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Abstract. In this work, we introduce an iterative algorithm for solving the split feasibility problem on zeros of the sum of monotone operators and fixed point sets and also solving the fixed point problem of a nonexpansive mapping. This algorithm is a modification of the method based on the inertial and Mann viscosity-type methods. By assuming the existence of solutions, we show the strong convergence theorems of the constructed sequences. Finally, we also apply the proposed algorithm to related problems in Hilbert spaces.

1. Introduction

The split feasibility problem (SFP) was introduced by Censor and Elfving [8]. This is the problem of finding a point $x^* \in \mathbb{R}^n$ such that

$$x^* \in C \cap L^{-1}Q, \tag{1.1}$$

where C and Q are nonempty closed convex subsets of $\mathbb{R}^n$, and L is an $n \times n$ matrix. There are many applications of the problem (1.1), in various fields of science and technology such as in signal processing, medical image reconstruction, and intensity-modulated radiation therapy; see [5, 6, 8, 9] and the references therein. The popular iterative algorithm for solving the problem (1.1) is the following CQ algorithm, suggested by Byrne [5]: for arbitrary $x_1 \in \mathbb{R}^n$,

$$x_{n+1} = P_C\left(x_n - \gamma L^\top (I - P_Q) L x_n\right), \quad \forall n \in \mathbb{N}, \tag{1.2}$$

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where $\gamma \in \left(0, 2/\|L\|^2\right)$, L is a real $m \times n$ matrix and $L^\top$ is the transpose of the matrix L. Subsequently, in 2010, Xu [38] considered SFP in infinite-dimensional Hilbert spaces and proposed the CQ algorithm by using a bounded linear operator $L : H_1 \to H_2$ instead of the matrix L and also used L^* (the adjoint operator of L). In addition, by considering the algorithm (1.2), López et al. [16] suggested to use the stepsizes γ_n without the norm of operator L,

$$\gamma_n = \frac{\psi_n \|(I - P_Q)Lx_n\|^2}{2\|L^*(I - P_Q)Lx_n\|^2}, \tag{1.3}$$

where $0 < \psi_n < 4$ and $L^*(I - P_Q)Lx_n \neq 0$. They point out that the higher dimensions of L may be hard to compute the operator norm and it may affect the computing in the iteration process. For example, the CPU time, and the algorithm with stepsizes (1.3) gives faster results.

For a Hilbert space H, let $B : H \to 2^H$ be a set-valued operator. Martinet [21] introduced the variational inclusion problem (VIP), the problem of finding a point $x^* \in H$ such that

$$0 \in Bx^*. \tag{1.4}$$

The popular method for solving the problem (1.4) is called the proximal point algorithm: for a given $x_1 \in H$,

$$x_{n+1} = J^B_{\lambda_n} x_n, \quad \forall n \in \mathbb{N}, \tag{1.5}$$

where $\{\lambda_n\} \subset (0, \infty)$ and $J^B_{\lambda_n}$ is the resolvent of the maximal monotone operator B; see [14, 20, 37] for more details. Later, by the concept of SFP and VIP, Byrne et al. [7] proposed the split null point problem (SNPP): let $B_1 : H_1 \to 2^{H_1}$ and $B_2 : H_2 \to 2^{H_2}$ be set-valued mappings. The SNPP is the problem of finding a point

$$x^* \in B_1^{-1}0 \cap L^{-1}B_2^{-1}0. \tag{1.6}$$

They considered the following iterative algorithm: for $\lambda > 0$ and an arbitrary $x_1 \in H_1$,

$$x_{n+1} = J^{B_1}_\lambda \left(x_n - \gamma L^* \left(I - J^{B_2}_\lambda\right) Lx_n\right), \quad \forall n \in \mathbb{N}, \tag{1.7}$$

where $\gamma \in \left(0, 2/\|L\|^2\right)$. Under suitable control conditions, they obtained the weakly convergence results. Moreover, by considering a more general problem, Takahashi et al. [33] studied the problem as follows: find $x^* \in H_1$ such that

$$x^* \in B^{-1}0 \cap L^{-1}F(T), \tag{1.8}$$

where $B : H_1 \to 2^{H_1}$ is a maximal monotone operator and $T : H_2 \to H_2$ is a nonexpansive mapping. They proposed the following algorithm: for any $x_1 \in H_1$,

$$x_{n+1} = J_{\lambda_n}^B \left(x_n - \gamma_n L^*(I - T)Lx_n\right), \quad \forall n \in \mathbb{N}, \tag{1.9}$$

where $\{\lambda_n\}$ and $\{\gamma_n\}$ satisfy some suitable control conditions, and provide the weak convergence theorem of the algorithm (1.9) to the solution set of the problem (1.8).

A type of generalization of (1.4) is the following problem: find $x^* \in H$ such that

$$0 \in Ax^* + Bx^*, \tag{1.10}$$

where $A : H \to H$ is a single-valued operator and $B : H \to 2^H$ is a set-valued operator. When A and B are monotone operators, the elements in the solution set of the problem (1.10) are called the zeros of the sum of monotone operators. There are many kinds of real world problems that arise in the form of problem (1.10); see [4, 22, 28, 35] and the references therein. In 2018, Zhu et al. [40] considered a problem of finding a point $x^* \in H$ such that

$$x^* \in F(S) \cap (A + B)^{-1}0 \cap L^{-1}F(T) =: \Omega, \tag{1.11}$$

where $S : H_1 \to H_1$ and $T : H_2 \to H_2$ are nonexpansive mappings. They proposed the following method by using the viscosity algorithm [23]: for any $x_1 \in H_1$,

$$\begin{aligned} u_n &= J_{\lambda_n}^B \left((I - \lambda_n A) - \gamma_n L^*(I - T)L\right) x_n, \\ x_{n+1} &= \alpha_n f(x_n) + (1 - \alpha_n)Su_n, \quad \forall n \in \mathbb{N}, \end{aligned} \tag{1.12}$$

when $f : H_1 \to H_1$ is a contraction mapping, and showed that, by some suitable conditions, the sequence $\{x_n\}$ converges strongly to a point $z \in \Omega$, where $z = P_\Omega f(z)$. Recently, there are many authors who have studied the problems related to the fixed point and inclusion problems; see [11–13, 36] for more imformation.

On the other hand, the study of the inertial technique was first presented by Polyak in 1964, to speed up the rate of convergence; see [27]. This technique is a two-step iterative method, in which each iteration involves the previous two iterates. Recently, many authors used the inertial method because of the faster convergence rate of the algorithm; see [2, 26, 29, 34] for more information.

In 2001, Alvarez and Attouch [1] proposed the following inertial proximal point method for finding the solution of the problem (1.4): for arbitrary x_0, $x_1 \in H$,

$$\begin{aligned} y_n &= x_n + \mu_n(x_n - x_{n-1}), \\ x_{n+1} &= J_{\lambda_n}^B y_n, \quad \forall n \in \mathbb{N}, \end{aligned} \tag{1.13}$$

where $\{\lambda_n\}$ and $\{\mu_n\}$ satisfy some suitable conditions with

$$\sum_{n=1}^{\infty} \mu_n \|x_n - x_{n-1}\|^2 < \infty,$$

and present the weakly convergence results.

In 2015, Lorenz and Pock [17] considered the monotone inclusion problem (1.10) and proposed the inertial forward-backward algorithm for solving the problem (1.10): for arbitrary x_0, $x_1 \in H$,

$$\begin{aligned} y_n &= x_n + \mu_n(x_n - x_{n-1}), \\ x_{n+1} &= J^B_{\lambda_n}(I - \lambda_n A)y_n, \quad \forall n \in \mathbb{N}, \end{aligned} \tag{1.14}$$

where A and B are monotone operators. By suitable conditions of $\{\lambda_n\}$ and $\{\mu_n\}$, they proved that the sequence $\{x_n\}$ converges weakly to the zeros of $A + B$.

Recently, Anh et al. [2] proposed the following iterative algorithm for solving the problem (1.6), by combining the inertial method, the algorithm (1.7) and Mann iteration [19]: for arbitrary x_0, $x_1 \in H_1$,

$$\begin{aligned} z_n &= x_n + \mu_n(x_n - x_{n-1}), \\ y_n &= J^{B_1}_{\lambda}\left(z_n - \gamma_n L^*\left(I - J^{B_2}_{\lambda}\right) L z_n\right), \\ x_{n+1} &= (1 - \theta_n - \alpha_n)x_n + \theta_n y_n, \quad \forall n \in \mathbb{N}, \end{aligned} \tag{1.15}$$

where $\{\mu_n\} \subset [0, \mu)$ for some $\mu > 0$, $\{\theta_n\} \subset (a, b) \subset (0, 1 - \alpha_n)$ and $\{\alpha_n\}$ satisfies $\lim\limits_{n\to\infty} \dfrac{\mu_n}{\alpha_n}\|x_n - x_{n-1}\| = 0$. They proved that the sequence $\{x_n\}$ converges strongly to a solution of the problem (1.6).

