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Generalizations of Related Fritz Carlson Type Inequalities for Fuzzy Integrals

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Abstract. In this paper, We review general related inequalities to Carlson-type inequalities for the Sugeno integral on an abstract fuzzy measure space (X, Σ). Some examples are given to illustrate the validity of main results.

1. Introduction

The theory of fuzzy measures and fuzzy integrals was introduced by Sugeno [40] as a tool for modelling non-determinstic problems. The properties and applications of the Sugeno integral have been studied by many authors including Pap [34], Ralescu and Adams [35], Wang and Klir [41] among others. Ralescu and Adams [35] studied several equivalent definitions of fuzzy integrals, while Pap [34] and Wang and Klir [41], provided an overview of fuzzy measure theory. Fuzzy measure and Sugeno integrals have also been successfully applied to various fields by many researchers [24, 25, 27, 44].

The integral inequalities are useful tools in several theoretical and applied fields. For instance, integral inequalities play a role in the development of a time scales calculus [33]. The study of inequalities for Sugeno integral was initiated by Román-Flores et. al. and then followed by the authors [1–3, 5, 21, 26, 28–32, 38, 39]. In [39] Román-Flores et al. studied some properties of Sugeno integral for strictly monotone real functions, they also provided some Yong type inequalities. Based on these results, Flores-Franulič and Román- Flores [23]

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provided some Chebyshev type inequalities for Sugeno integral of continuous and strictly monotone real functions based on Lebesgue measure. Some other classical inequalities have also been generalized to Sugeno integral by them and others (see, [6–8, 20, 22, 26, 31, 32, 42]).

Later on, Ouyang and Fang [29] generalized the main results of [28] to the case of monotone real functions. Based on these results, Ouyang et. al. further generalized the fuzzy Chebyshev type inequalities to the case of arbitrary fuzzy measure-based Sugeno integrals [26, 29]. In fact, they proved the following result:

Theorem 1.1. *Let $f, g \in \mathcal{F}^{\mu}(X)$ and μ be an arbitrary fuzzy measure such that both $⨍_A f d\mu$ and $⨍_A g d\mu$ are finite. And let $\star : [0,\infty)^2 \to [0,\infty)$ be continuous and non-decreasing in both arguments and bounded from above by minimum. If f, g are comonotone, then the inequality*

$$⨍_A f \star g d\mu \geq \left(⨍_A f d\mu\right) \star \left(⨍_A g d\mu\right) \tag{1.1}$$

holds.

In view of the fact that

$$⨍_A f \star g d\mu \leq \left(⨍_A f d\mu\right) \star \left(⨍_A g d\mu\right) \tag{1.2}$$

holds for comonotone functions $f, g \in \mathcal{F}^{\mu}(X)$ wherever $\star \geq max$, it is of great interest to determine the operator $\star$ such that

$$⨍_A f \star g d\mu = \left(⨍_A f d\mu\right) \star \left(⨍_A g d\mu\right) \tag{1.3}$$

holds for any comonotone functions f, g and for fuzzy measure μ and any measurable set A. Ouyang et al. [31] proved that there are only 18 operators such that (1.3) holds, including that four well-known operators: minimum, maximum, PF and PL, where PF for frist projection is defined as the operator $\star$ such that $x \star y = x$ for each pair (x, y) and PL for last projection is defined as the operator $\star$ such that $x \star y = y$ for each pair (x, y).

Recently, Daraby and Arabi [22] proved two related inequalities to Fritz Carlson type inequality for the Sugeno integral. In fact, they proved the following two theorems:

Theorem 1.2. *Let $f : [0,1] \to [0,\infty)$ be a strictly increasing function and μ be the Lebesgue measure on $\mathbb{R}$. And let $\star : [0,\infty)^2 \to [0,\infty)$ be continuous and nondecreasing in both arguments and bounded from below* *by maximum. Then, the inequality*

(1.4)

$$\frac{1}{2}\fint_0^1 f(x)\star f(x)d\mu(x) \leq \left(\fint_0^1 x^2 f^2(x)d\mu(x)\right)^{\frac{1}{2}} \star \left(\fint_0^1 f^2(x)d\mu(x)\right)^{\frac{1}{2}}$$

holds.

Theorem 1.3. *Let $f:[0,1]\to[0,\infty)$ be a strictly increasing function and μ be the Lebesgue measure on $\mathbb{R}$. And let $\star$ be a binary operation such that the Sugeno integral possesses comonotonic-$\star$-property. Then, the inequality*

(1.5)

$$\left(\frac{1}{2}\right)\star\left(\fint_0^1 f(x)\star f(x)d\mu(x)\right) \leq \left(\fint_0^1 x^2\star f^2(x)d\mu(x)\right)^{\frac{1}{2}}\star\left(\fint_0^1 f^2(x)d\mu(x)\right)^{\frac{1}{2}}$$

holds.

2. Preliminaries

Definition 2.1. Let Σ be a σ-algebra of subsets of X and let $\mu:\Sigma\to[0,\infty)$ be a non-negative, extended real-valued set function. We say that μ is a fuzzy measure iff:

(FM1) $\mu(\emptyset)=0$;
(FM2) $E,F\in\sum$ and $E\subseteq F$ imply $\mu(E)\leq\mu(F)$ (monotonicity);
(FM3) $E_p\subseteq\sum$, $E_1\subseteq E_2\subseteq\ldots$ imply $\lim\mu(E_p)=\mu(\bigcup_{i=1}^{\infty}E_p)$ (continuity from below);
(FM4) $E_p\subseteq\sum$, $E_1\supseteq E_2\supseteq\ldots,\mu(E_1)<\infty$ imply $\lim\mu(E_p)=\mu(\bigcap_{i=1}^{\infty}E_p)$ (continuity from above).

Let f be a non-negative real-valued function on X. For $\alpha>0$, we denote the α-level of f by $F_\alpha=\{x\in X\mid f(x)\geq\alpha\}=\{f\geq\alpha\}$. The support of f is shown by $F_0=\{x\in X\mid f(x)>0\}=\text{supp}(f)$. We know that

$$\alpha\leq\beta \quad\Rightarrow\quad \{f\geq\beta\}\subseteq\{f\geq\alpha\}.$$

Definition 2.2. Let (X,Σ,μ) be a fuzzy measure space and $\mathcal{B}$ be the Borel field on $(-\infty,\infty)$.

A non-negative real-valued function $f:X\to[0,\infty)$ is measurable if and only if

$$f^{-1}(B)=\{x\mid f(x)\in B\}\in\Sigma$$

for any Borel set $B\in\mathcal{B}$.

