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A New and Faster Iterative Scheme Including Generalized α-nonexpansive Mappings in Banach Spaces

Asghar Rahimi[1], Ali Rezaei[2], Bayaz Daraby[3] and Mostafa Ghasemi[4*]

Abstract. In this paper, we proposed a new iterative process to approximate fixed point of generalized α-nonexpansive mappings and show that the coefficient used in the proposed iterative process play a fundamental role in the rate of convergence. We compare the speed of convergence of new iterative process with other well-known iterative process by using numerical examples. Finally, by using new iterative process, we obtained some weak and strong convergence theorems for generalized α-nonexpansive mappings in a Banach space.

1. Introduction

Let E be a Banach space and D be a nonempty subset of E. A mapping $K : D \to D$ is said to be nonexpansive whenever $\|Kw - Ku\| \leq \|w - u\|$ for all $w, u \in D$. It is called quasi-nonexpansive mapping if $F(K) \neq \emptyset$ and $\|Kw - p\| \leq \|w - p\|$ for all $w \in D$ and $p \in F(K)$, where $F(K)$ is the set of fixed points of K i.e., $F(K) = \{w \in D : Kw = w\}$. It is well known that if D is a closed, bounded and convex subset of a uniformly convex Banach space E, then $F(K)$ is nonempty. A number of generalizations of nonexpansive mappings have been considered by some mathematicians in recent years. Suzuki in [1] introduced the concept of generalized nonexpansive mappings which is a condition on mappings called condition(C) and obtained some existence and convergence theorems for such mappings. A mapping $K : D \to D$ is said to

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satisfy condition(C) if

$$\frac{1}{2}\left\|w-Kw\right\| \leq \left\|w-u\right\| \quad \Rightarrow \quad \left\|Kw-Ku\right\| \leq \left\|w-u\right\|,$$

for all $w, u \in D$. It is obvious that every mapping satisfying condition(C) with a fixed point is a quasi-nonexpansive mapping [1].

Recently, Aoyama and Kohsaka in [2], introduced the class of α-nonexpansive mappings in Banach spaces which includes nonexpansive mappings and obtained a fixed point theorem for such mappings. A mapping $K : D \to D$ is said to be α-nonexpansive if for a given real number $\alpha < 1$,

$$\left\|Kw-Ku\right\|^2 \leq \alpha\left\|Kw-u\right\|^2 + \alpha\left\|w-Ku\right\|^2 + (1-2\alpha)\left\|w-u\right\|^2,$$

for all $w, u \in D$.

Ariza-Puiz et al. in [3] showed that the concept of α-nonexpansive is trivial for $\alpha < 0$. It is obvious that every α-nonexpansive mapping with a fixed point is quasi-nonexpansive. In general the mapping satisfying in condition (C) and α-nonexpansive mapping are not continuous mappings (see [1] and [4]). Very recently Pant and Shukla in [4], introduced the class of generalized α-nonexpansive mappings which includes the mapping satisfying in condition (C). A mapping $K : D \to D$ is said to be generalized α-nonexpansive if there exists an $\alpha \in [0, 1)$ such that

$$\frac{1}{2}\left\|w-Kw\right\| \leq \left\|w-u\right\|$$

then

$$\left\|Kw-Ku\right\| \leq \alpha\left\|Kw-u\right\| + \alpha\left\|Ku-w\right\| + (1-2\alpha)\left\|w-u\right\|, \ \forall w, u \in D.$$

There exists some iteration processes which is often used to approximate fixed points of nonexpansive mappings such as Picard iteration, Mann iteration [5], Ishikawa iteration [6], Noor iteration [7], Agarwal iteration [8] and Abbas iteration [9].

Recently in 2015, Thakur et al. [10] introduced the following iteration process:

$$\begin{cases} w_1 = w \in D, \\ v_n = (1-\xi_n)w_n + \xi_n K w_n, \\ u_n = K((1-\delta_n)w_n + \delta_n v_n), \\ w_{n+1} = K u_n, \qquad n \in \mathbb{N}, \end{cases} \tag{1.1}$$

where $\{\delta_n\}$ and $\{\xi_n\}$ are in $(0, 1)$. They proved some weak and strong convergence theorems in a uniformly convex Banach space by using this iteration process for the mapping satisfying condition(C).

After this in 2018, Ullah and Arshad [11] introduced the following M-iteration process:

$$\begin{cases} w_1 = w \in D, \\ v_n = (1-\delta_n)w_n + \delta_n K w_n, \\ u_n = K v_n, \\ w_{n+1} = K u_n, \qquad n \in \mathbb{N}, \end{cases} \tag{1.2}$$

where $\{\delta_n\}$ is in $(0,1)$. They used this process to obtained convergence results for the mapping satisfying condition (C).

In this paper, at the first we prove that, under conditions on M-iteration process coefficients, M-iteration process (1.2) converges faster than Thakur iteration processes (1.1) for contraction mappings. Also motivated by [9], [10] and [11], we introduce a new iteration process and show that this iteration process converges faster than M-iteration process (1.2) for contraction mappings. We prove some weak and strong convergence theorems for generalized α-nonexpansive mappings in uniformly convex Banach spaces by using our new iteration process. Finally, we provide an example of generalized α-nonexpansive mappings for numerical comparison of new iteration process with the other existing iteration processes.

2. Preliminaries

Throughout this paper, we denote by $\mathbb{N}$ the set of positive integers and by $\mathbb{R}$ the set of real numbers. Assume that E be a Banach space. We denote the weak convergence and the strong convergence of $\{w_n\}$ to $w \in E$ by $w_n \rightharpoonup w$ and $w_n \to w$, respectively. Let $S_E = \{w \in E : \|w\| = 1\}$ be a unit sphere on E, for all $w, u \in S_E$ and $\mu \in (0,1)$ such that $w \neq u$ if $\|(1-\mu)w + \mu u\| < 1$, then E is called strictly convex. A Banach space E is called uniformly convex if for each $0 < \varepsilon \leq 2$, there is a $\alpha > 0$ such that, for $w, u \in E$ with $\|w\| \leq 1$, $\|u\| \leq 1$ and $\|w - u\| \geq \varepsilon$, $\|w + u\| \leq 2(1-\alpha)$ holds. It is well known that a Banach space E satisfies the Opial property [12], if for each $\{w_n\}$ in E such that $w_n \rightharpoonup w$ and $w \neq u$, then

$$\liminf_{n\to\infty} \|w_n - w\| < \liminf_{n\to\infty} \|w_n - u\| .$$

We know that Picard, Mann [5], Ishikawa [6], Noor [7], Agarwal [8] and Abbas [9] iteration processes are defined as follows respectively:

$$\begin{cases} w_1 = w \in D, \\ w_{n+1} = K w_n, \qquad n \in \mathbb{N}, \end{cases} \tag{2.1}$$

$$\begin{cases} w_1 = w \in D, \\ w_{n+1} = (1-\delta_n)w_n + \delta_n K w_n, \qquad n \in \mathbb{N}, \end{cases} \tag{2.2}$$

$$\tag{2.3} \begin{cases} w_1 = w \in D, \\ u_n = (1-\xi_n)w_n + \xi_n K w_n, \\ w_{n+1} = (1-\delta_n)w_n + \delta_n K u_n, & n \in \mathbb{N}, \end{cases}$$

$$\tag{2.4} \begin{cases} w_1 = w \in D, \\ v_n = (1-\lambda_n)w_n + \lambda_n K w_n, \\ u_n = (1-\xi_n)w_n + \xi_n K v_n, \\ w_{n+1} = (1-\delta_n)w_n + \delta_n K u_n, & n \in \mathbb{N}, \end{cases}$$

$$\tag{2.5} \begin{cases} w_1 = w \in D, \\ u_n = (1-\xi_n)w_n + \xi_n K w_n, \\ w_{n+1} = (1-\delta_n)K w_n + \delta_n K u_n, & n \in \mathbb{N}, \end{cases}$$

and

$$\tag{2.6} \begin{cases} w_1 = w \in D, \\ v_n = (1-\lambda_n)w_n + \lambda_n K w_n, \\ u_n = (1-\xi_n)K w_n + \xi_n K v_n, \\ w_{n+1} = (1-\delta_n)K v_n + \delta_n K u_n, & n \in \mathbb{N}, \end{cases}$$

where $\{\delta_n\}$, $\{\xi_n\}$ and $\{\lambda_n\}$ are in $(0,1)$.

Let D be a nonempty closed convex subset of a Banach space E and let $\{w_n\}$ be a bounded sequence in E. For $w \in E$, we set

$$r(w, \{w_n\}) = \limsup_{n\to\infty} \|w - w_n\|.$$

The asymptotic radius [13] of $\{w_n\}$ relative to D is given by

$$r(D, \{w_n\}) = \inf\{r(w, \{w_n\}) : w \in D\}.$$

The asymptotic center [13] of $\{w_n\}$ relative to D is the set

$$A(D, \{w_n\}) = \{w \in D : r(w, \{w_n\}) = r(D, \{w_n\}\}.$$

It is known that in a uniformly convex Banach space, $A(D, \{w_n\})$ consists of exactly one point. Also, $A(D, \{w_n\})$ is nonempty and convex when D is weakly compact and convex (for more details see [13] and [14]).

Proposition 2.1 ([4])**.** *Let D be a nonempty subset of a Banach space E and $K : D \to D$ a generalized α-nonexpansive mapping. Then $F(K)$ is closed. Moreover, if E is strictly convex and D is convex, then $F(K)$ is also convex.*

Proposition 2.2 ([4])**.** *Every mapping satisfying condition (C) is a generalized α-nonexpansive mapping, but the converse is not true.*

Proposition 2.3 ([4])**.** *Let D be a nonempty subset of a Banach space E and $K : D \to D$ a generalized α-nonexpansive mapping with a fixed point $u \in D$. Then K is quasi-nonexpansive.*

Proposition 2.4 ([4]). *Let D be a nonempty subset of a Banach space E and $K : D \to D$ a generalized α-nonexpansive mapping. Then for all $w, u \in D$.*

$$\|w - K(u)\| \leq \frac{(3+\alpha)}{(1-\alpha)}\|w - K(w)\| + \|w - u\|.$$

Proposition 2.5 ([4] (Demiclosedness principle)). *Let D be a nonempty closed subset of a Banach space E with the Opial's property and $K : D \to D$ generalized α-nonexpansive mapping. If $\{w_n\}$ converges weakly to a point v and $\lim_{n\to\infty} \|K(w_n) - w_n\| = 0$, then $K(v) = v$. That is, $I - K$ is demiclosed at zero, where I is the identity mapping on E.*

Theorem 2.6 ([1]). *Let D be a weakly compact convex subset of a uniformly convex Banach space E. Let K be a mapping on D. Assume that K satisfies condition (C). Then K has a fixed point.*

Lemma 2.7 ([15]). *Let E be a uniformly convex Banach space and $0 < a \leq l_n \leq b < 1$ for all $n \in \mathbb{N}$. Let $\{w_n\}$ and $\{u_n\}$ be the two sequences such that $\limsup_{n\to\infty} \|w_n\| \leq r$, $\limsup_{n\to\infty} \|u_n\| \leq r$ and $\lim_{n\to\infty} \|l_n w_n + (1 - l_n)u_n\| = r$ hold for some $r \geq 0$. Then*

$$\lim_{n\to\infty} \|w_n - u_n\| = 0.$$

Definition 2.8 ([16]). Let $\{w_n\}$ and $\{u_n\}$ be two fixed point iteration processes that both converging to the same fixed point p and

$$\begin{cases} w_n - p \leq a_n, & \forall n \in \mathbb{N} \\ u_n - p \leq b_n, & \forall n \in \mathbb{N} \end{cases}$$

If $\{a_n\}$ and $\{b_n\}$ be two sequences of real numbers converging to a and b, respectively and

$$\lim_{n\to\infty} \frac{|a_n - a|}{|b_n - b|} = 0,$$

then we say that $\{w_n\}$ converges faster than $\{u_n\}$.

3. Rate of Convergence and New Iteration Process

In this section, we compare the speed of convergence for the state of (1.2) in M-iteration process and the state of (1.1) in Thakur et.al. iteration process in different modes for contraction mappings. Also, we introduce a new iteration process and prove that new iteration process in different modes, convergence faster than all the above mentioned iteration processes for contraction mappings.