Very recently, Tan et al. [34] introduced the modified inertial Mann viscosity algorithm for solving fixed point problems. Let T be nonexpansive mapping, for arbitrary x_0, $x_1 \in H$,

$$\begin{aligned} z_n &= x_n + \mu_n(x_n - x_{n-1}), \\ y_n &= \beta_n z_n + (1 - \beta_n)T z_n, \\ x_{n+1} &= \alpha_n f(x_n) + (1 - \alpha_n)y_n, \quad \forall n \in \mathbb{N}, \end{aligned} \tag{1.16}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ in $(0, 1)$, and $\{\alpha_n\}$ satisfies $\lim\limits_{n\to\infty} \dfrac{\mu_n}{\alpha_n}\|x_n - x_{n-1}\| = 0$ and some suitable conditions. They showed that the sequence $\{x_n\}$ converges strongly to $z = P_{F(T)}f(z)$.

In this paper, motivated and inspired by the above literature, we are going to consider a problem (1.11). We aim to suggest a modified iterative algorithm, which is generated by using the inertial method and Mann viscosity-type algorithm, for solving the considered problem. In our results, we provide some suitable conditions to guarantee that the constructed sequence $\{x_n\}$ converges strongly to a point in Ω.

2. Preliminaries

Throughout this paper, we denote by $\mathbb{N}$ and $\mathbb{R}$ for the set of natural numbers and real numbers, respectively. Let H be a real Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and the norm $\|\cdot\|$. Let $\{x_n\}$ be a sequence in H, we denote the strong convergence and weak convergence of the sequence $\{x_n\}$ to x by $x_n \to x$ and $x_n \rightharpoonup x$, respectively.

Let $T : H \to H$ be a mapping. Then, T is said to be a Lipschitz mapping if there exists $\alpha \geq 0$ such that

$$\|Tx - Ty\| \leq \alpha \|x - y\|, \quad \forall x, y \in H.$$

The number α is called a Lipschitz constant. If $\alpha \in [0, 1)$, then T is a contraction mapping, and T is a nonexpansive mapping if $\alpha = 1$.

Moreover, we say that T is firmly nonexpansive if

$$\langle Tx - Ty, x - y \rangle \geq \|Tx - Ty\|^2, \quad \forall x, y \in H.$$

The set of fixed points of a self-mapping T will be denoted by $F(T)$, that is $F(T) = \{x \in H : Tx = x\}$. We note that if T is nonexpansive, then $F(T)$ is closed and convex.

Let $A : H \to H$ be a single-valued mapping. For a positive real number β, we will say that A is β-inverse strongly monotone (β-ism) if

$$\langle Ax - Ay, x - y \rangle \geq \beta \|Ax - Ay\|^2, \quad \forall x, y \in H.$$

Now, we collect some important properties for our proof.

Lemma 2.1 ([4, 39]). *We have*

(i) *If $A : H \to H$ is β-ism and $\lambda \in (0, \beta]$, then $T := I - \lambda A$ is firmly nonexpansive.*

(ii) *A mapping $T : H \to H$ is nonexpansive if and only if $I - T$ is $\frac{1}{2}$-ism.*

Let $B : H \to 2^H$ be a set-valued mapping. The effective domain of B is denoted by $D(B)$, that is, $D(B) = \{x \in H : Bx \neq \emptyset\}$. Recall that B is said to be monotone if

$$\langle x - y, u - v \rangle \geq 0, \quad \forall x, y \in D(B),\, u \in Bx,\, v \in By.$$

A monotone mapping B is said to be maximal if its graph is not properly contained in the graph of any other monotone operator. To a maximal monotone operator $B : H \to 2^H$ and $\lambda > 0$, its resolvent J_λ^B is defined by

$$J_\lambda^B := (I + \lambda B)^{-1} : H \to D(B).$$

Notice that the resolvent J_λ^B is a single-valued and firmly nonexpansive mapping, and $F\left(J_\lambda^B\right) = B^{-1}0 \equiv \{x \in H : 0 \in Bx\}, \forall \lambda > 0$; see [32, 33].

Lemma 2.2 ([3])**.** *Let C be a nonempty, closed, and convex subset of a real Hilbert space H and $A : C \to H$ be an operator. If $B : H \to 2^H$ is a maximal monotone operator, then $F\left(J_\lambda^B(I - \lambda A)\right) = (A + B)^{-1}0$.*

The following fundamental results are needed in our proof.

Let C be a nonempty closed convex subset of H. For every point $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that

$$\|x - P_C x\| \leq \|x - y\|, \quad \forall y \in C.$$

P_C is called a metric projection of H onto C; see [31]. The following property of P_C is well known and useful:

$$\langle x - P_C x, y - P_C x\rangle \leq 0, \quad \forall x \in H, y \in C.$$

For each $x, y, z \in H$, the following facts are valid for inner product spaces,

$$\|\alpha x + (1-\alpha)y\|^2 = \alpha \|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)\|x - y\|^2, \tag{2.1}$$

and

$$\begin{aligned}\|\alpha x + \beta y + \gamma z\|^2 &= \alpha\|x\|^2 + \beta\|y\|^2 + \gamma\|z\|^2 \\ &\quad - \alpha\beta\|x - y\|^2 - \alpha\gamma\|x - z\|^2 - \beta\gamma\|y - z\|^2,\end{aligned} \tag{2.2}$$

for any $\alpha, \beta, \gamma \in [0, 1]$ such that $\alpha + \beta + \gamma = 1$; see [25, 32].

We also use the following lemmas for proving the main theorems.

Lemma 2.3 ([30])**.** *Let C be a closed convex subset of a Hilbert space H and $T : C \to C$ be a nonexpansive mapping. Then, $U := I - T$ is demiclosed, that is, $x_n \rightharpoonup x_0$ and $Ux_n \to y_0$ imply $Ux_0 = y_0$.*

Lemma 2.4 ([15, 37])**.** *Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the following relation:*

$$a_{n+1} \leq (1 - \alpha_n)a_n + \alpha_n b_n + c_n, \quad \forall n \in \mathbb{N},$$

where $\{\alpha_n\}$, $\{b_n\}$ and $\{c_n\}$ are sequences of real numbers satisfying

(i) $\{\alpha_n\} \subset [0, 1]$, $\sum_{i=1}^{\infty} \alpha_n = \infty$;
(ii) $\limsup_{n\to\infty} b_n \leq 0$;
(iii) $c_n \geq 0$, $\sum_{i=1}^{\infty} c_n < \infty$.

Then, $a_n \to 0$ as $n \to \infty$.

3. Main Results

In this section, we start by introducing the assumptions and the modified inertial algorithm that will be used to provide our main results.

(A1) $A : H_1 \to H_1$ is a η-inverse strongly monotone operator;
(A2) $B : H_1 \to 2^{H_1}$ is a maximal monotone operator;
(A3) $L : H_1 \to H_2$ is a bounded linear operator;
(A4) $T : H_2 \to H_2$ is a nonexpansive mapping;
(A5) $S : H_1 \to H_1$ is a nonexpansive mapping;
(A6) $f : H_1 \to H_1$ is a contraction mapping with coefficient $\nu \in (0, 1)$.

Algorithm 3.1. Let $\{\alpha_n\}$, $\{\delta_n\}$, $\{\theta_n\}$ and $\{\beta_n\}$ be sequences in $(0, 1)$ with $\alpha_n + \delta_n + \theta_n = 1$ and the initial $x_0, x_1 \in H_1$ be arbitrary, define

$$\begin{aligned}
z_n &= x_n + \mu_n(x_n - x_{n-1}),\\
w_n &= J^B_\lambda(I - \lambda A)\left(z_n - \gamma_n L^*(I - T)Lz_n\right),\\
y_n &= \beta_n z_n + (1 - \beta_n)w_n,\\
x_{n+1} &= \alpha_n f(x_n) + \delta_n x_n + \theta_n S y_n, \quad \forall n \in \mathbb{N},
\end{aligned}$$

where $\{\mu_n\} \subset [0, \mu)$ with $\mu \in [0, 1)$, $\lambda \in (0, \eta)$ and $\{\gamma_n\}$ is depend on $\psi_n \in [a, b] \subset (0, 1)$ by

$$\gamma_n = \begin{cases} \frac{\psi_n\|(I-T)Lz_n\|^2}{\|L^*(I-T)Lz_n\|^2}, & \text{if } L^*(I - T)Lz_n \neq 0;\\ \gamma, & \text{if otherwise,}\end{cases}$$

where γ is any nonnegative value.

Remark 3.2. The sequence $\{\gamma_n\}$ is bounded; see [26] for more detail.

Now, we will present the strong convergence theorem, by using the above assumptions and Algorithm 3.1.