Suppose that $\mathcal{F}(X)=\{f:X\to[0,\infty)\mid f \text{ is a fuzzy measurable function}\}$.

Definition 2.3 (See [41]). Let μ be a fuzzy measure on (X, Σ). If $f \in \mathcal{F}(X)$ and $A \in \Sigma$, then the Sugeno integral (or fuzzy integral) of f on A, with respect to the fuzzy measure μ, is defined as

$$\fint_A f d\mu = \bigvee_{\alpha \geq 0} (\alpha \wedge \mu(A \cap F_\alpha)), \tag{2.1}$$

where $\vee$, $\wedge$ denotes the operation sup and inf on $[0, \infty)$, respectively. In particular, if $A = X$ then:

$$\begin{aligned} \fint_X f d\mu &= \fint f d\mu \\ &= \bigvee_{\alpha \geq 0} (\alpha \wedge \mu(F_\alpha)). \end{aligned} \tag{2.2}$$

The following proposition gives the most elementary properties of the fuzzy integral and can be found in [41].

Proposition 2.4. *Let $(X, \mathcal{F}, \mu)$ be a fuzzy measure space, with A, $B \in \Sigma$ and $f, g \in \mathcal{F}$. We have*

(1) $\fint_A f d\mu \leq \mu(A)$.
(2) $\fint_A k d\mu \leq k \wedge \mu(A)$, *for k nonnegative constant.*
(3) *If $f \leq g$ on A, then* $\fint_A f d\mu \leq \fint_A g d\mu$.
(4) *If $A \subset B$, then* $\fint_A f d\mu \leq \fint_B f d\mu$.
(5) *If $\mu(A) < \infty$, then* $\fint_A f d\mu \geq \alpha \Leftrightarrow \mu(A \cap \{f \geq \alpha\}) \geq \alpha$.
(6) $\mu(A \cap \{f \geq \alpha\}) \leq \alpha \Rightarrow \fint_A f d\mu \leq \alpha$.
(7) $\fint_A f d\mu = \alpha \Leftrightarrow \mu(A \cap F_\alpha) \geq \alpha \geq \mu(A \cap F_{\alpha^+})$, *where*

$$\mu(A \cap F_{\alpha^+}) = \lim_{\varepsilon \to 0} \mu(A \cap F_{\alpha+\varepsilon}).$$

(8) $\fint_A f d\mu < \alpha \Leftrightarrow$ *there exists $\gamma < \alpha$ such that* $\mu(A \cap \{f \geq \gamma\}) < \alpha$.
(9) $\fint_A f d\mu > \alpha \Leftrightarrow$ *there exists $\gamma > \alpha$ such that* $\mu(A \cap \{f \geq \gamma\}) > \alpha$.

Remark 2.5. Let $F(\alpha) = \mu(A \cap F_\alpha)$, from parts (5) and (6) of the above Proposition, it very important to note that

$$F(\alpha) = \alpha \quad \Rightarrow \quad \fint_A f d\mu = \alpha. \tag{2.3}$$

Thus, from a numerical point of view, the Sugeno integral can be calculated by solving the equation $F(\alpha) = \alpha$.

Recall that two functions $f, g : X \to \mathbb{R}$ are said to be comonotone commuting property if, for all $(x, y) \in X^2$,$(f(x)-f(y))(g(x)-g(y)) \geq 0$. Clearly, if f and g are comonotone, then, for any real numbers p, q, either $F_p \subset G_q$ or $G_q \subset F_p$.

In [43] Xu and Ouyang proved the following Lemma:

Lemma 2.6 (Xu and Ouyang [43])**.** *Let (X, Σ, μ) be a fuzzy measure space, let $A \in \Sigma$ and let $f : X \to \mathbb{R}$ be a measureable function such that $⨍_A f d\mu \leq 1$. Then, for any $s \geq 1$, we have*

$$⨍_A f^s d\mu \geq \left(⨍_A f d\mu\right)^s . \tag{2.4}$$

Caballero and Sadarangani [5] proved a Carlson inequality for the Sugeno integral. In fact they proved the following result:

Theorem 2.7. *Let $f : [0, 1] \to [0, \infty]$ be a non-decreasing function and μ be the Lebesgue measure on $\mathbb{R}$, Then*

$$⨍_0^1 f(x)d\mu(x) \leq \sqrt{2}\left(⨍_0^1 x^2 f^2(x)d\mu(x)\right)^{\frac{1}{4}} \left(⨍_0^1 f^2(x)d\mu(x)\right)^{\frac{1}{4}} . \tag{2.5}$$

Xu and Ouyang [43] presented a general version of Carlson's inequality for the Sugeno integral as follow:

Theorem 2.8. *Let (X, Σ, μ) be a fuzzy measure space, let $f_i : X \to \mathbb{R}$,$i = 1, 2, 3$ be measurable functions such that $⨍_A f_i d\mu \leq 1$. If any two functions of f_i,$i = 1, 2, 3$ are comonotone, then for any $p, q \geq 1$, we have*

$$⨍_A f_1 d\mu \leq \frac{1}{\sqrt{C}}\left(⨍_A f_1^p f_2^p d\mu\right)^{\frac{1}{2p}} \left(⨍_A f_1^q f_3^q d\mu\right)^{\frac{1}{2q}} , \tag{2.6}$$

where $C = \left(⨍_A f_2 d\mu\right)\left(⨍_A f_3 d\mu\right)$.

Theorem 2.9 (Fuzzy Chebyshev's inequality [23])**.** *Let $f, g : [0, 1] \to \mathbb{R}$ be two real-valued functions and let μ be the Lebesgue measure on $\mathbb{R}$. If f, g are both continuous and strictly increasing functions, then the inequality*

$$⨍_A fg d\mu \geq \left(⨍_A f d\mu\right)\left(⨍_A g d\mu\right) \tag{2.7}$$

holds.

The following theorem shows the Hermite-Hadamard inequality for fuzzy integrals that appears in [3] and we use it for estimating fuzzy integrals.

Theorem 2.10. *Let $g : [0, 1] \to [0, \infty)$ be a convex function such that $g(0) < g(1)$ and μ be the Lebesgue measure on $\mathbb{R}$. Then*

$$⨍ g d\mu \leq \min\left\{\frac{g(1)}{1 + g(1) - g(0)}, 1\right\}. \tag{2.8}$$

Definition 2.11 (see [31])**.** Let $\star : [0, \infty)^2 \to [0, \infty)$ be a binary operation. We say that the Sugeno integral possesses comonotonic-$\star$-property, if (1.3) holds for any fuzzy measure space (X, Σ, μ) and any measurable set A, and for any comonotone function $f, g : X \to [0, \infty)$.