Theorem 3.1. *Let D be a nonempty closed convex subset of a Banach space E and $w_1 \in D$. Let $K : D \to D$ be a contraction mapping with a contraction factor $t \in (0, 1)$ and fixed point p. Consider the state of*

(1.2) in M-iteration process and also the following state in this iteration process.

$$\begin{cases} v_n = \delta_n w_n + (1-\delta_n)Kw_n, \\ u_n = Kv_n, \\ w_{n+1} = Ku_n, \qquad n \in \mathbb{N}. \end{cases} \tag{3.1}$$

Consider the state of (1.1) in the Thakur et.al. iteration process and also the following states in this iteration process

$$\begin{cases} v_n = \xi_n w_n + (1-\xi_n)Kw_n, \\ u_n = K((1-\delta_n)w_n + \delta_n v_n), \\ w_{n+1} = Ku_n, \qquad n \in \mathbb{N}, \end{cases} \tag{3.2}$$

$$\begin{cases} v_n = \xi_n w_n + (1-\xi_n)Kw_n, \\ u_n = K(\delta_n w_n + (1-\delta_n)v_n), \\ w_{n+1} = Ku_n, \qquad n \in \mathbb{N}, \end{cases} \tag{3.3}$$

and

$$\begin{cases} v_n = (1-\xi_n)w_n + \xi_n Kw_n, \\ u_n = K(\delta_n w_n + (1-\delta_n)v_n), \\ w_{n+1} = Ku_n, \qquad n \in \mathbb{N}. \end{cases} \tag{3.4}$$

If $1-\delta_n < \delta_n$ and $1-\xi_n < \xi_n$ for all $n \in \mathbb{N}$, that's mean $\delta_n, \xi_n \in (\frac{1}{2}, 1)$, then the state of (1.2) in M-iteration process converges faster than (3.1) in the M-iteration process and also (1.1), (3.2), (3.3) and (3.4) in the Thakur et.al. iteration process. Also the state of (1.1) in the Thakur et.al. iteration process convergence faster than (3.1) in M-iteration process

Proof. Let's consider the iteration (1.2) in M-iteration process and we have

$$\begin{aligned} \|v_n - p\| &= \|(1-\delta_n)w_n + \delta_n Kw_n - p\| \\ &\leq (1-\delta_n)\|w_n - p\| + t\delta_n\|w_n - p\| \\ &= [1-(1-t)\delta_n]\|w_n - p\|, \end{aligned}$$

so that

$$\begin{aligned} \|u_n - p\| &= \|Kv_n - p\| \\ &\leq t[1-(1-t)\delta_n]\|w_n - p\|. \end{aligned} \tag{3.5}$$

Hence, from (3.5) we have

$$\begin{aligned} \|w_{n+1} - p\| &= \|Ku_n - p\| \\ &\leq t^2[1-(1-t)\delta_n]\|w_n - p\|. \end{aligned} \tag{3.6}$$

Since $\delta_n \in (\frac{1}{2}, 1)$, hence $-(1-t) < -(1-t)\delta_n < -\frac{1}{2}(1-t)$ thus

$$1-(1-t) < 1-(1-t)\delta_n \tag{3.7}$$

$$< 1 - \frac{1}{2}(1-t).$$

It follows from (3.6) and (3.7) that

$$\|w_{n+1} - p\| \leq t^{2n} \left[1 - \frac{1}{2}(1-t)\right]^n \|w_1 - p\|.$$

Let $a_n = t^{2n} \left[1 - \frac{1}{2}(1-t)\right]^n \|w_1 - p\|$. Now, let's consider the iteration (3.1) in M-iteration process and we have

$$\begin{aligned}\|v_n - p\| &= \|\delta_n w_n + (1-\delta_n) K w_n - p\| \\ &\leq \delta_n \|w_n - p\| + t(1-\delta_n)\|w_n - p\| \\ &= [t + (1-t)\delta_n]\|w_n - p\|,\end{aligned}$$

so that

$$\begin{aligned}\|u_n - p\| &= \|K v_n - p\| \\ &\leq t[t + (1-t)\delta_n]\|w_n - p\|.\end{aligned} \tag{3.8}$$

Hence, from (3.7) and (3.8), we have

$$\begin{aligned}\|w_{n+1} - p\| &= \|K u_n - p\| \\ &\leq t^2[t + (1-t)\delta_n]\|w_n - p\| \\ &\leq t^2[1 + (1-t)\delta_n]\|w_n - p\| \\ &= t^2[1 - (1-t)\delta_n + 2(1-t)\delta_n]\|w_n - p\| \\ &\leq t^2[1 - (1-t)\delta_n + 2(1-t)]\|w_n - p\| \\ &\leq t^2 \left[1 - \frac{1}{2}(1-t) + 2(1-t)\right] \|w_n - p\|.\end{aligned} \tag{3.9}$$

It follows from (3.9) that

$$\|w_{n+1} - p\| \leq t^{2n} \left[1 - \frac{1}{2}(1-t) + 2(1-t)\right]^n \|w_1 - p\|.$$

Let $b_n = t^{2n} \left[1 - \frac{1}{2}(1-t) + 2(1-t)\right]^n \|w_1 - p\|$. Then, we conclude that

$$\begin{aligned}\lim_{n\to\infty} \frac{a_n}{b_n} &= \lim_{n\to\infty} \frac{t^{2n} \left[1 - \frac{1}{2}(1-t)\right]^n \|w_1 - p\|}{t^{2n} \left[1 - \frac{1}{2}(1-t) + 2(1-t)\right]^n \|w_1 - p\|} \\ &= 0,\end{aligned}$$

so the iteration (1.2) convergence faster than the iteration (3.1). Now, let's consider the iteration (1.1) in Thakur et.al. iteration process and we have

$$\begin{aligned}\|v_n - p\| &= \|(1-\xi_n) w_n + \xi_n K w_n - p\| \\ &\leq (1-\xi_n)\|w_n - p\| + t\xi_n \|w_n - p\|\end{aligned}$$

$$= [1 - (1-t)\xi_n]\|w_n - p\|,$$

so that

$$\begin{aligned} \|u_n - p\| &= \|K((1-\delta_n)w_n + \delta_n v_n) - p\| \\ &\leq t(1-\delta_n)\|w_n - p\| + t\delta_n\|v_n - p\| \\ &\leq [t(1-\delta_n) + t\delta_n(1-(1-t)\xi_n)]\|w_n - p\| \\ &= t[1-(1-t)\delta_n\xi_n]\|w_n - p\|. \end{aligned} \tag{3.10}$$

Hence, from (3.10), we have

$$\begin{aligned} \|w_{n+1} - p\| &= \|Ku_n - p\| \\ &\leq t^2[1-(1-t)\delta_n\xi_n]\|w_n - p\|. \end{aligned}$$

Since $\delta_n, \xi_n \in (\frac{1}{2}, 1)$, hence $-(1-t) < -(1-t)\delta_n\xi_n < -\frac{1}{4}(1-t)$ thus

$$\begin{aligned} 1-(1-t) &< 1-(1-t)\delta_n\xi_n \\ &< 1 - \frac{1}{4}(1-t). \end{aligned} \tag{3.11}$$