Theorem 3.3. *Let H_1 and H_2 be two real Hilbert spaces. Let $\{x_n\}$ be generated by Algorithm 3.1. Suppose that the assumptions (A1)-(A6) hold, $\Omega \neq \emptyset$ and the following control conditions are satisfied:*

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(ii) $0 < a \leq \delta_n$ *and* $0 < a \leq \theta_n$;
(iii) $0 < b_1 \leq \beta_n \leq b_2 < 1$;
(iv) $\lim_{n\to\infty} \frac{\mu_n}{\alpha_n}\|x_n - x_{n-1}\| = 0$.

Then, $\{x_n\}$ converges strongly to $p \in \Omega$, where $p = P_\Omega f(p)$.

Proof. First, we prove that the sequence $\{x_n\}$ is bounded. Let $z \in \Omega$. Then $z \in F(S)$, $z \in (A + B)^{-1}0$ and $z \in L^{-1}F(T)$, imply that $Sz = z$, $J^B_\lambda(I - \lambda A)z = z$ and $TLz = Lz$.

Since T is nonexpansive, by Lemma 2.1(ii) we have $(I-T)$ is $\frac{1}{2}$-ism. That is,

$$\langle (I-T)Lz_n - (I-T)Lz, Lz_n - Lz\rangle \geq \frac{1}{2}\left\|(I-T)Lz_n - (I-T)Lz\right\|^2,$$

for each $n \in \mathbb{N}$. By $TLz = Lz$, the inequality is reduced to

$$\langle Lz_n - TLz_n, Lz_n - Lz\rangle \geq \frac{1}{2}\left\|(I-T)\,Lz_n\right\|^2. \tag{3.1}$$

By using (3.1), we obtain the following relations

$$\begin{aligned}
\|w_n - z\|^2 &= \left\|J_\lambda^B(I-\lambda A)\left(z_n - \gamma_n L^*(I-T)Lz_n\right) - z\right\|^2 \\
&\leq \left\|(z_n - z) - \gamma_n L^*(I-T)Lz_n\right\|^2 \\
&\leq \|z_n - z\|^2 - 2\gamma_n \langle z_n - z, L^*(I-T)Lz_n\rangle \\
&\quad + \gamma_n^2 \left\|L^*(I-T)Lz_n\right\|^2 \\
&= \|z_n - z\|^2 - 2\gamma_n \langle Lz_n - Lz, Lz_n - TLz_n\rangle \\
&\quad + \gamma_n^2 \left\|L^*(I-T)Lz_n\right\|^2 \\
&\leq \|z_n - z\|^2 - \gamma_n \left\|(I-T)Lz_n\right\|^2 + \gamma_n^2 \left\|L^*(I-T)Lz_n\right\|^2 \\
&= \|z_n - z\|^2 - \gamma_n \left(\left\|(I-T)Lz_n\right\|^2 - \gamma_n \left\|L^*(I-T)Lz_n\right\|^2\right),
\end{aligned} \tag{3.2}$$

for each $n \in \mathbb{N}$. By the definition of γ_n, we get

$$\gamma_n \left(\left\|(I-T)Lz_n\right\|^2 - \gamma_n \left\|L^*(I-T)Lz_n\right\|^2\right) \geq 0.$$

Thus, from (3.2), we have

$$\|w_n - z\| \leq \|z_n - z\|. \tag{3.3}$$

Next, by the definition of y_n and (3.3), we obtain

$$\begin{aligned}
\|y_n - z\| &\leq \beta_n \|z_n - z\| + (1-\beta_n)\|w_n - z\| \\
&\leq \beta_n \|z_n - z\| + (1-\beta_n)\|z_n - z\| \\
&= \|z_n - z\|,
\end{aligned} \tag{3.4}$$

for each $n \in \mathbb{N}$. Now, by condition (iv), we see that

$$\begin{aligned}
\|z_n - z\| &\leq \|x_n - z\| + \mu_n \|x_n - x_{n-1}\| \\
&= \|x_n - z\| + \alpha_n \frac{\mu_n}{\alpha_n}\|x_n - x_{n-1}\| \\
&\leq \|x_n - z\| + \alpha_n K_1,
\end{aligned} \tag{3.5}$$

for some $K_1 > 0$. It follows by using (3.5) that

$$\|y_n - z\| \leq \|x_n - z\| + \alpha_n K_1. \tag{3.6}$$

We use (3.6) and the definition of x_{n+1}, we have

$$\begin{aligned}
\|x_{n+1}-z\| &= \|\alpha_n f(x_n)+\delta_n x_n+\theta_n Sy_n - z\| \\
&\le \alpha_n \|f(x_n)-z\| + \delta_n\|x_n-z\| + \theta_n\|Sy_n-z\| \\
&\le \alpha_n \|f(x_n)-f(z)\| + \alpha_n \|f(z)-z\| + \delta_n\|x_n-z\| \\
&\quad + \theta_n\|y_n-z\| \\
&\le \alpha_n\nu \|x_n-z\| + \alpha_n \|f(z)-z\| + \delta_n\|x_n-z\| \\
&\quad + \theta_n (\|x_n-z\| + \alpha_n K_1) \\
&\le (\alpha_n\nu+\delta_n+\theta_n) \|x_n-z\| + \alpha_n (\|f(z)-z\| + K_1) \\
&= (1-\alpha_n(1-\nu)) \|x_n-z\| \\
&\quad + \alpha_n(1-\nu) \left(\frac{\|f(z)-z\|+K_1}{1-\nu}\right) \\
&\le \max\left\{\|x_n-z\|, \frac{\|f(z)-z\|+K_1}{1-\nu}\right\} \\
&\quad \vdots \\
&\le \max\left\{\|x_1-z\|, \frac{\|f(z)-z\|+K_1}{1-\nu}\right\},
\end{aligned} \tag{3.7}$$

for each $n\in\mathbb{N}$. Thus, we get $\{\|x_n-z\|\}$ is a bounded sequence, implies that $\{x_n\}$ is bounded. Consequently, $\{z_n\}$, $\{w_n\}$, $\{y_n\}$ and $\{f(x_n)\}$ are also bounded.

Next, we note that $P_\Omega f(\cdot)$ is a contraction mapping. Let p be a unique fixed point of $P_\Omega f(\cdot)$, that is $p = P_\Omega f(p)$. For each $n\in\mathbb{N}$,

$$\begin{aligned}
\|z_n-p\|^2 &= \|x_n+\mu_n(x_n-x_{n-1})-p\|^2 \\
&\le \|x_n-p\|^2 + \mu_n^2\|x_n-x_{n-1}\|^2 + 2\mu_n \langle x_n-p, x_n-x_{n-1}\rangle \\
&\le \|x_n-p\|^2 + \mu_n^2\|x_n-x_{n-1}\|^2 + 2\mu_n\|x_n-p\|\|x_n-x_{n-1}\|.
\end{aligned} \tag{3.8}$$

By using (3.8), we have

$$\begin{aligned}
\|x_{n+1}-p\|^2 &= \langle \alpha_n f(x_n)+\delta_n x_n+\theta_n Sy_n - p, x_{n+1}-p\rangle \\
&= \alpha_n \langle f(x_n)-f(p), x_{n+1}-p\rangle + \alpha_n \langle f(p)-p, x_{n+1}-p\rangle \\
&\quad + \delta_n \langle x_n-p, x_{n+1}-p\rangle + \theta_n \langle Sy_n-p, x_{n+1}-p\rangle \\
&\le \frac{\alpha_n}{2}\left(\|f(x_n)-f(p)\|^2 + \|x_{n+1}-p\|^2\right) \\
&\quad + \frac{\delta_n}{2}\left(\|x_n-p\|^2 + \|x_{n+1}-p\|^2\right) \\
&\quad + \frac{\theta_n}{2}\left(\|Sy_n-p\|^2 + \|x_{n+1}-p\|^2\right)
\end{aligned}$$

$$
\begin{aligned}
&+\alpha_n \langle f(p)-p, x_{n+1}-p\rangle \\
&\leq \left(\frac{\alpha_n \nu^2}{2}+\frac{\delta_n}{2}\right)\|x_n-p\|^2+\frac{\theta_n}{2}\|z_n-p\|^2 \\
&+\frac{\alpha_n+\delta_n+\theta_n}{2}\|x_{n+1}-p\|^2 \\
&+\alpha_n \langle f(p)-p, x_{n+1}-p\rangle \\
&\leq \left(\frac{\alpha_n \nu^2+\delta_n+\theta_n}{2}\right)\|x_n-p\|^2+\frac{1}{2}\|x_{n+1}-p\|^2 \\
&+\frac{\theta_n \mu_n^2}{2}\|x_n-x_{n-1}\|^2+\theta_n \mu_n \|x_n-p\|\|x_n-x_{n-1}\| \\
&+\alpha_n \langle f(p)-p, x_{n+1}-p\rangle,
\end{aligned}
$$

for each $n \in \mathbb{N}$. Then,

(3.9)

