\alpha+m^\alpha. \tag{3.17}$$

If $1\le\alpha\le 2$, then $\frac{1}{2}\le\frac{\alpha}{2}\le 1$. Then by Theorem 3.5 and Lowner-Heinz theorem [5], we have

$$\Phi^\alpha\Bigg(A\nabla_\nu B + rMm\left(A^{-1}+A^{-1}\sharp_\mu B^{-1}-2\left(A^{-1}\sharp_{\frac{\mu}{2}}B^{-1}\right)\right) + \left(\frac{\nu}{\mu}\right)Mm\Big(A^{-1}\nabla_\mu B^{-1}-A^{-1}\sharp_\mu B^{-1}\Big)\Bigg) \le \left(\frac{K(h)}{K\left(\sqrt{h'^{\mu}},2\right)^{r'}}\right)^\alpha\Phi^{-\alpha}(A\sharp_\nu B). \tag{3.18}$$

By applying the same argument, we get

$$M^{\frac{p}{2}}m^{\frac{p}{2}}\Bigg\|\Phi^{\frac{p}{2}}\Bigg(A\nabla_\nu B + rMm\left(A^{-1}+A^{-1}\sharp_\mu B^{-1}-2\left(A^{-1}\sharp_{\frac{\mu}{2}}B^{-1}\right)\right)$$

$$+\left(\frac{\nu}{\mu}\right)Mm\left(A^{-1}\nabla_{\mu}B^{-1}-A^{-1}\sharp_{\mu}B^{-1}\right)\Phi^{-\frac{p}{2}}(A\sharp_{\nu}B)\Big)\Big\|$$

$$\leq\frac{1}{4}\Bigg\|\frac{K\left(\sqrt{h'},2\right)^{\frac{r'p}{4}}}{K^{\frac{p}{4}}(h)}\Phi^{\frac{p}{2}}\Big(A\nabla_{\nu}B+rMm\Big(A^{-1}+A^{-1}\sharp_{\mu}B^{-1}$$

$$-2\left(A^{-1}\sharp_{\frac{\mu}{2}}B^{-1}\right)\Big)+\left(\frac{\nu}{\mu}\right)Mm\Big(A^{-1}\nabla_{\mu}B^{-1}-A^{-1}\sharp_{\mu}B^{-1}\Big)\Big)$$

$$+\left(\frac{M^2m^2K(h)}{K\left(\sqrt{h'^{\mu}},2\right)^{r'}}\right)^{\frac{p}{4}}\Phi^{-\frac{p}{2}}(A\sharp_{\nu}B)\Bigg\|^2\quad\text{(by Lemma 3.4 (i))}$$

$$\leq\frac{1}{4}\Bigg\|\Bigg(\frac{K\left(\sqrt{h'^{\mu}},2\right)^{\frac{r'\alpha}{2}}}{K^{\frac{\alpha}{2}}(h)}\Phi^{\alpha}\Big(A\nabla_{\nu}B+rMm\Big(A^{-1}+A^{-1}\sharp_{\mu}B^{-1}$$

$$-2\left(A^{-1}\sharp_{\frac{\mu}{2}}B^{-1}\right)\Big)+\left(\frac{\nu}{\mu}\right)Mm\Big(A^{-1}\nabla_{\mu}B^{-1}-A^{-1}\sharp_{\mu}B^{-1}\Big)\Big)$$

$$+\frac{M^{\alpha}m^{\alpha}K^{\frac{\alpha}{2}}(h)}{K\left(\sqrt{h'^{\mu}},2\right)^{\frac{r'\alpha}{2}}}\Phi^{-\alpha}(A\sharp_{\nu}B)\Bigg)^{\frac{p}{2\alpha}}\Bigg\|^2\quad\text{(by Lemma 3.4 (ii))}$$

$$=\frac{1}{4}\Bigg\|\frac{K\left(\sqrt{h'^{\mu}},2\right)^{\frac{r'\alpha}{2}}}{K^{\frac{\alpha}{2}}(h)}\Phi^{\alpha}\Big(A\nabla_{\nu}B+rMm\Big(A^{-1}+A^{-1}\sharp_{\mu}B^{-1}$$

$$-2\left(A^{-1}\sharp_{\frac{\mu}{2}}B^{-1}\right)\Big)+\left(\frac{\nu}{\mu}\right)Mm\Big(A^{-1}\nabla_{\mu}B^{-1}-A^{-1}\sharp_{\mu}B^{-1}\Big)\Big)$$

$$+\frac{M^{\alpha}m^{\alpha}K^{\frac{\alpha}{2}}(h)}{K\left(\sqrt{h'^{\mu}},2\right)^{\frac{r'\alpha}{2}}}\Phi^{-\alpha}(A\sharp_{\nu}B)\Big)\Bigg\|^{\frac{p}{\alpha}}.$$

So, (3.15) is proved. By utilizing the same ideas as used in the proof of inequality (3.15), we can reach inequality (3.16). □

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