3. Main Results

In this section, we prove some general related inequalities to Carlson type inequality for Sugeno integrals.

Theorem 3.1. *Let $f : [0,1] \to [0,\infty)$ be a strictly increasing function and μ be the Lebesgue measure on $\mathbb{R}$. And let $\star : [0,\infty)^2 \to [0,\infty)$ be continuous and nondecreasing in both arguments and bounded from below by maximum. Then, the inequality*

(3.1)
$$\frac{1}{2}\fint_0^1 f(x)\star f(x)d\mu(x) \le \left(\fint_0^1 x^2 f^2(x)d\mu(x)\right)^{\frac{1}{2}} \star \left(\fint_0^1 f^2(x)d\mu(x)\right)^{\frac{1}{2}}.$$

holds.

Proof. Since $\fint_0^1 xd\mu(x) = \frac{1}{2}$, then

$$\frac{1}{4}\left(\fint_0^1 f(x)d\mu(x)\right)^2 = \left(\fint_0^1 xd\mu(x)\right)^2\left(\fint_0^1 f(x)d\mu(x)\right)^2. \tag{3.2}$$

Using inequality (2.4) we have

$$\left(\fint_0^1 xd\mu(x)\right)^2\left(\fint_0^1 f(x)d\mu(x)\right)^2 \le \left(\fint_0^1 xf(x)d\mu(x)\right)^2. \tag{3.3}$$

Taking $f = g$ in Inequality (1.1), we have

$$\fint_0^1 f\star f d\mu \le \left(\fint_0^1 fd\mu\right)\star\left(\fint_0^1 fd\mu\right).$$

It follows that

(3.4)
$$\begin{aligned}\frac{1}{4}\left(\fint_0^1 f(x)\star f(x)d\mu(x)\right)^2 &\le \frac{1}{4}\left[\left(\fint_0^1 f(x)d\mu(x)\right)\star\left(\fint_0^1 f(x)d\mu(x)\right)\right]^2\\ &= \left[\frac{1}{4}\left(\fint_0^1 f(x)d\mu(x)\right)^2\right]\star\left(\fint_0^1 f(x)d\mu(x)\right)^2\end{aligned}$$

from (3.2) and (3.4), we conclude that

(3.5)
$$\begin{aligned}\frac{1}{4}\left(\fint_0^1 f(x)\star f(x)d\mu(x)\right)^2 \le &\left[\left(\fint_0^1 xd\mu(x)\right)^2\left(\fint_0^1 f(x)d\mu(x)\right)^2\right]\\ &\star\left(\fint_0^1 f(x)d\mu(x)\right)^2.\end{aligned}$$

Using (3.3), (3.5) and Jensen's Inequality we have

(3.6)

$$\begin{aligned}\frac{1}{4}\left(\fint_0^1 f(x)\star f(x)d\mu(x)\right)^2 &\leq \left(\fint_0^1 xf(x)d\mu(x)\right)^2\star\left(\fint_0^1 f(x)d\mu(x)\right)^2\\ &\leq \left(\fint_0^1 x^2f^2(x)d\mu(x)\right)\star\left(\fint_0^1 f^2(x)d\mu(x)\right).\end{aligned}$$

This inequality implies that

$$\frac{1}{2}\fint_0^1 f(x)\star f(x)d\mu(x)\leq\left(\fint_0^1 x^2f^2(x)d\mu(x)\right)^{\frac{1}{2}}\star\left(\fint_0^1 f^2(x)d\mu(x)\right)^{\frac{1}{2}}.$$

This completes the proof. □

Example 3.2. Suppose that $f(x) = x^{\frac{1}{2}}$ for all $x \in [0,1]$, μ be the Lebesgue measure on $\mathbb{R}$ and $\star = \sup$. Then $\star$ satisfies conditions of Theorem 3.1. We show that the following inequality holds.

$$\frac{1}{2}\fint_0^1 \sup\left(x^{\frac{1}{2}},x^{\frac{1}{2}}\right)d\mu(x)\leq\sup\left(\left(\fint_0^1 x^2xd\mu(x)\right)^{\frac{1}{2}},\left(\fint_0^1 xd\mu(x)\right)^{\frac{1}{2}}\right).$$

We know that

$$\fint_0^1 x^{\frac{1}{2}}d\mu(x)\simeq 0.618.$$

Using (2.5) we can estimate $\fint_0^1 x^3d\mu(x)$ i.e.

$$\left(\fint_0^1 x^3d\mu(x)\right)^{\frac{1}{2}}\leq 0.707.$$

Also a simple calculation reveals that

$$\left(\fint_0^1 xd\mu(x)\right)^{\frac{1}{2}}\simeq 0.707.$$

Hence

$$\begin{aligned}\frac{1}{2}\fint_0^1 \sup\left(x^{\frac{1}{2}},x^{\frac{1}{2}}\right)d\mu(x)&\simeq 0.309\\ &\leq 0.707\\ &\simeq\sup\left(\left(\fint_0^1 x^2xd\mu(x)\right)^{\frac{1}{2}},\left(\fint_0^1 xd\mu(x)\right)^{\frac{1}{2}}\right).\end{aligned}$$

Therefore, for this example Inequality (3.1) holds.

Example 3.3. Let $f(x) = x$ for all $x \in [0,1]$, μ be the Lebesgue measure on $\mathbb{R}$. Let $\star = S_D$ be the drastic sum:

$$S_D(x,y) = \begin{cases} 1 & \text{if} \quad \min\{x,y\} > 0, \\ \max\{x,y\} & \text{if} \quad \min\{x,y\} = 0. \end{cases}$$

Then we show that

$$\frac{1}{2}\fint_0^1 S_D(x,x)d\mu(x) \le S_D\left(\left(\fint_0^1 x^4 d\mu(x)\right)^{\frac{1}{2}}, \left(\fint_0^1 x^2 d\mu(x)\right)^{\frac{1}{2}}\right).$$

Using (2.5), we have

$$\left(\fint_0^1 x^4 d\mu(x)\right)^{\frac{1}{2}} \le 0.707,$$

and a simple calculation shows that

$$\left(\fint_0^1 x^2 d\mu(x)\right)^{\frac{1}{2}} \simeq 0.618.$$

Also from definition of S_D we have

$$S_D(0.707, 0.618) = 1.$$

Hence if $x > 0$, then $S_D(x,x) = 1$. It follows that

$$\begin{aligned} \frac{1}{2}\fint_0^1 S_D(x,x)d\mu(x) &= \frac{1}{2} \\ &\le 1 \\ &= S_D\left(\left(\fint_0^1 x^4 d\mu(x)\right)^{\frac{1}{2}}, \left(\fint_0^1 x^2 d\mu(x)\right)^{\frac{1}{2}}\right). \end{aligned}$$

If $x = 0$, then $S_D(x,x) = 0$. It follows that

$$\begin{aligned} \frac{1}{2}\fint_0^1 S_D(x,x)d\mu(x) &= 0 \\ &\le 1 \\ &= S_D\left(\left(\fint_0^1 x^4 d\mu(x)\right)^{\frac{1}{2}}, \left(\fint_0^1 x^2 d\mu(x)\right)^{\frac{1}{2}}\right). \end{aligned}$$

Therefore, for this example Inequality (3.1) holds.