$$
\begin{aligned}
\|x_{n+1}-p\|^2 &\leq \left(1-\alpha_n(1-\nu^2)\right)\|x_n-p\|^2+\mu_n^2\|x_n-x_{n-1}\|^2 \\
&\quad +2\mu_n\|x_n-p\|\|x_n-x_{n-1}\|+2\alpha_n\langle f(p)-p, x_{n+1}-p\rangle \\
&\leq \left(1-\alpha_n(1-\nu^2)\right)\|x_n-p\|^2 \\
&\quad +\mu_n\|x_n-x_{n-1}\|\left(\mu_n\|x_n-x_{n-1}\|+2\|x_n-p\|\right) \\
&\quad +2\alpha_n\langle f(p)-p, x_{n+1}-p\rangle \\
&\leq \left(1-\alpha_n(1-\nu^2)\right)\|x_n-p\|^2 \\
&\quad +K_2\mu_n\|x_n-x_{n-1}\|+2\alpha_n\langle f(p)-p, x_{n+1}-p\rangle,
\end{aligned}
$$

where $K_2 = 3\sup_n\{\mu\|x_n-x_{n-1}\|, \|x_n-p\|\} > 0$. Thus,

$$
\|x_{n+1}-p\|^2 \leq \left(1-\alpha_n(1-\nu^2)\right)\|x_n-p\|^2+\alpha_n(1-\nu^2)T_n, \tag{3.10}
$$

where $T_n = \dfrac{K_2}{1-\nu^2}\dfrac{\mu_n}{\alpha_n}\|x_n-x_{n-1}\|+\dfrac{2}{1-\nu^2}\langle f(p)-p, x_{n+1}-p\rangle$.

Now, we consider the following two cases.

Case 1: Suppose that there exists $n_0 \in \mathbb{N}$ such that $\{\|x_n-p\|\}$ is monotonically non-increasing. Since $\{\|x_n-p\|\}$ is bounded, it is a convergent sequence.

Consider the following relation, by using (2.1) and (3.2), we have

(3.11)

$$
\begin{aligned}
\|y_n-p\|^2 &= \|\beta_n z_n+(1-\beta_n)w_n-p\|^2 \\
&= \beta_n\|z_n-p\|^2+(1-\beta_n)\|w_n-p\|^2 \\
&\quad -\beta_n(1-\beta_n)\|z_n-w_n\|^2 \\
&\leq \beta_n\|z_n-p\|^2+(1-\beta_n)\|w_n-p\|^2
\end{aligned}
$$

$$
\begin{aligned}
&\leq \beta_n\|z_n-p\|^2+(1-\beta_n)\|z_n-p\|^2\\
&\quad -(1-\beta_n)\gamma_n\left(\|(I-T)Lz_n\|^2-\gamma_n\left\|L^*(I-T)Lz_n\right\|^2\right)\\
&=\|z_n-p\|^2\\
&\quad -(1-\beta_n)\gamma_n\left(\|(I-T)Lz_n\|^2-\gamma_n\left\|L^*(I-T)Lz_n\right\|^2\right),
\end{aligned}
$$

for each $n\in\mathbb{N}$. Furthermore, from (3.5) we have

$$
\begin{aligned}
\|z_n-p\|^2&=\|x_n-p\|^2+2\alpha_nK_1\|x_n-p\|+\alpha_n^2K_1^2\\
&=\|x_n-p\|^2+\alpha_n\left(2K_1\|x_n-p\|+\alpha_nK_1^2\right)\\
&=\|x_n-p\|^2+\alpha_nK_3,
\end{aligned}
\tag{3.12}
$$

where $K_3=\sup_n\left\{2K_1\|x_n-p\|+\alpha_nK_1^2\right\}>0$, for each $n\in\mathbb{N}$.

By using (2.2), (3.11) and (3.12), we obtain

(3.13)

$$
\begin{aligned}
&\|x_{n+1}-p\|^2\\
&\quad=\|\alpha_n\left(f(x_n)-p\right)+\delta_n(x_n-p)+\theta_n(Sy_n-p)\|^2\\
&\quad\leq \alpha_n\left\|f(x_n)-p\right\|^2+\delta_n\|x_n-p\|^2+\theta_n\|Sy_n-p\|^2\\
&\qquad -\alpha_n\delta_n\left\|f(x_n)-x_n\right\|^2-\alpha_n\theta_n\left\|f(x_n)-Sy_n\right\|^2\\
&\qquad -\delta_n\theta_n\|x_n-Sy_n\|^2\\
&\quad\leq \alpha_n\left\|f(x_n)-p\right\|^2+\delta_n\|x_n-p\|^2+\theta_n\|Sy_n-p\|^2\\
&\quad\leq \alpha_n\left\|f(x_n)-p\right\|^2+\delta_n\|x_n-p\|^2+\theta_n\|y_n-p\|^2\\
&\quad\leq \alpha_n\left\|f(x_n)-p\right\|^2+\delta_n\|x_n-p\|^2+\theta_n\|z_n-p\|^2\\
&\qquad -\theta_n(1-\beta_n)\gamma_n\left(\|(I-T)Lz_n\|^2-\gamma_n\left\|L^*(I-T)Lz_n\right\|^2\right)\\
&\quad\leq \alpha_n\left\|f(x_n)-p\right\|^2+\delta_n\|x_n-p\|^2+\theta_n\|x_n-p\|^2+\alpha_n\theta_nK_3\\
&\qquad -\theta_n(1-\beta_n)\gamma_n\left(\|(I-T)Lz_n\|^2-\gamma_n\left\|L^*(I-T)Lz_n\right\|^2\right),
\end{aligned}
$$

for each $n\in\mathbb{N}$. Then,

(3.14)

$$
\begin{aligned}
&\theta_n(1-\beta_n)\gamma_n\left(\|(I-T)Lz_n\|^2-\gamma_n\left\|L^*(I-T)Lz_n\right\|^2\right)\\
&\qquad\leq \|x_n-p\|^2-\|x_{n+1}-p\|^2+\alpha_n\left\|f(x_n)-p\right\|^2+\alpha_n\theta_nK_3\\
&\qquad\leq \|x_n-p\|^2-\|x_{n+1}-p\|^2+\alpha_n\left(\left\|f(x_n)-p\right\|^2+K_3\right).
\end{aligned}
$$

Consequently, by condition (i), (ii) and (iii), we get

$$\gamma_n\left(\|(I-T)Lz_n\|^2-\gamma_n\|L^*(I-T)Lz_n\|^2\right)\to 0,$$

as $n\to\infty$. Furthermore, by the definition of γ_n, we have

$$\gamma_n\left(\|(I-T)Lz_n\|^2-\gamma_n\|L^*(I-T)Lz_n\|^2\right)=\psi_n(1-\psi_n)\frac{\|(I-T)Lz_n\|^4}{\|L^*(I-T)Lz_n\|^2}.$$

This implies

$$\psi_n(1-\psi_n)\frac{\|(I-T)Lz_n\|^4}{\|L^*(I-T)Lz_n\|^2}\to 0,$$

as $n\to\infty$. Since $\psi_n\in[a,b]\subset(0,1)$, it follows that

$$\lim_{n\to\infty}\frac{\|(I-T)Lz_n\|^2}{\|L^*(I-T)Lz_n\|}=0. \tag{3.15}$$

In addition, by the fact $\|L^*(I-T)Lz_n\|\le\|L^*\|\,\|(I-T)Lz_n\|$, implies

$$\|(I-T)Lz_n\|\le\|L^*\|\frac{\|(I-T)Lz_n\|^2}{\|L^*(I-T)Lz_n\|},$$

for each $n\in\mathbb{N}$. Thus, by (3.15), we get

$$\lim_{n\to\infty}\|(I-T)Lz_n\|=0, \tag{3.16}$$

and also

$$\lim_{n\to\infty}\|L^*(I-T)Lz_n\|=0. \tag{3.17}$$

From the following relation

$$\begin{aligned}\|x_{n+1}-p\|^2\le{}&\alpha_n\|f(x_n)-p\|^2+\delta_n\|x_n-p\|^2+\theta_n\|Sy_n-p\|^2\\&-\alpha_n\delta_n\|f(x_n)-p\|^2-\alpha_n\theta_n\|f(x_n)-Sy_n\|^2\\&-\delta_n\theta_n\|x_n-Sy_n\|^2,\end{aligned} \tag{3.18}$$

for each $n\in\mathbb{N}$, which implies that

$$\begin{aligned}\delta_n\theta_n\|x_n-Sy_n\|^2\le{}&\alpha_n\|f(x_n)-p\|^2+\delta_n\|x_n-p\|^2\\&+\theta_n\|y_n-p\|^2-\|x_{n+1}-p\|^2.\end{aligned} \tag{3.19}$$