It follows from (3.10) and (3.11) that

$$\|w_{n+1} - p\| \leq t^{2n}\left[1 - \frac{1}{4}(1-t)\right]^n \|w_1 - p\|$$

Let $c_n = t^{2n}\left[1 - \frac{1}{4}(1-t)\right]^n \|w_1 - p\|$. Then, we conclude that

$$\begin{aligned} \lim_{n\to\infty} \frac{a_n}{c_n} &= \lim_{n\to\infty} \frac{t^{2n}\left[1-\frac{1}{2}(1-t)\right]^n \|w_1 - p\|}{t^{2n}\left[1-\frac{1}{4}(1-t)\right]^n \|w_1 - p\|} \\ &= 0, \end{aligned}$$

so the iteration (1.2) converges faster than the iteration (1.1). Also, we conclude that iteration (1.1) in Thakur et.al. iteration process converges faster than the iteration (3.1) in M-iteration process, because

$$\begin{aligned} \lim_{n\to\infty} \frac{c_n}{b_n} &= \lim_{n\to\infty} \frac{t^{2n}\left[1-\frac{1}{4}(1-t)\right]^n \|w_1 - p\|}{t^{2n}\left[1-\frac{1}{2}(1-t) + 2(1-t)\right]^n \|w_1 - p\|} \\ &= 0. \end{aligned}$$

By applying the same argument, we can conclude that the iteration (1.2) in M-iteration process converges faster than the iterations (3.2), (3.3) and (3.4) in Thakur et.al. iteration process. □

Remark 3.2. Similar to proof of Theorem 3.1, it can easily be proven that the state (1.1) in the Thakur et.al. iteration process converges faster than the other states possible in this iteration process for contraction mappings. Also the state (3.1) in the M-iteration process converges faster than the states (3.2), (3.3) and (3.4) in the Thakur et.al. iteration process when $1 - \delta_n < \delta_n$ and $1 - \xi_n < \xi_n$ for all $n \in \mathbb{N}$.

Now, the question arises whether there is a faster iteration process than iteration processes (1.1) in the Thakur et.al. iteration and (1.2) in M-iteration? as an answer, we define the following iteration process.

$$\begin{cases} w_1 = w \in D \\ w_{n+1} = K^n((1-\delta_n)w_n + \delta_n K w_n), & n \in \mathbb{N}, \end{cases} \tag{3.12}$$

where $\delta_n \in (0,1)$ and $K^n = KoK^{n-1}$.

Theorem 3.3. *Let D and K be as in Theorem 3.1. Consider the state of (3.12) in new iteration process and also the following state in this iteration process.*

$$\begin{cases} w_1 = w \in D \\ w_{n+1} = K^n(\delta_n w_n + (1-\delta_n) K w_n), & n \in \mathbb{N}, \end{cases} \tag{3.13}$$

If $1-\delta_n < \delta_n$ for all $n \in \mathbb{N}$, that's mean $\delta_n \in (\frac{1}{2}, 1)$, then the state (3.12) of new iteration process converges faster than (3.13) in the new iteration process and (3.13) in the new iteration process is faster than (1.2) in the M-iteration process.

Proof. Let's consider the iteration (3.12) in the new iteration process, we have

$$\begin{aligned} \|w_{n+1} - p\| &= \|K^n((1-\delta_n)w_n + \delta_n K w_n) - p\| \\ &\le t^n \|(1-\delta_n)w_n + \delta_n K w_n - p\| \\ &\le t^n((1-\delta_n)\|w_n - p\| + \delta_n \|K w_n - p\|) \\ &\le t^n(1-(1-t)\delta_n)\|w_n - p\|. \end{aligned} \tag{3.14}$$

Since $\delta_n \in \left(\frac{1}{2}, 1\right)$, hence $-(1-t) < -(1-t)\delta_n < -\frac{1}{2}(1-t)$ thus

$$\begin{aligned} 1-(1-t) &< 1-(1-t)\delta_n \\ &< 1-\frac{1}{2}(1-t). \end{aligned} \tag{3.15}$$

It follows from (3.14) and (3.15) that

$$\begin{aligned} \|w_{n+1} - p\| &\le t^n \left[1 - \frac{1}{2}(1-t)\right] \|w_n - p\| \\ &\le t^n t^{n-1} \left[1 - \frac{1}{2}(1-t)\right]^2 \|w_{n-1} - p\| \\ &\vdots \\ &\le t^n t^{n-1} \cdots t^2 t^1 \left[1 - \frac{1}{2}(1-t)\right]^n \|w_1 - p\| \\ &= t^{\frac{n(n+1)}{2}} \left[1 - \frac{1}{2}(1-t)\right]^n \|w_1 - p\|. \end{aligned}$$

Let $d_n = t^{\frac{n(n+1)}{2}} \left[1 - \frac{1}{2}(1-t)\right]^n \|w_1 - p\|$. Now, let's consider the iteration (3.13) in the new iteration process and we have

$$\begin{aligned}
\|w_{n+1} - p\| &= \|K^n(\delta_n w_n + (1-\delta_n)Kw_n) - p\| \\
&\leq t^n \|\delta_n w_n + (1-\delta_n)Kw_n - p\| \\
&\leq t^n(\delta_n \|w_n - p\| + (1-\delta_n)\|Kw_n - p\|) \\
&\leq t^n(t + (1-t)\delta_n)\|w_n - p\| \\
&\leq t^n(1 + (1-t)\delta_n)\|w_n - p\| \\
&= t^n(1 - (1-t)\delta_n + 2(1-t)\delta_n)\|w_n - p\|
\end{aligned} \tag{3.16}$$

It follows from (3.15) and (3.16) that

$$\|w_{n+1} - p\| \leq t^{\frac{n(n+1)}{2}} \left[1 - \frac{1}{2}(1-t) + 2(1-t)\right]^n \|w_1 - p\|.$$

Let $f_n = t^{\frac{n(n+1)}{2}} \left[1 - \frac{1}{2}(1-t) + 2(1-t)\right]^n \|w_1 - p\|$. Then, we conclude that

$$\begin{aligned}
\lim_{n\to\infty} \frac{d_n}{f_n} &= \lim_{n\to\infty} \frac{t^{\frac{n(n+1)}{2}} \left[1 - \frac{1}{2}(1-t)\right]^n \|w_1 - p\|}{t^{\frac{n(n+1)}{2}} \left[1 - \frac{1}{2}(1-t) + 2(1-t)\right]^n \|w_1 - p\|} \\
&= 0.
\end{aligned}$$

So the iteration (3.12) in new iteration process converges faster than the iteration (3.13) in this iteration process.