Example 3.4. Let $f(x) = 2x$ for all $x \in [0,1]$, μ be the Lebesgue measure on $\mathbb{R}$. Let $\star = S_M$ be the maximum t-conorm:

$$S_M(x,y) = \max\{x,y\}.$$

Then we show that

$$\frac{1}{2}\fint_0^1 S_M(2x,2x)d\mu(x) \le S_M\left(\left(\fint_0^1 4x^4 d\mu(x)\right)^{\frac{1}{2}}, \left(\fint_0^1 4x^2 d\mu(x)\right)^{\frac{1}{2}}\right).$$

We know that

$$\fint_0^1 2x d\mu(x) = \frac{2}{3}$$

and

$$\left(\fint_0^1 4x^2 d\mu(x)\right)^{\frac{1}{2}} = 0.78.$$

Also by using (2.5) we have

$$\fint_0^1 4x^4 d\mu(x) \le 0.89.$$

Consequently we have

$$\begin{aligned}\frac{1}{2}\fint_0^1 S_M(2x,2x)d\mu(x) &\simeq 0.33 \\ &\le 0.89 \\ &\simeq S_M\left(\left(\fint_0^1 4x^4 d\mu(x)\right)^{\frac{1}{2}}, \left(\fint_0^1 4x^2 d\mu(x)\right)^{\frac{1}{2}}\right).\end{aligned}$$

If in Theorem 3.1 we suppose that the Sugeno integral possesses comonotonic-$\star$-property as defined in Definition 2.11, then we get another general related to Carlson's inequality for fuzzy integrals, as follows:

Theorem 3.5. *Let $f : [0,1] \to [0,\infty)$ be a strictly increasing function and μ be the Lebesgue measure on $\mathbb{R}$. And let $\star$ be a binary operation such that the Sugeno integral possesses comonotonic-$\star$-property. Then, the inequality*

(3.7)

$$\frac{1}{2}\star\fint_0^1 f(x)\star f(x)d\mu(x) \le \left(\fint_0^1 x^2\star f^2(x)d\mu(x)\right)^{\frac{1}{2}} \star \left(\fint_0^1 f^2(x)d\mu(x)\right)^{\frac{1}{2}}$$

holds.

Proof. Using (1.3) we have

$$\fint_0^1 x\star f(x)d\mu(x) = \left(\fint_0^1 x d\mu(x)\right)\star\left(\fint_0^1 f(x)d\mu(x)\right). \tag{3.8}$$

Also we know that

$$\fint_0^1 x d\mu(x) = \frac{1}{2}. \tag{3.9}$$

Replacing (3.9) in inequality (3.8) we obtain

$$\fint_0^1 x \star f(x) d\mu(x) = \left(\frac{1}{2}\right) \star \left(\fint_0^1 f(x) d\mu(x)\right). \tag{3.10}$$

Using Jensen's Inequality [43] we have

$$\left(\fint_0^1 x \star f(x) d\mu(x)\right)^2 \leq \fint_0^1 x^2 \star f^2(x) d\mu(x). \tag{3.11}$$

By inequality (1.2), (3.10), (3.11) and taking

$$\left(\frac{1}{4}\right) \star \left[\fint_0^1 f(x) \star f(x) d\mu(x)\right]^2 = a$$

we get

$$\begin{aligned} a &= \left(\frac{1}{4}\right) \star \left[\left(\fint_0^1 f(x) d\mu(x)\right) \star \left(\fint_0^1 f(x) d\mu(x)\right)\right]^2 \\ &\leq \left[\left(\frac{1}{4}\right) \star \left(\fint_0^1 f(x) d\mu(x)\right)^2\right] \star \left(\fint_0^1 f(x) d\mu(x)\right)^2 \\ &= \left(\fint_0^1 x \star f(x) d\mu(x)\right)^2 \star \left(\fint_0^1 f(x) d\mu(x)\right)^2 \\ &\leq \left(\fint_0^1 x^2 \star f^2(x) d\mu(x)\right) \star \left(\fint_0^1 f^2(x) d\mu(x)\right). \end{aligned} \tag{3.12}$$

It follows that

$$\left(\frac{1}{2}\right) \star \left(\fint_0^1 f(x) \star f(x) d\mu(x)\right) \leq \left(\fint_0^1 x^2 \star f^2(x) d\mu(x)\right)^{\frac{1}{2}} \star \left(\fint_0^1 f^2(x) d\mu(x)\right)^{\frac{1}{2}}.$$

This completes the proof. □

Example 3.6. Let $f(x) = x^{\frac{1}{2}}$ and $\star = \wedge$. Then simple calculations show that

$$\fint_0^1 f(x) \wedge f(x) d\mu(x) = \fint_0^1 x^{\frac{1}{2}} d\mu(x) \simeq 0.618,$$

$$\left(\fint_0^1 x^2 d\mu(x)\right)^{\frac{1}{2}} \simeq 0.618.$$

Also

$$\left(\fint_0^1 x^2 \wedge f^2(x) d\mu(x)\right)^{\frac{1}{2}} = \left(\fint_0^1 x d\mu(x)\right)^{\frac{1}{2}} \simeq 0.707$$

and

$$\left(\fint_0^1 f^2(x) d\mu(x)\right)^{\frac{1}{2}} = \left(\fint_0^1 x d\mu(x)\right)^{\frac{1}{2}} \simeq 0.707.$$

Hence inequality (3.7) holds as follows:

$$\begin{aligned}
&\left(\frac{1}{2}\right) \star \left(⨍_0^1 f(x) \star f(x) d\mu(x)\right) \\
&\quad = 0.5 \\
&\quad \leq 0.618 \\
&\quad \simeq \left(⨍_0^1 x^2 \star f^2(x) d\mu(x)\right)^{\frac{1}{2}} \star \left(⨍_0^1 f^2(x) d\mu(x)\right)^{\frac{1}{2}}.
\end{aligned}$$

Example 3.7. Let $f(x) = x$ and $\star = PF$. Then we have

$$PF\left[\frac{1}{2}, ⨍_0^1 x d\mu(x)\right] \leq PF\left[\left(⨍_0^1 x^2 d\mu(x)\right)^{\frac{1}{2}}, \left(⨍_0^1 x^2 d\mu(x)\right)^{\frac{1}{2}}\right].$$

It follows that

$$\begin{aligned}
PF\left[\frac{1}{2}, ⨍_0^1 x d\mu(x)\right] &= \frac{1}{2} \\
&\leq 0.618 \\
&= \left(⨍_0^1 x^2 d\mu(x)\right)^{\frac{1}{2}}.
\end{aligned}$$

Hence inequality (3.7) holds.