Moreover, by (3.4) and (3.12), we know that

$$\|y_n-p\|^2\le\|z_n-p\|^2\le\|x_n-p\|^2+\alpha_nK_3.$$

Then, from (3.19), we obtain

$$\begin{aligned}\delta_n\theta_n\|x_n-Sy_n\|^2&\le\alpha_n\|f(x_n)-p\|^2+\delta_n\|x_n-p\|^2\\&\quad+\theta_n\|x_n-p\|^2+\alpha_n\theta_nK_3-\|x_{n+1}-p\|^2\\&\le\alpha_n\left(\|f(x_n)-p\|^2+K_3\right)\end{aligned} \tag{3.20}$$

$$+ \|x_n - p\|^2 - \|x_{n+1} - p\|^2 .$$

Hence, by conditions (i) and (ii), we have

$$\lim_{n\to\infty} \|x_n - Sy_n\| = 0. \tag{3.21}$$

Next, we will show that $\lim_{n\to\infty} \|y_n - x_n\| = 0$. Consider

$$\begin{aligned} \|y_n - x_n\| &= \|\beta_n z_n + (1-\beta_n)w_n - x_n\| \\ &\le \beta_n\|z_n - x_n\| + (1-\beta_n)\|w_n - x_n\|, \end{aligned} \tag{3.22}$$

for each $n \in \mathbb{N}$. In the first term of (3.22), we obtain

$$\begin{aligned} \|z_n - x_n\| &= \|x_n + \mu_n(x_n - x_{n-1}) - x_n\| \\ &\le \alpha_n \frac{\mu_n}{\alpha_n} \|x_n - x_{n-1}\| . \end{aligned} \tag{3.23}$$

By conditions (i) and (iv), we get

$$\lim_{n\to\infty} \|z_n - x_n\| = 0. \tag{3.24}$$

And, in the second term of (3.22), we use the following relation

$$\|w_n - x_n\| \le \|w_n - z_n\| + \|z_n - x_n\|, \tag{3.25}$$

for each $n \in \mathbb{N}$. It remains to show that $\|w_n - z_n\| \to 0$, as $n \to \infty$. Consider

$$\begin{aligned} \|y_n - p\|^2 &= \|\beta_n(z_n - p) + (1-\beta_n)(w_n - p)\|^2 \\ &\le \beta_n\|z_n - p\|^2 + (1-\beta_n)\|w_n - p\|^2 \\ &\quad - \beta_n(1-\beta_n)\|z_n - w_n\|^2 \\ &\le \|z_n - p\|^2 - \beta_n(1-\beta_n)\|z_n - w_n\|^2, \end{aligned} \tag{3.26}$$

for each $n \in \mathbb{N}$. From the relation in (3.13), we use (3.12) and (3.26), it follows that

$$\begin{aligned} \|x_{n+1} - p\|^2 &\le \alpha_n \|f(x_n) - p\|^2 + \delta_n\|x_n - p\|^2 + \theta_n\|y_n - p\|^2 \\ &\le \alpha_n \|f(x_n) - p\|^2 + \delta_n\|x_n - p\|^2 + \theta_n\|z_n - p\|^2 \\ &\quad - \theta_n\beta_n(1-\beta_n)\|z_n - w_n\|^2 \\ &\le \alpha_n \|f(x_n) - p\|^2 + \delta_n\|x_n - p\|^2 + \theta_n\|x_n - p\|^2 \\ &\quad + \theta_n\alpha_n K_3 - \theta_n\beta_n(1-\beta_n)\|z_n - w_n\|^2, \end{aligned} \tag{3.27}$$

which implies that

(3.28)

$$\begin{aligned} &\theta_n\beta_n(1-\beta_n)\|z_n - w_n\|^2 \\ &\qquad \le \alpha_n \|f(x_n) - p\|^2 + \theta_n\alpha_n K_3 + \|x_n - p\|^2 - \|x_{n+1} - p\|^2 \end{aligned}$$

$$\leq \alpha_n \left(\|f(x_n) - p\|^2 + K_3 \right) + \|x_n - p\|^2 - \|x_{n+1} - p\|^2 .$$

By using condition (i), (ii) and (iii), we get

$$\lim_{n\to\infty} \|z_n - w_n\| = 0. \tag{3.29}$$

From (3.25), by using (3.24) and (3.29), we obtain

$$\lim_{n\to\infty} \|w_n - x_n\| = 0. \tag{3.30}$$

Substituting (3.24) and (3.30) in (3.22) we obtain

$$\lim_{n\to\infty} \|y_n - x_n\| = 0. \tag{3.31}$$

Moreover, by using (3.21) and (3.31), we get

$$\|y_n - Sy_n\| \leq \|y_n - x_n\| + \|x_n - Sy_n\| \to 0, \tag{3.32}$$

as $n \to \infty$.

Using the definition of x_{n+1}, we have

$$\begin{aligned} \|x_{n+1} - x_n\| &= \|\alpha_n f(x_n) + \delta_n x_n + \theta_n S y_n - x_n\| \\ &\leq \alpha_n \|f(x_n) - x_n\| + \theta_n \|Sy_n - x_n\| , \end{aligned} \tag{3.33}$$

for each $n \in \mathbb{N}$. By using (3.21) and condition (i), we obtain

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{3.34}$$

Now, the sequence $\{x_n\}$ is bounded on H_1, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ converges weakly to $x^* \in H_1$. Next, we will show that $x^* \in \Omega$.

To show $x^* \in F(S)$, we use Lemma 2.3. Since $y_{n_i} \rightharpoonup x^*$, from (3.32) we have $x^* \in F(S)$.

Next, we show that $x^* \in (A+B)^{-1}0$. Consider

$$\begin{aligned} &\left\|x^* - J_\lambda^B (I - \lambda A) x^*\right\|^2 \\ &\quad \leq \left\langle x^* - J_\lambda^B (I - \lambda A) x^*, x^* - x_{n_i} \right\rangle \\ &\qquad + \left\langle x^* - J_\lambda^B (I - \lambda A) x^*, x_{n_i} - J_\lambda^B (I - \lambda A) x_{n_i} \right\rangle \\ &\qquad + \left\langle x^* - J_\lambda^B (I - \lambda A) x^*, J_\lambda^B (I - \lambda A) x_{n_i} - J_\lambda^B (I - \lambda A) x^* \right\rangle , \end{aligned} \tag{3.35}$$

for each $i \in \mathbb{N}$. We see that

$$\begin{aligned} &\left\|w_n - J_\lambda^B (I - \lambda A) x_n\right\| \\ &\quad \leq \left\|J_\lambda^B (I - \lambda A) \left(z_n - \gamma_n L^*(I - T) L z_n\right) - J_\lambda^B (I - \lambda A) x_n\right\| \\ &\quad \leq \|z_n - x_n\| + \gamma_n \|L^*(I - T) L z_n\| , \end{aligned}$$

for each $n \in \mathbb{N}$. By (3.17) and (3.23), we get

$$\lim_{n\to\infty} \left\| w_n - J_\lambda^B (I - \lambda A) x_n \right\| = 0. \tag{3.36}$$

Observe that the following inequality

$$\left\| x_n - J_\lambda^B (I - \lambda A) x_n \right\| \leq \|x_n - z_n\| + \|z_n - w_n\| + \|w_n - J_\lambda^B (I - \lambda A) x_n\|,$$

for each $n \in \mathbb{N}$. By using (3.23), (3.29) and (3.36) we have

$$\lim_{n\to\infty} \left\| x_n - J_\lambda^B (I - \lambda A) x_n \right\| = 0.$$

Also the subsequence $\{x_{n_i}\}$, we have

$$\lim_{i\to\infty} \left\| x_{n_i} - J_\lambda^B (I - \lambda A) x_{n_i} \right\| = 0. \tag{3.37}$$

From above (3.35), by using (3.37) and together with $x_{n_i} \rightharpoonup x^*$, we obtain

$$\lim_{i\to\infty} \left\| x^* - J_\lambda^B (I - \lambda A) x^* \right\| = 0.$$

It follows that, $x^* = J_\lambda^B (I - \lambda A) x^*$ and hence $x^* \in (A + B)^{-1} 0$.

Next, we show that $Lx^* \in F(T)$. Similarly, consider

$$\begin{aligned} \|Lx^* - TLx^*\|^2 \leq\ & \langle Lx^* - TLx^*, Lx^* - Lx_{n_i} \rangle \\ & + \langle Lx^* - TLx^*, Lx_{n_i} - TLx_{n_i} \rangle \\ & + \langle Lx^* - TLx^*, TLx_{n_i} - TLx^* \rangle, \end{aligned} \tag{3.38}$$

for each $i \in \mathbb{N}$. Now, we have the following inequality,

$$\begin{aligned} \|(I - T)Lx_n\| &\leq \|(I - T)Lx_n - (I - T)Lz_n\| + \|(I - T)Lz_n\| \\ &\leq \|Lx_n - Lz_n\| + \|TLx_n - TLz_n\| + \|(I - T)Lz_n\| \\ &\leq 2\|L\| \|x_n - z_n\| + \|(I - T)Lz_n\|, \end{aligned}$$

for each $n \in \mathbb{N}$. Then, by (3.16) and (3.23), we have

$$\lim_{n\to\infty} \|(I - T)Lx_n\| = 0.$$

And, we also have

$$\lim_{i\to\infty} \|(I - T)Lx_{n_i}\| = 0.$$

In addition, by using the linearity and continuity of L, $Lx_{n_i} \rightharpoonup Lx^*$, as $i \to \infty$. Thus, from (3.38) we get

$$\lim_{i\to\infty} \|Lx^* - TLx^*\| = 0.$$

Therefore, $Lx^* = TLx^*$, implies $Lx^* \in F(T)$. Consequently, we obtain $x^* \in \Omega$.