By considering the iteration (3.13) in the new iteration process, we have

$$\begin{aligned}
\|w_{n+1} - p\| &= \|K^n(\delta_n w_n + (1-\delta_n)Kw_n) - p\| \\
&\leq t^n \|\delta_n w_n + (1-\delta_n)Kw_n - p\| \\
&\leq t^n(\delta_n \|w_n - p\| + (1-\delta_n)\|Kw_n - p\|) \\
&\leq t^n(t + (1-t)\delta_n)\|w_n - p\| \\
&\leq t^n \|w_n - p\|.
\end{aligned}$$

It follows that $\|w_{n+1} - p\| \leq t^{\frac{n(n+1)}{2}} \|w_1 - p\|$. Let $g_n = t^{\frac{n(n+1)}{2}} \|w_1 - p\|$. Then, we conclude that

$$\begin{aligned}
0 &\leq \frac{g_n}{a_n} \\
&= \frac{t^{\frac{n(n+1)}{2}} \|w_1 - p\|}{t^{2n} \left[1 - \frac{1}{2}(1-t)\right]^n \|w_1 - p\|} \\
&= \frac{t^{\frac{n^2-3n}{2}}}{\frac{t+1}{2}}
\end{aligned}$$

$$\leq \frac{t^{\frac{n^2-3n}{2}}}{t}$$
$$= t^{\frac{n^2-5n}{2}}.$$

It follows that $\lim\limits_{n\to\infty} \frac{g_n}{a_n} = 0$. Therefore, the iteration (3.13) in new iteration process converges faster than the iteration (1.2) in M-iteration process. □

Now, we present two numerical example to reveal the proof of these theorems.

Example 3.4. Let $D = [0, 4]$ and $K : D \to D$ be a mapping defined by $K(w) = \frac{3}{2+w}$ for any $w \in D$. Choose $\delta_n = \xi_n = 0.7$ and also the initial value $w_1 = 4$. It is obvious that $w^* = 1$ is fixed point of K and also K is a contraction mapping with constant contraction $\frac{3}{4}$. Table 1 shows the behavior all the iteration processes of (1.1), (1.2), (3.1), (3.2), (3.3), (3.4), (3.12) and (3.13) to fixed point of K in 8 iterations.

TABLE 1. Comparison of converges rate iteration processes to fixed point of T

Step	iteration (3.12)	iteration (3.13)	iteration (1.2)	iteration (1.1)	iteration (3.1)	iteration iteration (3.2)	iteration (3.3)	(3.4)
1	4.000000	4.000000	4.000000	4.000000	4.000000	4.000000	4.000000	4.000000
2	0.999779	0.999401	1.054455	1.111063	1.151163	1.167406	1.186848	1.167406
3	**1**	**1**	1.000428	1.004312	1.009956	1.013108	1.017660	1.013108
4	1	1	1.000003	1.000166	1.000663	1.001047	1.001721	1.001047
5	1	1	**1**	1.000006	1.000044	1.000084	1.000168	1.000084
6	1	1	1	**1**	1.000003	1.000007	1.000016	1.000007
7	1	1	1	1	**1**	1.000001	1.000002	1.000001
8	1	1	1	1	1	**1**	**1**	**1**

In Table 1, we see that the iterations (3.12) and (3.13) in new iteration process converges faster than the other iteration processes.

Example 3.5. Let $D = [0, 2]$ and $K : D \to D$ be a mapping defined by $K(w) = \frac{1}{3}w + \frac{2}{3}$ for any $w \in D$. Choose $\delta_n = \xi_n = \lambda_n = 1 - \frac{1}{(3n)^{\frac{1}{2}}}$ and also the initial value $w_1 = 2$. It is obvious that $w^* = 1$ is fixed point of K and also K is a contraction mapping with constant contraction $\frac{1}{3}$. Table 2 shows the convergence of all the iteration processes (2.2), (2.3), (2.4), (2.5), (2.6), (1.1), (1.2) and (3.12) to the fixed point of K.

Table 2. Comparison of converges rate

Step	Mann (2.2)	Ishikawa (2.3)	Noor (2.4)	Agarwal (2.5)	Abbas (2.6)	Thakur (1.1)	M (1.2)	New (3.12)
1	2.00000000	2.00000000	2.00000000	2.00000000	2.00000000	2.00000000	2.00000000	2.00000000
2	1.71823351	1.67853734	1.67294481	1.29363716	1.17959266	1.09787905	1.07980372	**1**
3	1.43488957	1.35805281	1.34477259	1.07502951	1.02384972	1.00833661	1.00536901	1
4	1.24160532	1.16355499	1.14992169	1.01759951	1.00271540	1.00065183	1.00033142	1
5	1.12703207	1.06760427	1.05797204	1.00388760	1.00027960	1.00004800	1.00001936	1
6	1.06421038	1.02590487	1.02046109	1.00082048	1.00002676	1.00000338	1.00000109	1
7	1.03149315	1.00934277	1.00670275	1.00016699	1.00000242	1.00000023	1.00000006	1
8	1.01507930	1.00320451	1.00206177	1.00003298	1.00000021	1.00000002	**1**	1
9	1.00707846	1.00105318	1.00060062	1.00000635	1.00000002	**1**	1	1
10	1.00326766	1.00033356	1.00016680	1.00000120	**1**	1	1	1
11	1.00148694	1.00010226	1.00004439	1.00000022	1	1	1	1
12	1.00066821	1.00003045	1.00001137	1.00000004	1	1	1	1
13	1.00029698	1.00000883	1.00000281	1.00000001	1	1	1	1
14	1.00013070	1.00000250	1.00000067	**1**	1	1	1	1
15	1.00005701	1.00000069	1.00000016	1	1	1	1	1
16	1.00002467	1.00000019	1.00000004	1	1	1	1	1
17	1.00001060	1.00000005	1.00000001	1	1	1	1	1
18	1.00000452	1.00000001	**1**	1	1	1	1	1
19	1.00000192	**1**	1	1	1	1	1	1
20	1.00000081	1	1	1	1	1	1	1
21	1.00000034	1	1	1	1	1	1	1
22	1.00000014	1	1	1	1	1	1	1
23	1.00000006	1	1	1	1	1	1	1
24	1.00000002	1	1	1	1	1	1	1
25	1.00000001	1	1	1	1	1	1	1
26	**1**	1	1	1	1	1	1	1

As shown in Figer 1, the speed of convergence of the new iteration process (3.12) is faster than the speed of convergence of other iteration processes.

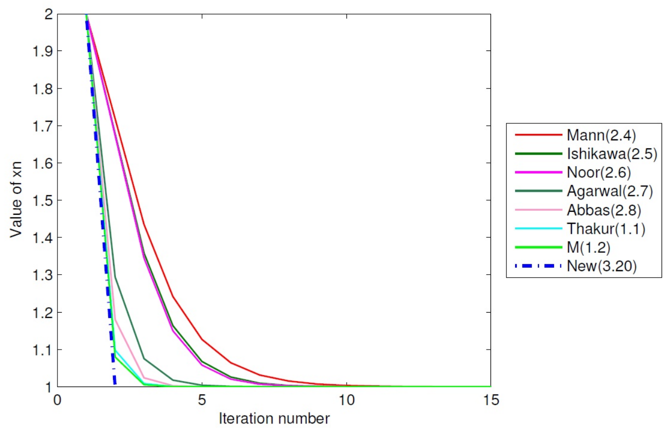

FIGURE 1. Convergence behavior of different iteration processes.