Theorem 3.8. *Let (X, Σ, μ) be a fuzzy measure space, let $f_i : X \to \mathbb{R}$,$i = 1, 2, 3$ be measurable functions such that $⨍_A f_i d\mu \leq 1$. And let $\star : [0, \infty)^2 \to [0, \infty)$ be continuous and non-decreasing in both arguments and bounded from below by maximum. If any two functions of f_i,$i = 1, 2, 3$ are comonotone, then for any $p, q \geq 1$, we have*

$$C ⨍_A f_1 \star f_1 d\mu \leq \left(⨍_A f_1^p f_2^p d\mu\right)^{\frac{1}{p}} \star \left(⨍_A f_1^q f_3^q d\mu\right)^{\frac{1}{q}}, \tag{3.13}$$

where $C = \left(⨍_A f_2 d\mu\right)\left(⨍_A f_3 d\mu\right)$.

Proof. Frist, we show that $⨍_A f_1 d\mu \leq 1$ and $⨍_A f_2 d\mu \leq 1$ imply that $⨍_A f_1 f_2 d\mu \leq 1$. In fact by (7) of Proposition 2.4, we have $\mu\left(A \cap F_{1+}^{(1)}\right) \leq 1$ and $\mu\left(A \cap F_{1+}^{(2)}\right) \leq 1$, where

$$\mu\left(A \cap F_{1+}^{(i)}\right) = \lim_{\varepsilon \to 0} \mu\left(A \cap \{x \mid f_i(x) \geq 1 + \varepsilon\}\right).$$

Thus the comonotonicity of f_1 and f_2 implies that

$$\mu\left(A \cap \left(F_{1+}^{(1)} \cup F_{1+}^{(2)}\right)\right) \leq 1.$$

Noting by fact that $\{x \mid f_1(x)f_2(x) > 1\} \subset \{x \mid f_1(x) > 1\} \cup \{x \mid f_2(x) > 1\}$, we have that

$$\mu\left(A \cap \{x \mid f_1(x)f_2(x) \geq 1+\varepsilon\}\right) \leq 1,$$

for any $\varepsilon > 0$. Again, by (7) of Proposition 2.4, we conclude that $⨍_A f_1 f_2 d\mu \leq 1$.

Now, by Lemma 2.6, for $p, q \geq 1$, we have

$$\left(⨍_A f_1 f_2 d\mu\right)^p \leq ⨍_A f_1^p f_2^p d\mu \tag{3.14}$$

and

$$\left(⨍_A f_1 f_3 d\mu\right)^q \leq ⨍_A f_1^q f_3^q d\mu. \tag{3.15}$$

Since $\star$ is non-decreasing,

$$\left(⨍_A f_1 f_2 d\mu\right) \star \left(⨍_A f_1 f_3 d\mu\right) \leq \left(⨍_A f_1^p f_2^p d\mu\right)^{\frac{1}{p}} \star \left(⨍_A f_1^q f_3^q d\mu\right)^{\frac{1}{q}}. \tag{3.16}$$

Since the usual product satisfies $. \mid_{[0,1]^2} \leq \min$, by (1.1),

$$⨍_A f_1 f_2 d\mu \geq \left(⨍_A f_1 d\mu\right)\left(⨍_A f_2 d\mu\right) \tag{3.17}$$

and

$$⨍_A f_1 f_3 d\mu \geq \left(⨍_A f_1 d\mu\right)\left(⨍_A f_3 d\mu\right). \tag{3.18}$$

Since $\star \geq \max$, by (1.2),

$$⨍_A f_1 \star f_1 d\mu \leq \left(⨍_A f_1 d\mu\right) \star \left(⨍_A f_1 d\mu\right) \tag{3.19}$$

Thus

$$\begin{aligned}
\left(⨍_A f_2 d\mu ⨍_A f_3 d\mu\right)\left(⨍_A f_1 \star f_1 d\mu\right) &\leq \left(⨍_A f_2 d\mu ⨍_A f_3 d\mu\right)\left(⨍_A f_1 d\mu \star ⨍_A f_1 d\mu\right) \\
((3.17),(3.18) \Rightarrow) &\leq \left(⨍_A f_2 d\mu ⨍_A f_3 d\mu\right)\left(\frac{⨍_A f_1 f_2 d\mu}{⨍_A f_2 d\mu} \star \frac{⨍_A f_1 f_3 d\mu}{⨍_A f_3 d\mu}\right) \\
(\alpha, \beta \leq 1 \Rightarrow \alpha\beta(a \star b) \leq \alpha a \star \beta b) &\leq ⨍_A f_1 f_2 d\mu \star ⨍_A f_1 f_3 d\mu \\
((3.16) \Rightarrow) &\leq \left(⨍_A f_1^p f_2^p d\mu\right)^{\frac{1}{p}} \star \left(⨍_A f_1^q f_3^q d\mu\right)^{\frac{1}{q}}.
\end{aligned}$$

Thus we have

$$\left(⨍_A f_2 d\mu ⨍_A f_3 d\mu\right)\left(⨍_A f_1 \star f_1 d\mu\right) \leq \left(⨍_A f_1^p f_2^p d\mu\right)^{\frac{1}{p}} \star \left(⨍_A f_1^q f_3^q d\mu\right)^{\frac{1}{q}}$$

If we denote $C = \left(⨍_A f_2 d\mu\right)\left(⨍_A f_3 d\mu\right)$, then we get the desired result:

$$C ⨍_A f_1 \star f_1 d\mu \leq \left(⨍_A f_1^p f_2^p d\mu\right)^{\frac{1}{p}} \star \left(⨍_A f_1^q f_3^q d\mu\right)^{\frac{1}{q}}. \qquad \square$$