Finally, we prove that $\{x_n\}$ converges strongly to $p = P_\Omega f(p)$. Now, we know that the sequence $\{x_n\}$ is bounded, and we have from (3.34) that $\|x_{n+1} - x_n\| \to 0$, as $n \to \infty$. With loss of generality, we may assume that a subsequence $\{x_{n_i+1}\}$ of $\{x_{n+1}\}$ converges weakly to $x^* \in H_1$. Thus, we obtain

(3.39)
$$\begin{aligned}\limsup_{n\to\infty} \frac{2}{1-\nu^2}\langle f(p)-p, x_{n+1}-p\rangle &= \lim_{i\to\infty} \frac{2}{1-\nu^2}\langle f(p)-p, x_{n_i+1}-p\rangle \\ &= \frac{2}{1-\nu^2}\langle f(p)-p, x^*-p\rangle \\ &\leq 0.\end{aligned}$$

By using (3.39) and together with condition (iv) we get

$$\limsup_{n\to\infty}\left(\frac{K_2}{1-\nu^2}\frac{\mu_n}{\alpha_n}\|x_n-x_{n-1}\| + \frac{2}{1-\nu^2}\langle f(p)-p, x_{n+1}-p\rangle\right) \leq 0.$$

From (3.9), by using Lemma 2.4, we can conclude that $\|x_n - p\| \to 0$, as $n \to \infty$. Thus $x_n \to p$, as $n \to \infty$.

Case 2: Suppose that $\{\|x_n - p\|\}$ is not monotonically decreasing sequence. Set $\Gamma_n = \|x_n - p\|$, $\forall n \in \mathbb{N}$ and let $\tau : \mathbb{N} \to \mathbb{N}$ be a mapping for all $n \geq n_0$ (for some n_0 large enough) by

$$\tau(n) := \max\{k \in \mathbb{N} : k \leq n, \quad \Gamma_k \leq \Gamma_{k+1}\}.$$

Then, we have $\{\tau(n)\}$ is a nondecreasing sequence, with $\tau(n) \to \infty$ as $n \to \infty$ and

$$0 \leq \Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}, \quad \forall n \geq n_0.$$

Obviously, we get $\left\|x_{\tau(n)} - p\right\|^2 - \left\|x_{\tau(n)+1} - p\right\|^2 \leq 0$, for each $n \geq n_0$. From the relation in (3.14), we obtain

(3.40)
$$\begin{aligned}&\theta_{\tau(n)}(1-\beta_{\tau(n)})\gamma_{\tau(n)}\left(\left\|(I-T)Lz_{\tau(n)}\right\|^2 - \gamma_{\tau(n)}\left\|L^*(I-T)Lz_{\tau(n)}\right\|^2\right) \\ &\qquad \leq \left\|x_{\tau(n)}-p\right\|^2 - \left\|x_{\tau(n)+1}-p\right\|^2 + \alpha_{\tau(n)}\left(\left\|f(x_{\tau(n)})-p\right\|^2 + K_3\right) \\ &\qquad \leq \alpha_{\tau(n)}\left(\left\|f(x_{\tau(n)})-p\right\|^2 + K_3\right),\end{aligned}$$

for each $n \geq n_0$. Similar as in **Case 1**, we obtain

$$\begin{aligned}&\lim_{n\to\infty}\left\|(I-T)Lz_{\tau(n)}\right\| = 0, \\ &\lim_{n\to\infty}\left\|L^*(I-T)Lz_{\tau(n)}\right\| = 0, \\ &\lim_{n\to\infty}\left\|x_{\tau(n)+1} - x_{\tau(n)}\right\| = 0\end{aligned}$$

and

$$\limsup_{n\to\infty}\left(\frac{K_2}{1-\nu^2}\frac{\mu_{\tau(n)}}{\alpha_{\tau(n)}}\|x_{\tau(n)}-x_{\tau(n)-1}\| + \frac{2}{1-\nu^2}\left\langle f(p)-p, x_{\tau(n)+1}-p\right\rangle\right) \leq 0.$$

Since the sequence $\{x_{\tau(n)}\}$ is bounded, we can find a subsequence of $\{x_{\tau(n)}\}$, still denoted by $\{x_{\tau(n)}\}$, which converges weakly to $x^* \in F(S) \cap (A+B)^{-1}0 \cap L^{-1}F(T)$. From (3.9), it follows that

$$\left\|x_{\tau(n)+1} - p\right\|^2 \leq \left(1 - \alpha_{\tau(n)}(1-\nu^2)\right)\left\|x_{\tau(n)} - p\right\|^2 + \alpha_{\tau(n)}(1-\nu^2)T_{\tau(n)},$$

where $T_{\tau(n)} = \dfrac{K_2}{1-\nu^2}\dfrac{\mu_{\tau(n)}}{\alpha_{\tau(n)}}\|x_{\tau(n)} - x_{\tau(n)-1}\| + \dfrac{2}{1-\nu^2}\left\langle f(p) - p, x_{\tau(n)+1} - p\right\rangle$,

for each $n \geq n_0$. Consequently, we have

$$\begin{aligned}\alpha_{\tau(n)}(1-\nu^2)\left\|x_{\tau(n)} - p\right\|^2 &\leq \left\|x_{\tau(n)} - p\right\|^2 - \left\|x_{\tau(n)+1} - p\right\|^2 \\ &\quad + \alpha_{\tau(n)}(1-\nu^2)T_{\tau(n)} \\ &\leq \alpha_{\tau(n)}(1-\nu^2)T_{\tau(n)}\end{aligned} \tag{3.41}$$

By $\alpha_{\tau(n)}(1-\nu^2) > 0$, from (3.41) we get

$$\limsup_{n\to\infty}\left\|x_{\tau(n)} - p\right\|^2 \leq 0.$$

This implies that

$$\lim_{n\to\infty}\left\|x_{\tau(n)} - p\right\|^2 = 0,$$

and also

$$\lim_{n\to\infty}\left\|x_{\tau(n)} - p\right\| = 0. \tag{3.42}$$

Now, we use $\lim_{n\to\infty}\left\|x_{\tau(n)+1} - x_{\tau(n)}\right\| = 0$ and (3.42), it follows that

$$\left\|x_{\tau(n)+1} - p\right\| \leq \left\|x_{\tau(n)+1} - x_{\tau(n)}\right\| + \left\|x_{\tau(n)} - p\right\| \to 0, \tag{3.43}$$

as $n \to \infty$. Furthermore, if $\tau(n) < n$, we see that $\Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}$, because $\Gamma_j \geq \Gamma_{j+1}$ for $\tau(n)+1 \leq j \leq n$. As a consequence, we have

$$\begin{aligned}0 &\leq \Gamma_n \\ &\leq \max\left\{\Gamma_{\tau(n)}, \Gamma_{\tau(n)+1}\right\} \\ &= \Gamma_{\tau(n)+1},\end{aligned}$$

for each $n \geq n_0$. By using (3.43), we can conclude that $\lim_{n\to\infty}\Gamma_n = 0$. Therefore, we obtain the sequence $\{x_n\}$ converges strongly to p. This completes the proof. $\square$

Remark 3.4. (a) ([29]) The condition (iv) is easily implemented in numerical computation because we can find the valued of $\|x_n - x_{n-1}\|$ before choosing μ_n. Indeed, we can choose the parameter μ_n such that $0 \leq \mu_n \leq \bar{\mu}_n$, where

$$\bar{\mu}_n = \begin{cases} \min\left\{\mu, \dfrac{\omega_n}{\|x_n - x_{n-1}\|}\right\}, & \text{if } x_n \neq x_{n-1}; \\ \mu, & \text{if otherwise,}\end{cases}$$

where ω_n is a positive sequence such that $\omega_n = o(\alpha_n)$.

(b) The following choice is the special case of (a); we choose $\alpha_n = \dfrac{1}{n+1}$, $\omega_n = \dfrac{1}{(n+1)^2}$ and $\mu = \dfrac{n-1}{n+\nu-1} \in [0,1)$. Then, we have

$$\bar{\mu}_n = \begin{cases} \min\left\{\dfrac{n-1}{n+\nu-1}, \dfrac{1}{(n+1)^2\|x_n - x_{n-1}\|}\right\}, & \text{if } x_n \neq x_{n-1}; \\ \dfrac{n-1}{n+\nu-1}, & \text{if otherwise.} \end{cases}$$

(c) If $S := I$ (the identity operator) and $A := 0$ (the zero operator), then Problem (1.11) reduces to Problem (1.8). And, if $L := I$ and $A := 0$, then we observe that Problem (1.11) reduces to a type of common fixed points of nonexpansive mappings; see [18] for more information.