Open problem. Is it possible to develop an iteration process which rate of convergence is even faster than the iteration process (3.12)?

4. Convergence Theorems in Uniformly Convex Banach Spaces

In this section, we prove some weak and strong convergence theorems, using new iteration process, for generalized α-nonexpansive mappings in uniformly convex Banach spaces. We first have following lemma.

Lemma 4.1. *Let D be a nonempty closed convex subset of a Banach space E and $K : D \to D$ be a generalized α-nonexpansive mapping. Suppose that the sequence $\{w_n\}$ be generated by (3.12), then $\lim_{n\to\infty} \|w_n - p\|$ exists for all $p \in F(K)$.*

Proof. Let $p \in F(K)$, by using Proposition 2.3, we have

$$\begin{aligned}
\|w_{n+1} - p\| &= \|K^n((1-\delta_n)w_n + \delta_n K w_n) - p\| \\
&\leq \|(1-\delta_n)w_n + \delta_n K w_n - p\| \\
&\leq (1-\delta_n)\|w_n - p\| + \delta_n\|Kw_n - p\| \\
&\leq (1-\delta_n)\|w_n - p\| + \delta_n\|w_n - p\| \\
&= \|w_n - p\|.
\end{aligned}$$

This implies that $\{\|w_n - p\|\}$ is bounded and nonincreasing for all $p \in F(K)$. Hence $\lim_{n\to\infty} \|w_n - p\|$ exists. □

Theorem 4.2. *Let D be a nonempty closed convex subset of a uniformly convex Banach space E and let $K : D \to D$ be a generalized α-nonexpansive mapping. Suppose that the sequence $\{w_n\}$ be generated by (3.12). Then $F(K) \neq \emptyset$ if and only if $\{w_n\}$ is bounded and* $\lim_{n\to\infty} \|Kw_n - w_n\| = 0$.

Proof. Suppose $F(K) \neq \emptyset$ and let $p \in F(K)$. Then, by Lemma 4.1, $\lim_{n\to\infty} \|w_n - p\|$ exists and $\{w_n\}$ is bounded. Put

$$\lim_{n\to\infty} \|w_n - p\| = r. \tag{4.1}$$

So, by using Proposition 2.3, we have

$$\begin{aligned} \limsup_{n\to\infty} \|Kw_n - p\| &\leq \limsup_{n\to\infty} \|w_n - p\| \\ &= r. \end{aligned} \tag{4.2}$$

It follows that

$$\begin{aligned} r &= \lim_{n\to\infty} \|w_{n+1} - p\| \\ &= \lim_{n\to\infty} \|K^n((1-\delta_n)w_n + \delta_n Kw_n) - p\| \\ &\leq \lim_{n\to\infty} \|(1-\delta_n)w_n + \delta_n Kw_n - p\| \\ &= \lim_{n\to\infty} \|(1-\delta_n)(w_n - p) + \delta_n(Kw_n - p)\| \\ &\leq \lim_{n\to\infty} ((1-\delta_n)\|w_n - p\|) + \limsup_{n\to\infty}(\delta_n\|Kw_n - p\|) \\ &\leq \lim_{n\to\infty} ((1-\delta_n)\|w_n - p\|) + \limsup_{n\to\infty}(\delta_n\|w_n - p\|) \\ &= \lim_{n\to\infty} \|w_n - p\| \\ &= r. \end{aligned} \tag{4.3}$$

Hence, from (4.3), we have

$$r = \lim_{n\to\infty} \|(1-\delta_n)(w_n - p) + \delta_n(Kw_n - p)\|. \tag{4.4}$$

Now, from (4.1), (4.2), (4.4) and by using Lemma 2.7, we have that $\lim_{n\to\infty} \|Kw_n - w_n\| = 0$. Conversely, suppose that $\{w_n\}$ is bounded and $\lim_{n\to\infty} \|Kw_n - w_n\| = 0$. Let $p \in A(D, \{w_n\})$. By Proposition 2.4, we have

$$\begin{aligned} r\left(Kp, \{w_n\}\right) &= \lim_{n\to\infty} \sup \|w_n - Kp\| \\ &\leq \left(\frac{3+\alpha}{1-\alpha}\right) \lim_{n\to\infty} \sup \|Kw_n - w_n\| + \lim_{n\to\infty} \sup \|w_n - p\| \end{aligned}$$

$$= \lim_{n\to\infty} \sup \|w_n - p\|$$
$$= r(p, \{w_n\}).$$

Hence, we have $Kp \in A(D, \{w_n\})$. Since E is uniformly convex, $A(D, \{w_n\})$ consist of a unique member. Hence, we have $Kp = p$. □

By using Theorem 4.2, we have the following weak convergence theorem.

Theorem 4.3. *Let D, E, K and $\{w_n\}$ be as in Theorem 4.2. Suppose E has the Opial's property and $F(K) \neq \emptyset$. Then, $\{w_n\}$ converges weakly to a fixed point of K.*

Proof. By Theorem 4.2, the sequence $\{w_n\}$ is bounded and $\lim_{n\to\infty} \|Kw_n - w_n\| = 0$. Since E is uniformly convex, E is reflexive. So, there exists a subsequence $\{w_{n_j}\}$ of $\{w_n\}$ such that $\{w_{n_j}\}$ converges weakly to some $v_1 \in D$. By Proposition(2.5), $v_1 \in F(K)$. It is sufficient to show that $\{w_n\}$ converges weakly to v_1. In fact, if $\{w_n\}$ does not converges weakly to v_1. Then there exists a subsequence $\{w_{n_k}\}$ of $\{w_n\}$ and $v_2 \in D$ such that $w_{n_k} \rightharpoonup v_2$ and $v_1 \neq v_2$.

Again, by Proposition 2.5, $v_2 \in F(K)$. Since $\lim_{n\to\infty} \|w_n - p\|$ exists for all $p \in F(K)$. By the Opial property, we have

$$\begin{aligned}
\lim_{n\to\infty} \|w_n - v_1\| &= \lim_{j\to\infty} \|w_{n_j} - v_1\| \\
&< \lim_{j\to\infty} \|w_{n_j} - v_2\| \\
&= \lim_{n\to\infty} \|w_n - v_2\| \\
&= \lim_{k\to\infty} \|w_{n_k} - v_2\| \\
&< \lim_{k\to\infty} \|w_{n_k} - v_1\| \\
&= \lim_{n\to\infty} \|w_n - v_1\|.
\end{aligned}$$

This is a contradiction. So, we have $v_1 = v_2$. Thus $\{w_n\}$ converges weakly to $v_1 \in F(K)$. □

Now, we prove a strong convergence theorem for mappings stisfying condition (C).