Example 3.9. Suppose that $f_1(x) = 1$, $f_2(x) = \sqrt{x}$, $f_3(x) = x$, $A = [0,1]$, $p = 2$, $q = 1$ and μ be the Lebesgue measure, then simple calculation shows that

$$\begin{aligned}
&⨍_0^1 f_1(x) d\mu = ⨍_0^1 1 d\mu = 1,\\
&⨍_0^1 f_2(x) d\mu = ⨍_0^1 \sqrt{x} d\mu \approx 0.618,\\
&⨍_0^1 f_3(x) d\mu = ⨍_0^1 x d\mu = 0.5,\\
&C \approx 0.618 \times 0.5 \approx 0.309,\\
&\left(⨍_0^1 f_1^2(x) f_2^2(x) d\mu\right)^{\frac{1}{2}} = \left(⨍_0^1 x d\mu\right)^{\frac{1}{2}} = (0.5)^{\frac{1}{2}} \approx 0.707,\\
&⨍_0^1 f_1(x) f_3(x) d\mu = ⨍_0^1 x d\mu = 0.5;
\end{aligned}$$

Now if we let $\star = \vee$, then we have

$$\begin{aligned}
C ⨍_0^1 f_1 \vee f_1 d\mu &= C ⨍_0^1 f_1 d\mu\\
&\approx 0.309\\
&\leq 0.707\\
&\approx \left(⨍_0^1 f_1^2 f_2^2 d\mu\right)^{\frac{1}{2}} \vee \left(⨍_0^1 f_1 f_3 d\mu\right).
\end{aligned}$$

Let $f_2(x) = x$, $f_3(x) = 1$, $\forall x \in X$ and $p = q = 2$. Then we have the following corollary, which is, in fact, Theorem 1.2:

Corollary 3.10. *Let $f : [0,1] \to [0,\infty)$ be a strictly increasing function and μ be the Lebesgue measure on $\mathbb{R}$. And let $\star : [0,\infty)^2 \to [0,\infty)$ be continuous and nondecreasing in both arguments and bounded from below by maximum. Then, the inequality*

$$\frac{1}{2} ⨍_0^1 f(x) \star f(x) d\mu \leq \left(⨍_0^1 x^2 f^2(x) d\mu\right)^{\frac{1}{2}} \star \left(⨍_0^1 f^2(x) d\mu\right)^{\frac{1}{2}}$$

holds.

Proof. $C = \left(⨍_0^1 x d\mu\right)\left(⨍_0^1 1 d\mu\right) = \frac{1}{2} \times 1 = \frac{1}{2}$, by replacing this and $f_2(x) = x$, $f_3(x) = 1$, $p = q = 2$ in (3.1) we have desired result. $\square$

In the following example we show that comonotonicity of $f_i, i = 1,2,3$ in Theorem 3.8 is necessary:

Example 3.11. Let $f_1(x) = x, f_2(x) = f_3(x) = 2 - x, p = q = 1, \star = \vee$, $A = [0,2]$ and μ is the Lebesgue measure. Then

$$\fint_0^2 f_1 d\mu = \fint_0^2 f_2 d\mu = \fint_0^2 f_3 d\mu = 1, \quad C = 2,$$
$$\fint_0^2 f_1 f_2 d\mu = \fint_0^2 f_1 f_3 d\mu = \fint_0^2 x(2-x) d\mu = 2\sqrt{2} - 2.$$

Thus

$$\begin{aligned} C \fint_0^2 f_1 \vee f_1 d\mu &= 2 \\ &> 2\sqrt{2} - 2 \\ &= \left(\fint_0^2 f_1 f_2 d\mu \right) \vee \left(\fint_0^2 f_1 f_3 d\mu \right). \end{aligned}$$

The following example shows that, if we omit the condition $\fint_A f_i d\mu \leq 1, i = 1,2,3$ in Theorem 3.8, then (3.13) may not hold:

Example 3.12. Let $A = [0,3], f_i(x) = x, i = 1,2,3$ and $p = q = 1, \star = \vee$ and μ is the Lebesgue measure. Then

$$\fint_0^3 f_1 d\mu = \fint_0^3 f_2 d\mu = \fint_0^3 f_3 d\mu = \fint_0^3 x d\mu = \frac{3}{2}, \quad C = (\frac{3}{2})^2,$$
$$\fint_0^3 f_1 f_2 d\mu = \fint_0^3 f_1 f_3 d\mu = \fint_0^3 x^2 d\mu = \frac{7 - \sqrt{13}}{2}.$$

Thus

$$\begin{aligned} C \fint_0^3 f_1 \vee f_1 d\mu &= \left(\frac{3}{2}\right)^3 \\ &= 3.375 \\ &> 1.697 \\ &\approx \frac{7-\sqrt{13}}{2} \\ &= \left(\fint_0^3 f_1 f_2 d\mu \right) \vee \left(\fint_0^3 f_1 f_3 d\mu \right). \end{aligned}$$

Now, if we change $f_1^p f_2^p$ and $f_1^q f_3^q$ in Theorem 3.1 to $f_1^p \star f_2^p$ and $f_1^q \star f_3^q$, respectively, then we have next Theorem.

Theorem 3.13. *Let (X, Σ, μ) be a fuzzy measure space, let $f_i : X \to \mathbb{R}$, $i = 1,2,3$ be measurable functions such that $\fint_A f_i d\mu \leq 1$. And let*

$\star : [0,\infty)^2 \to [0,\infty)$ *be a binary operation such that the Sugeno integral possesses comonotonic-$\star$-property. Then we have*

$$C \fint_A f_1 \star f_1 d\mu \leq \left(\fint_A f_1^p \star f_2^p d\mu\right)^{\frac{1}{p}} \star \left(\fint_A f_1^q \star f_3^q d\mu\right)^{\frac{1}{q}}, \tag{3.20}$$

where $C = (\fint_A f_2 d\mu)(\fint_A f_3 d\mu)$.