4. Applications

In this section, we discuss some applications of the problem (1.11) via Theorem 3.3.

4.1. Split feasibility problem.

Now, we consider the normal cone to C at $u \in C$ is defined as

$$N_C(u) = \{z \in H : \langle z, y - u\rangle \leq 0, \quad \forall y \in C\},$$

where C is a nonempty closed convex subset of H. Notice that N_C is a maximal monotone operator. Then, when we set $B := N_C : H_1 \to 2^{H_1}$, we get $J_\lambda^B =: P_C$. It follows that $F(J_\lambda^B) = F(P_C) = C$. By the setting $T =: P_Q$, we can verify that the problem (1.8) is reduced to the split feasibility problem (1.1). By considering $A := 0$ (the zero operator), the problem (1.11) is reduced to a problem of finding a point

$$x^* \in F(S) \cap C \cap L^{-1}Q =: \Omega_{S,C,Q}$$

Thus, we obtain the following result.

Algorithm 4.1. Let $\{\alpha_n\}$, $\{\delta_n\}$, $\{\theta_n\}$ and $\{\beta_n\}$ be sequences in $(0,1)$ with $\alpha_n + \delta_n + \theta_n = 1$ and the initial $x_0, x_1 \in H_1$ be arbitrary, define

$$\begin{aligned} z_n &= x_n + \mu_n(x_n - x_{n-1}), \\ w_n &= P_C\left(z_n - \gamma_n L^*(I - P_Q)Lz_n\right), \\ y_n &= \beta_n z_n + (1-\beta_n)w_n, \\ x_{n+1} &= \alpha_n f(x_n) + \delta_n x_n + \theta_n S y_n, \quad \forall n \in \mathbb{N}, \end{aligned}$$

where $\{\mu_n\} \subset [0, \mu)$ with $\mu \in [0, 1)$ and $\{\gamma_n\}$ is depend on $\psi_n \in [a, b] \subset (0, 1)$ by

$$\gamma_n = \begin{cases} \frac{\psi_n \left\|(I-P_Q)Lz_n\right\|^2}{\left\|L^*(I-P_Q)Lz_n\right\|^2}, & \text{if } L^*(I - P_Q)Lz_n \neq 0; \\ \gamma, & \text{if otherwise,} \end{cases}$$

where γ is any nonnegative value.

Theorem 4.2. *Let C and Q be nonempty colsed convex subsets of real Hilbert spaces H_1 and H_2, respectively. Let $\{x_n\}$ be generated by Algorithm 4.1. Suppose that the assumptions (A3), (A5) and (A6) hold, $\Omega_{S,C,Q} \neq \emptyset$ and the following control conditions are satisfied:*

(i) $\lim\limits_{n\to\infty} \alpha_n = 0$ *and* $\sum\limits_{n=1}^{\infty} \alpha_n = \infty$*;*

(ii) $0 < a \leq \delta_n$ *and* $0 < a \leq \theta_n$*;*

(iii) $0 < b_1 \leq \beta_n \leq b_2 < 1$*;*

(iv) $\lim\limits_{n\to\infty} \dfrac{\mu_n}{\alpha_n} \|x_n - x_{n-1}\| = 0.$

Then, $\{x_n\}$ converges strongly to $p \in \Omega_{S,C,Q}$, where $p = P_{\Omega_{S,C,Q}} f(p)$.

4.2. **Split monotone variational inclusion problem.** We consider a $\tilde{\eta}$-inverse strongly monotone $\tilde{A} : H_1 \to H_1$ and maximal monotone operator $\tilde{B} : H_2 \to 2^{H_2}$. By setting $T := J_\lambda^{\tilde{B}}(I - \lambda\tilde{A})$, we obtain $F(T) := (\tilde{A} + \tilde{B})^{-1}0$. In this case, we can verify that the problem (1.11) is reduced to the problem of finding a common solution of the split monotone variational inclusion problem [24] and the fixed point problem. That is, we consider a problem of finding a point

$$x^* \in F(S) \cap (A + B)^{-1}0 \cap L^{-1}(\tilde{A} + \tilde{B})^{-1}0 =: \Omega_{S,A,B}$$

By the above setting, we get the result follows from Theorem 3.3.

Algorithm 4.3. Let $\{\alpha_n\}$, $\{\delta_n\}$, $\{\theta_n\}$ and $\{\beta_n\}$ be sequences in $(0, 1)$ with $\alpha_n + \delta_n + \theta_n = 1$ and the initial $x_0, x_1 \in H_1$ be arbitrary, define

$$\begin{aligned} z_n &= x_n + \mu_n(x_n - x_{n-1}), \\ w_n &= J_\lambda^B(I - \lambda A)\left(z_n - \gamma_n L^*(I - J_\lambda^{\tilde{B}}(I - \lambda\tilde{A}))Lz_n\right), \\ y_n &= \beta_n z_n + (1 - \beta_n)w_n, \\ x_{n+1} &= \alpha_n f(x_n) + \delta_n x_n + \theta_n S y_n, \quad \forall n \in \mathbb{N}, \end{aligned}$$

where $\{\mu_n\} \subset [0,\mu)$ with $\mu \in [0,1)$, $\lambda \in (0, \min\{\eta, \tilde{\eta}\})$ and $\{\gamma_n\}$ is depend on $\psi_n \in [a,b] \subset (0,1)$ by

$$\gamma_n = \begin{cases} \frac{\psi_n \left\| (I - J_\lambda^{\tilde{B}}(I-\lambda\tilde{A}))Lz_n \right\|^2}{\left\| L^*(I - J_\lambda^{\tilde{B}}(I-\lambda\tilde{A}))Lz_n \right\|^2}, & \text{if } L^*(I - J_\lambda^{\tilde{B}}(I-\lambda\tilde{A}))Lz_n \neq 0; \\ \gamma, & \text{if otherwise,} \end{cases}$$

where γ is any nonnegative value.

Theorem 4.4. *Let H_1 and H_2 be two real Hilbert spaces. Let $\tilde{A} : H_1 \to H_1$ be a $\tilde{\eta}$-ism and $\tilde{B} : H_2 \to 2^{H_2}$ be maximal monotone operator. Let $\{x_n\}$ be generated by Algorithm 4.1. Suppose that the assumptions (A1)-(A3) and (A5)-(A6) hold, $\Omega_{S,A,B} \neq \emptyset$ and the following control conditions are satisfied:*

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(ii) $0 < a \leq \delta_n$ *and* $0 < a \leq \theta_n$;
(iii) $0 < b_1 \leq \beta_n \leq b_2 < 1$;
(iv) $\lim_{n\to\infty} \frac{\mu_n}{\alpha_n} \|x_n - x_{n-1}\| = 0$.

Then, $\{x_n\}$ converges strongly to $p \in \Omega_{S,A,B}$, where $p = P_{\Omega_{S,A,B}} f(p)$.

4.3. **Split common fixed point problem.** Let $V : H_1 \to H_1$ be a nonexpansive mapping. Then, by Lemma 2.1(ii), we know that $A := I - V$ is a $\frac{1}{2}$-ism. Moreover, since $Ax^* = 0$ if and only if $x^* \in F(V)$. From case $B := 0$ (the zero operator), we get the problem (1.11) is reduced to the problem of finding a common solution of the split common fixed point problem [10] and the fixed point problem. That is, we consider a problem of finding a point

$$x^* \in F(S) \cap F(V) \cap L^{-1}F(T) =: \Omega_{S,V,T}.$$

By applying Theorem 3.3, we obtain the following result.

Algorithm 4.5. Let $\{\alpha_n\}$, $\{\delta_n\}$, $\{\theta_n\}$ and $\{\beta_n\}$ be sequences in $(0,1)$ with $\alpha_n + \delta_n + \theta_n = 1$ and the initial $x_0, x_1 \in H_1$ be arbitrary, define

$$\begin{aligned} z_n &= x_n + \mu_n(x_n - x_{n-1}), \\ w_n &= ((1-\lambda)I + \lambda V)\left(z_n - \gamma_n L^*(I-T)Lz_n\right), \\ y_n &= \beta_n z_n + (1-\beta_n)w_n, \\ x_{n+1} &= \alpha_n f(x_n) + \delta_n x_n + \theta_n S y_n, \quad \forall n \in \mathbb{N}, \end{aligned}$$

where $\{\mu_n\} \subset [0,\mu)$ with $\mu \in [0,1)$, $\lambda \in (0, \frac{1}{2})$ and $\{\gamma_n\}$ is depend on $\psi_n \in [a,b] \subset (0,1)$ by

$$\gamma_n = \begin{cases} \frac{\psi_n \|(I-T)Lz_n\|^2}{\|L^*(I-T)Lz_n\|^2}, & \text{if } L^*(I-T)Lz_n \neq 0; \\ \gamma, & \text{if otherwise,} \end{cases}$$

where γ is any nonnegative value.