Theorem 4.4. *Let D be a nonempty compact convex subset of a uniformly convex Banach space E and K be a self mapping on D satisfying in condition (C). If $\{w_n\}$ be generated by (3.12). Then the sequence $\{w_n\}$ converges strongly to a fixed point of K.*

Proof. By Theorem 2.6, $F(K) \neq \emptyset$. So by Theorem 4.2, $\{w_n\}$ is bounded and $\lim_{n\to\infty} \|Kw_n - w_n\| = 0$. Since D is compact, there exists a subsequence $\{w_{n_j}\}$ of $\{w_n\}$ such that $w_{n_j} \to v \in D$. Since the mapping satisfying in condition (C) is a generalized 0-nonexpansive mapping, then from Proposition 2.4, we have

$$\|w_{n_j} - Kv\| \leq 3\|Kw_{n_j} - w_{n_j}\| + \|w_{n_j} - v\|,$$

as $j \to \infty$, we conclude that $\{w_{n_j}\}$ converges to Kv, so $Kv = v$. Therefore, $w_n \to v$. □

Open problem. Is it possible that in general case, Theorem 4.4 is true for generalized α-nonexpansive mappings?

Now, by applying the special condition on the set of fixed points of generalized α-nonexpansive mappings and the sequence generated by (3.12), we obtain the following strong convergence theorem.

Theorem 4.5. *Let D, E, K and $\{w_n\}$ be as in Theorem 4.2. Suppose $F(K) \neq \emptyset$ and $\liminf_{n\to\infty} d(w_n, F(K)) = 0$ (where $d(w, F(K)) = \inf_{p\in F(K)} \|w - p\|$). Then, $\{w_n\}$ converges strongly to a fixed point of K.*

Proof. By Lemma 4.1, $\lim_{n\to\infty} \|w_n - p\|$ exists for all $p \in F(K)$. So, $\lim_{n\to\infty} d(w_n, F(K))$ exists, thus

$$\lim_{n\to\infty} d(w_n, F(K)) = 0.$$

So, there exist a subsequence $\{w_{n_j}\}$ of $\{w_n\}$ and $\{v_j\}$ in $F(K)$ such that $\|w_{n_j} - v_j\| \leq \frac{1}{2^j}$ for all $j \in \mathbb{N}$. By the proof of Lemma 4.1, $\{w_n\}$ is nonincreasing. So

$$\begin{aligned}\|w_{n_{j+1}} - v_j\| &\leq \|w_{n_j} - v_j\| \\ &\leq \frac{1}{2^j}.\end{aligned}$$

Hence

$$\begin{aligned}\|v_{j+1} - v_j\| &\leq \|v_{j+1} - w_{n_{j+1}}\| + \|w_{n_{j+1}} - v_j\| \\ &\leq \frac{1}{2^{j+1}} + \frac{1}{2^j} \\ &\leq \frac{1}{2^{j-1}} \to 0, \quad j \to \infty.\end{aligned}$$

So, we conclude that $\{v_j\}$ is a Cauchy sequence in $F(K)$ and so it converges to a point p. By Proposition 2.1, $F(K)$ is closed, then $p \in F(K)$. So, $\{w_{n_j}\}$ converges strongly to $p \in F(K)$ and since $\lim_{n\to\infty} \|w_n - p\|$ exists, hence $\{w_n\}$ converges strongly to $p \in F(K)$. □

Now, we give an example and examine the convergence behavior of sequences (2.2), (2.3), (2.4), (2.5), (2.6), (1.1), (1.2) and (3.12) for generalized α-nonexpansive mappings.

Example 4.6. Let the set $D = [0, \infty)$ be equipped with the usual norm $|.|$ and let $K : D \to D$ be defined as:

$$K(w) = \begin{cases} 0, & \text{if } w \in [0, \frac{1}{2}], \\ \frac{w}{2}, & \text{if } w \in (\frac{1}{2}, \infty). \end{cases}$$

Then, it is clear that K does not satisfy condition (C). In fact for $w = \frac{1}{2}$ and $u = \frac{8}{10}$ since $\frac{1}{2}|w - Kw| = \frac{1}{4} < |w - u| = \frac{3}{10}$ but $|Kw - Ku| = \frac{4}{10} > |w - u| = \frac{3}{10}$. Then K does not satisfy condition (C). While K is a generalized α-nonexpansive mapping. Actually for $\alpha = \frac{1}{3}$, If $w \in (\frac{1}{2}, \infty)$ and $u \in [0, \frac{1}{2}]$, we have

$$\begin{aligned} \frac{1}{3}|Kw - u| + \frac{1}{3}|w - Ku| + \frac{1}{3}|w - u| &= \frac{1}{3}\left|\frac{w}{2} - u\right| + \frac{1}{3}|w| + \frac{1}{3}|w - u| \\ &\geq \frac{1}{3}\left|\frac{w}{2}\right| + \frac{1}{3}|w| \\ &\geq \frac{1}{2}|w| \\ &= |Kw - Ku|. \end{aligned}$$

If $w, u \in (\frac{1}{2}, \infty)$, we have

$$\begin{aligned} \frac{1}{3}|Kw - u| + \frac{1}{3}|w - Ku| + \frac{1}{3}|w - u| &= \frac{1}{3}\left|\frac{w}{2} - u\right| + \frac{1}{3}\left|w - \frac{u}{2}\right| + \frac{1}{3}|w - u| \\ &\geq \frac{1}{3}\left|\frac{3}{2}w - \frac{3}{2}u\right| + \frac{1}{3}|w - u| \\ &= \frac{1}{2}|w - u| + \frac{1}{3}|w - u| \\ &\geq \frac{1}{2}|w - u| \\ &= |Kw - Ku|. \end{aligned}$$

Also, if $w, u \in [0, \frac{1}{2}]$, it is clear that

$$\begin{aligned} \frac{1}{3}|Kw - u| + \frac{1}{3}|w - Ku| + \frac{1}{3}|w - u| &\geq |Kw - Ku| \\ &= 0. \end{aligned}$$

So we showed that K is a generalized $\frac{1}{3}$-nonexpansive mapping. In this example, we make different choices of parameters δ_n, ξ_n and λ_n and set $\|w_n - w^*\| < 10^{-15}$ as our stoping criterion where $w^* = 0$ is fixed point of K. We obtained the influence of initial point for the new iteration (3.12) for $\delta_n = \frac{n}{(n+1)^{\frac{10}{9}}}$, $\xi_n = \frac{1}{(n+3)^{\frac{2}{3}}}$ and $\lambda_n = \frac{n}{(3n+1)}$.