Proof. Frist, since $\fint_A f_1 d\mu \leq 1$ and $\fint_A f_2 d\mu \leq 1$ from (1.3), $\fint_A f_1 \star f_2 d\mu = \fint_A f_1 d\mu \star \fint_A f_2 d\mu \leq 1$. Similarly $\fint_A f_1 \star f_3 \leq 1$. Therefore by Lemma 2.6, for $p, q \geq 1$, we have

$$\left(\fint_A f_1 \star f_2 d\mu\right)^p \leq \fint_A f_1^p \star f_2^p d\mu \tag{3.21}$$

and

$$\left(\fint_A f_1 \star f_3 d\mu\right)^q \leq \fint_A f_1^q \star f_3^q d\mu. \tag{3.22}$$

Since $\star$ is non-decreasing,
(3.23)

$$\left(\fint_A f_1 \star f_2 d\mu\right) \star \left(\fint_A f_1 \star f_3 d\mu\right) \leq \left(\fint_A f_1^p \star f_2^p d\mu\right)^{\frac{1}{p}} \star \left(\fint_A f_1^q \star f_3^q d\mu\right)^{\frac{1}{q}}.$$

Thus by using (1.3),

$$\begin{aligned}
\left(\fint_A f_2 d\mu \fint_A f_3 d\mu\right)\left(\fint_A f_1 \star f_1 d\mu\right) &= \left(\fint_A f_2 d\mu \fint_A f_3 d\mu\right)\left(\fint_A f_1 d\mu \star \fint_A f_1 d\mu\right) \\
(\alpha, \beta \leq 1 \Rightarrow \alpha\beta(a \star b) \leq \alpha a \star \beta b) &\leq \left(\fint_A f_1 d\mu \fint_A f_2 d\mu\right) \star \left(\fint_A f_1 d\mu \fint_A f_3 d\mu\right) \\
&\leq \left(\fint_A f_1 d\mu \star \fint_A f_2 d\mu\right) \star \left(\fint_A f_1 d\mu \star \fint_A f_3 d\mu\right) \\
((1.3) \Rightarrow) &= \left(\fint_A f_1 \star f_2 d\mu\right) \star \left(\fint_A f_1 \star f_3 d\mu\right) \\
((3.23) \Rightarrow) &\leq \left(\fint_A f_1^p \star f_2^p d\mu\right)^{\frac{1}{p}} \star \left(\fint_A f_1^q \star f_3^q d\mu\right)^{\frac{1}{q}}.
\end{aligned}$$

(Note that second inequality results from the fact,

$$\fint_A f_i d\mu \leq 1, \fint_A f_j d\mu \leq 1 \quad \Rightarrow \quad \fint_A f_i d\mu \fint_A f_j d\mu \leq \fint_A f_i d\mu \star \fint_A f_j d\mu)$$

So we have

$$\left(\fint_A f_2 d\mu \fint_A f_3 d\mu\right)\left(\fint_A f_1 \star f_1 d\mu\right) \leq \left(\fint_A f_1^p \star f_2^p d\mu\right)^{\frac{1}{p}} \star \left(\fint_A f_1^q \star f_3^q d\mu\right)^{\frac{1}{q}}$$

If we denote $C = \left(⨍_A f_2 d\mu\right)\left(⨍_A f_3 d\mu\right)$, then we get the desired result:

$$C ⨍_A f_1 \star f_1 d\mu \leq \left(⨍_A f_1^p f_2^p d\mu\right)^{\frac{1}{p}} \star \left(⨍_A f_1^q f_3^q d\mu\right)^{\frac{1}{q}}. \qquad \square$$

Example 3.14. Suppose that $f_1(x) = 1$, $f_2(x) = \sqrt{x}$, $f_3(x) = x$, $A = [0,1]$, $p = 2$, $q = 1$, μ be the Lebesgue measure and $\star \in \{\vee, \wedge, PF, PL\}$, then simple calculation shows that

$$⨍_0^1 f_1(x) d\mu = ⨍_0^1 1 d\mu = 1,$$
$$⨍_0^1 f_2(x) d\mu = ⨍_0^1 \sqrt{x} d\mu \approx 0.618,$$
$$⨍_0^1 f_3(x) d\mu = ⨍_0^1 x d\mu = 0.5,$$
$$C \approx 0.618 \times 0.5 = 0.309,$$
$$\left(⨍_0^1 f_1^2(x) d\mu\right)^{\frac{1}{2}} = \left(⨍_0^1 1 d\mu\right)^{\frac{1}{2}} = \left(⨍_0^1 1 d\mu\right)^{\frac{1}{2}} = (1)^{\frac{1}{2}} = 1,$$
$$\left(⨍_0^1 f_2^2(x) d\mu\right)^{\frac{1}{2}} = \left(⨍_0^1 x d\mu\right)^{\frac{1}{2}} = (0.5)^{\frac{1}{2}} \approx 0.707,$$
$$\left(⨍_0^1 f_1^2(x) \vee f_2^2(x) d\mu\right)^{\frac{1}{2}} = \left(⨍_0^1 1 \vee x d\mu\right)^{\frac{1}{2}} = \left(⨍_0^1 1 d\mu\right)^{\frac{1}{2}} = (1)^{\frac{1}{2}} = 1,$$
$$\left(⨍_0^1 f_1^2(x) \wedge f_2^2(x) d\mu\right)^{\frac{1}{2}} = \left(⨍_0^1 1 \wedge x d\mu\right)^{\frac{1}{2}} = \left(⨍_0^1 x d\mu\right)^{\frac{1}{2}} = (0.5)^{\frac{1}{2}} \approx 0.707,$$
$$⨍_0^1 f_1(x) \vee f_3(x) d\mu = ⨍_0^1 1 \vee x d\mu = ⨍_0^1 1 d\mu = 1,$$
$$⨍_0^1 f_1(x) \wedge f_3(x) d\mu = ⨍_0^1 1 \wedge x d\mu = ⨍_0^1 x d\mu = 0.5;$$

Now we consider 4 cases:

1) $\star = \vee$, then we have

$$\begin{aligned} C ⨍_0^1 f_1 \vee f_1 d\mu &= C ⨍_0^1 f_1 d\mu \\ &\approx 0.309 \\ &\leq 1 \\ &= \left(⨍_0^1 f_1^2 \vee f_2^2 d\mu\right)^{\frac{1}{2}} \vee \left(⨍_0^1 f_1 \vee f_3 d\mu\right). \end{aligned}$$

2) $\star = \wedge$, then we have

$$\begin{aligned} C \fint_0^1 f_1 \wedge f_1 d\mu &= C \fint_0^1 f_1 d\mu \\ &\approx 0.309 \\ &\leq 0.5 \\ &= \left(\fint_0^1 f_1^2 \wedge f_2^2 d\mu \right)^{\frac{1}{2}} \wedge \left(\fint_0^1 f_1 \wedge f_3 d\mu \right). \end{aligned}$$

3) $\star = PF$, then we have

$$\begin{aligned} C \fint_0^1 PF(f_1, f_1) d\mu &= C \fint_0^1 f_1 d\mu \\ &\approx 0.309 \\ &\leq 1 \\ &= \left(\fint_0^1 f_1^2 d\mu \right)^{\frac{1}{2}} \\ &= PF \left[\left(\fint_0^1 PF(f_1^2, f_2^2) d\mu \right)^{\frac{1}{2}}, \left(\fint_0^1 PF(f_1, f_3) d\mu \right) \right]. \end{aligned}$$