Theorem 4.6. *Let H_1 and H_2 be two real Hilbert spaces. Let $V : H_1 \to H_1$ be a nonexpansive mapping. Let $\{x_n\}$ be generated by Algorithm 4.5. Suppose that the assumptions (A3)-(A6) hold, $\Omega_{S,V,T} \neq \emptyset$ and the following control conditions are satisfied:*

(i) $\lim\limits_{n\to\infty} \alpha_n = 0$ *and* $\sum\limits_{n=1}^{\infty} \alpha_n = \infty$*;*
(ii) $0 < a \leq \delta_n$ *and* $0 < a \leq \theta_n$*;*
(iii) $0 < b_1 \leq \beta_n \leq b_2 < 1$*;*
(iv) $\lim\limits_{n\to\infty} \dfrac{\mu_n}{\alpha_n}\|x_n - x_{n-1}\| = 0.$

Then, $\{x_n\}$ converges strongly to $p \in \Omega_{S,V,T}$, where $p = P_{\Omega_{S,V,T}} f(p)$.

5. Conclusions

In this work, we present a new algorithm for finding a common solution of a class of split feasibility problems and fixed point problems of a nonexpansive mapping in Hilbert spaces, the problem (1.11). We suggest the modified algorithm including the inertial and Mann viscosity-type methods, Algorithm 3.1. By providing suitable control conditions to the process, we obtain the strong convergence theorem of the proposed algorithm (Theorem 3.3). In application of our results, we show that the proposed algorithms are applied to split feasibility problem, split monotone variational inclusion problem and split common fixed point problem.

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References

1. F. Alvarez and H. Attouch, *An inertial proximal method for maximal monotone operators via discretization of a nonlinear oscillator with damping. Wellposedness in optimization and related topics (Gargnano, 1999)*, Set-Valued Anal., 9 (2001), pp. 3–11.
2. P.K. Anh, D.V. Thong and V.T. Dung, *A strongly convergent Mann-type inertial algorithm for solving split variational inclusion problems*, Optim. Eng., 2020 (2020), 21 pages.
3. K. Aoyama, Y Kimura, W. Takahashi and M. Toyoda, *On a strongly nonexpansive sequence in Hilbert spaces*, J. Nonlinear Convex Anal., 8 (2007), pp. 471-489.

4. O.A. Boikanyo, *The viscosity approximation forward-backward splitting method for zeros of the sum of monotone operators*, Abstr. Appl. Anal., 2016 (2016), 10 pages.
5. C. Byrne, *Iterative oblique projection onto convex sets and the split feasibility problem*, Inverse Probl., 18 (2002), pp. 441-453.
6. C. Byrne, *A unified treatment of some iterative algorithms in signal processing and image reconstruction*, Inverse Probl., 20 (2004), pp. 103-120.
7. C. Byrne, Y. Censor, A. Gibali and S. Reich, *Weak and strong convergence of algorithms for the split common null point problem*, J. Nonlinear Convex Analysis, 13 (2012), pp. 759-775.
8. Y. Censor and T. Elfving, *A multiprojection algorithm using Bregman projections in product space*, Numer. Algorithms, 8 (1994), pp. 221-239.
9. Y. Censor, T. Bortfeld, B. Martin and A. Trofimov, *A unified approach for inversion problems in intensity- modulated radiation therapy*, Phys. Med. Biol., 51 (2006), pp. 2353-2365.
10. Y. Censor and A. Segal, *The split common fixed point problem for directed operators*, J. Convex Anal., 16 (2009), pp. 587-600.
11. J. Chen, A. Pitea, L.J. Zhu, *Split systems of nonconvex variational inequalities and fixed point problems on uniformly prox-regular sets*, Symmetry, 11 (2019), 1279, 13 pages.
12. V. Dadashi, M. Postolache, *Hybrid proximal point algorithm and applications to equilibrium problems and convex programming*, J. Optim. Theory Appl., 174 (2017), pp. 518-529.
13. V. Dadashi, M. Postolache, *Forward-backward splitting algorithm for fixed point problems and zeros of the sum of monotone operators*, Arab. J. Math., 9 (2020), pp. 89-99.
14. J. Eckstein and D.P. Bertsckas, *On the Douglas Rachford splitting method and the proximal point algorithm for maximal monotone operators*, Appl. Math. Mech.-Engl. Math. Programming, 55 (1992), pp. 293-318.
15. L.S. Liu, *Ishikawa and Mann iterative process with errors for nonlinear strongly accretive mappins in Banach spaces*, J. Math. Anal. Appl., 194 (1995), pp. 114-125.
16. G. López, V. Martín-Márquez, F. Wang and H.K. Xu, *Solving the split feasibility problem without prior knowledge of matrix norms*, Inverse Probl., 28 (2012), 18 pages.
17. D. Lorenz and T. Pock, *An inertial forward-backward algorithm for monotone inclusions*, J. Math. Imaging Vis., 51 (2015), pp. 311-325.

18. P.E. Maingé, *Approximation methods for common fixed points of nonexpansive mappings in Hilbert spaces*, J. Math. Anal. Appl., 325 (2007), pp. 69-479.
19. W.R. Mann, *Mean value methods in iteration*, Proc. Am. Math. Soc., 4 (1953), pp. 506–510.
20. G. Marino and H.K. Xu, *Convergence of generalized proximal point algorithm*, Comm. Pure Appl. Anal., 3 (2004), pp. 791-808.
21. B. Martinet, *Régularisation d'inéquations variationnelles par approximations successives*, Revue Francaise d'Informatique et de Recherche Opérationnelle, 3 (1970), pp. 154-158.
22. A. Moudafi and M. Thera, *Finding a zero of the sum of two maximal monotone operators*, J. Optimiz. Theory App., 94 (1997), pp. 425-448.
23. A. Moudafi, *Viscosity approximating methods for fixed point problems*, J. Math. Anal. Appl., 241 (2000), pp. 46-55.
24. A. Moudafi, *Split monotone variational inclusions*, J. Optimiz. Theory App., 150 (2011), pp. 275-283.
25. M.O. Osilike and D.I. Igbokwe, *Weak and strong convergence theorems for fixed points of pseudocontractions and solutions of monotone type operater equations*, Comput. Math. Appl., 40 (2000), pp. 559-567.
26. N. Petrot and M. Suwannaprapa, *Inertial viscosity-type algorithms for a class of split feasibility problems and fixed point problems in Hilbert spaces*, Linear Nonlinear Anal., 6 (2020), pp. 385-403.
27. B. Polyak, *Some methods of speeding up the convergence of iteration methods*, USSR Comput. Math. Math. Phys., 4 (1964), pp. 1-17.
28. X. Qin, S.Y. Cho and L. Wang, *A regularization method for treating zero points of the sum of two monotone operators*, Fixed Point Theory A., 2014 (2014), 10 pages.
29. S. Suantai, N. Pholasa and P. Cholamjiak, *The modified inertial relaxed CQ algorithm for solving the split feasibility problems*, J. Ind. Manag. Optim., 14 (2018), pp. 1595-1615.
30. W. Takahashi, *Nonlinear functional analysis: Fixed point theory and its applications*, Yokohama Publishers, Yokohama Japan, 2000.
31. W. Takahashi and M. Toyoda, *Weak convergence theorems for nonexpansive mappings and monotone mappings*, J. Optimiz. Theory App., 118 (2003), pp. 417-428.
32. W. Takahashi, *Introduction to nonlinear and convex analysis*, Yokohama Publishers, Yokohama Japan, 2009.
33. W. Takahashi, H.K. Xu and J.C. Yao, *Iterative methods for generalized split feasibility problems in Hilbert spaces*, Set-Valued Var. Anal., 23 (2015), pp. 205-221.

34. B. Tan, Z. Zhou, S. Li, *Strong Convergence of Modified inertial Mann algorithms for nonexpansive mappings*, Mathematics, 8 (2020), 11 pages.
35. P. Tseng, *A modified forward-backward splitting method for maximal monotone mappings*, Siam J. Control Optim., 38 (2000), pp. 431-446.
36. Y. Wang, J. Chen, A. Pitea, *The split equality fixed point problem of demicontractive operators with numerical example and application*, Symmetry, 12 (2020), 902, 14 pages.
37. H.K. Xu, *Iterative algorithms for nonlinear operators*, J. London Math. Soc., 66 (2002), pp. 240-256.
38. H.K. Xu, *Iterative methods for the split feasibility problem in infinite-dimensional Hilbert spaces*, Inverse Probl., 26 (2010), 17 pages.
39. H.K. Xu, *Averaged mappings and the gradient-projection algorithm*, J. Optim. Theory Appl., 150 (2011), pp. 360-378.
40. J. Zhu, J. Tang, S. Chang, *Strong convergence theorems for a class of split feasibility problems and fixed point problem in Hilbert spaces*, J. Inequal. Appl., 2018 (2018), 15 pages.

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