4) $\star = PL$, then we have

$$\begin{aligned} C \fint_0^1 PL(f_1, f_1) d\mu &= C \fint_0^1 f_1 d\mu \\ &\approx 0.309 \\ &\leq 0.5 \\ &= \fint_0^1 f_3 d\mu \\ &= PL \left[\left(\fint_0^1 PF(f_1^2, f_2^2) d\mu \right)^{\frac{1}{2}}, \left(\fint_0^1 PL(f_1, f_3) d\mu \right) \right]. \end{aligned}$$

Example 3.15. Suppose that $f_1(x) = x$, $f_2(x) = x^2$, $f_3(x) = \sqrt{x}$, $A = [0, 1]$, $p = q = 1$, μ be the Lebesgue measure and $\star \in \{\vee, \wedge, PF, PL\}$, then simple calculation shows that

$$\begin{aligned} \fint_0^1 f_1(x) d\mu &= \fint_0^1 x d\mu = 0.5, \\ \fint_0^1 f_2(x) d\mu &= \fint_0^1 x^2 d\mu \approx 0.118, \\ \fint_0^1 f_3(x) d\mu &= \fint_0^1 \sqrt{x} d\mu \approx 0.618, \end{aligned}$$

$$C \approx 0.118 \times 0.618 \approx 0.073,$$
$$⨍_0^1 f_1(x) \vee f_2(x) d\mu = ⨍_0^1 x \vee x^2 d\mu = ⨍_0^1 x d\mu = 0.5,$$
$$⨍_0^1 f_1(x) \wedge f_2(x) d\mu = ⨍_0^1 x \wedge x^2 d\mu = ⨍_0^1 x^2 d\mu \approx 0.118,$$
$$⨍_0^1 f_1(x) \vee f_3(x) d\mu = ⨍_0^1 x \vee \sqrt{x} d\mu = ⨍_0^1 \sqrt{x} d\mu \approx 0.618,$$
$$⨍_0^1 f_1(x) \wedge f_3(x) d\mu = ⨍_0^1 x \wedge \sqrt{x} d\mu = ⨍_0^1 x d\mu = 0.5;$$

Now we consider 4 cases:

1) $\star = \vee$, then we have

$$\begin{aligned} C ⨍_0^1 f_1 \vee f_1 d\mu &= C ⨍_0^1 f_1 d\mu \\ &\approx 0.036 \\ &\leq 0.618 \\ &\approx \left(⨍_0^1 f_1 \vee f_2 d\mu\right) \vee \left(⨍_0^1 f_1 \vee f_3 d\mu\right). \end{aligned}$$

2) $\star = \wedge$, then we have

$$\begin{aligned} C ⨍_0^1 f_1 \wedge f_1 d\mu &= C ⨍_0^1 f_1 d\mu \\ &\approx 0.036 \\ &\leq 0.118 \\ &\approx \left(⨍_0^1 f_1 \wedge f_2 d\mu\right) \wedge \left(⨍_0^1 f_1 \wedge f_3 d\mu\right). \end{aligned}$$

3) $\star = PF$, then we have

$$\begin{aligned} C ⨍_0^1 PF(f_1, f_1) d\mu &= C ⨍_0^1 f_1 d\mu \\ &\approx 0.036 \\ &\leq 0.5 \\ &= ⨍_0^1 f_1 d\mu \\ &= PF\left[\left(⨍_0^1 PF(f_1, f_2) d\mu\right), \left(⨍_0^1 PF(f_1, f_3) d\mu\right)\right]. \end{aligned}$$

4) $\star = PL$, then we have

$$\begin{aligned} C\fint_0^1 PL(f_1, f_1)d\mu &= C\fint_0^1 f_1 d\mu \\ &\approx 0.036 \\ &\leq 0.618 \\ &\approx \fint_0^1 f_3 d\mu \\ &= PL\left[\left(\fint_0^1 PL(f_1, f_2)d\mu\right), \left(\fint_0^1 PL(f_1, f_3)d\mu\right)\right]. \end{aligned}$$

The following example shows that, if we omit the condition comonotonicity of $f_i, i = 1, 2, 3$, then Inequality (3.20) may not hold:

Example 3.16. Let $A = [0, 2]$ and $f_1(x) = x, f_2(x) = f_3(x) = 2-x, p = q = 1, \star = \wedge$ and μ is the Lebesgue measure, then

$$\begin{aligned} &\fint_0^2 f_1(x)d\mu = \fint_0^2 f_2(x)d\mu = \fint_0^2 f_3(x)d\mu = 1, \quad C = 1, \\ &\fint_0^2 f_1(x) \wedge f_2(x)d\mu = \fint_0^2 f_1(x) \wedge f_3(x)d\mu = \frac{2}{3}. \end{aligned}$$

Thus

$$\begin{aligned} C\fint_0^2 f_1 \wedge f_1 d\mu &= 1 \\ &> \frac{2}{3} \\ &= \left(\fint_0^2 f_1 \wedge f_2 d\mu\right) \wedge \left(\fint_0^2 f_1 \wedge f_3 d\mu\right). \end{aligned}$$

The following example shows that the condition $\fint_A f_i d\mu \leq 1, i = 1, 2, 3$ in Theorem 3.13 is necessary:

Example 3.17. Let $A = [0, 5], f_i(x) = 5x, i = 1, 2, 3$ and $p = q = 1, \star = \vee$ and μ is the Lebesgue measure, then

$$\begin{aligned} &\fint_0^5 f_i(x)d\mu = \frac{25}{6}, \\ &C = \left(\frac{25}{6}\right)^2, \\ &\fint_0^5 f_1(x) \vee f_2(x)d\mu = \fint_0^5 f_1(x) \vee f_3(x)d\mu = \frac{25}{6}. \end{aligned}$$

Thus

$$C\fint_0^5 f_1 \vee f_1 d\mu = \left(\frac{25}{6}\right)^3$$
$$> \frac{25}{6}$$
$$= \left(\fint_0^5 f_1 \vee f_2 d\mu\right) \vee \left(\fint_0^5 f_1 \vee f_3 d\mu\right).$$

4. Conclusion

We have introduced general related inequalities to Carlson-type inequality for the Sugeno integral on an abstract fuzzy measure space (X, Σ). For further investigation we propose to consider the Carlson inequality for the Choquet integral and seminormed fuzzy integral.

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