TABLE 3. Influence of initial points for various iteration processes.

Number of iterations required to obtain fixed point.						
Initial points	Mann (2.2)	Ishikawa (2.3)	Noor (2.4)	Thakur (1.1)	M (1.2)	New (3.12)
2	34	34	33	2	2	**1**
8	38	37	37	3	3	**1**
44	43	41	41	4	4	**1**
262	48	45	45	5	5	**1**
1589	52	50	49	6	6	**1**
9707	58	55	54	8	7	**1**

Note: Items in bold show that the new iteration (3.12) converges faster than the others.

As shown in Table 4 and Figure 2, M (1.2), Thakur (1.1) and new (3.12) iterations are stable with respect to the choice of initial point and parameters. Also we can see that the average number of iterations of the new iteration process (3.12) is below as compare to the others iteration processes.

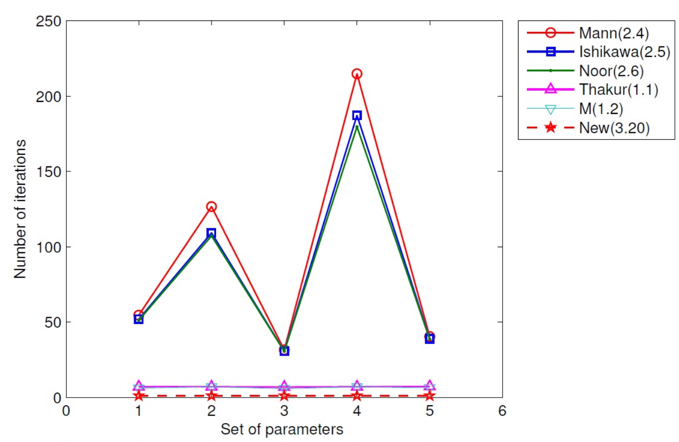

FIGURE 2. Average number of iterations under distinct parameters.

Table 4. Influence of parameters: comparison of various iteration processes.

Iterations	Initial points					
	10	10^2	10^3	10^4	10^5	10^6
For $\delta_n = \frac{n}{(n+1)^{\frac{10}{9}}}, \xi_n = \frac{1}{(n+3)^{\frac{2}{3}}}, \lambda_n = \frac{n}{(3n+1)}$						
Mann	39	45	51	58	64	70
Ishikawa	37	43	49	55	60	67
Noor	37	42	48	54	60	66
Thakur	3	5	6	8	9	11
M	3	4	5	7	8	9
New	**1**	**1**	**1**	**1**	**1**	**1**
for $\delta_n = \frac{n}{(n+7)^{\frac{17}{14}}}, \xi_n = \frac{n}{(n+2)}, \lambda_n = \frac{1}{(n)^{\frac{1}{3}}}$						
Mann	95	107	120	133	146	159
Ishikawa	89	97	105	113	121	130
Noor	88	96	103	111	119	126
Thakur	3	5	6	8	9	11
M	3	5	6	8	9	11
New	**1**	**1**	**1**	**1**	**1**	**1**
for $\delta_n = 1 - \frac{1}{\sqrt{5n+3}}, \xi_n = \frac{1}{n^3}, \lambda_n = 1 - \frac{1}{(2n+5)}$						
Mann	23	26	30	33	37	40
Ishikawa	22	25	29	32	36	40
Noor	22	25	29	32	36	39
Thakur	3	4	6	8	9	11
M	3	4	5	6	7	9
New	**1**	**1**	**1**	**1**	**1**	**1**
for $\delta_n = \frac{2n}{(9n+8)}, \xi_n = 1 - \frac{1}{(n+7)^2}, \lambda_n = 1 - \frac{6n}{(7n+3)^4}$						
Mann	166	185	205	224	244	265
Ishikawa	155	168	181	194	206	219
Noor	153	164	174	185	196	206
Thakur	3	5	6	8	9	11
M	3	5	6	8	9	11
New	**1**	**1**	**1**	**1**	**1**	**1**
for $\delta_n = \frac{n}{(n+5)}, \xi_n = \frac{n}{\sqrt{(36n^2+1)}}, \lambda_n = \sqrt{\frac{2n}{(4n+5)}}$						
Mann	32	35	39	42	45	49
Ishikawa	31	34	37	40	44	47
Noor	31	34	37	40	43	46
Thakur	3	5	6	8	10	11
M	3	4	6	7	8	10
New	**1**	**1**	**1**	**1**	**1**	**1**

5. Conclusions

New iteration process (3.12) is introduced for approximating fixed points of generalized α-nonexpansive mappings. It is shown that our new iteration process is moving faster than the leading iteration process (1.2) and iteration process (1.1). Strong and weak convergence of new iteration process to the fixed point of generalized α-nonexpansive mappings in the setting of uniformly convex Banach spaces are proved. Our new iteration process is now available for the engineers, computer scientists, physicists as well as mathematicians to solve different problems.

References

1. T. Suzuki, *Fixed point theorems and convergence theorems for some generalized nonexpansive mappings*, J. Math. Anal. Appl. 340 (2008), pp. 1088-1095.
2. K. Aoyama and F. Kohsaka, *Fixed point theorem for α-nonexpansive mappings in Banach spaces*, Nonlinear Anal. 74 (2011), pp. 4387-4391.
3. D. Ariza-Ruiz, C. Hermandez Linares, E. Llorens-Fuster and E. Moreno-Galvez, *On α-nonexpansive mappings in Banach spaces*, Carpathian. J. Math., 32 (2016), pp. 13-28.
4. R. Pant and R. Shukla, *Approximating fixed points of generalized α-nonexpansive mappings in Banach spaces*, Numer. Funct. Anal. Optim., 38 (2) (2017), pp. 248-266.
5. W.R. Mann, *Mean value methods in iteration*, Proc. Amer. Math. Soc., 4 (1953), pp. 506-510.
6. S. Ishikawa, *Fixed points by a new iteration method*, Proc. Amer. Math. Soc., 44 (1974), pp. 147-150.
7. M.A. Noor, *New approximation schemes for general variational inequalities*, J. Math. Anal. Appl., 251 (1) (2000), pp. 217-229.
8. R.P. Agarwal, D.O. Regan and D.R. Sahu, *Iterative construction of fixed points of nearly asymptotically nonexpansive mappings*, J. Nonlinear Convex Anal., 8 (1) (2007), pp. 61-79.
9. M. Abbas and T. Nazir, *A new faster iteration process applied to constrained minimization and feasibility problems*, Mat. Vesnik, 66 (2) (2014), pp. 223-234.
10. B.S. Thakur, D. Thakur and M. Postolache, *A new iterative scheme for numerical reckoning fixed points of Suzuki generalized nonexpansive mappings*, Appl. Math. Comput., 275 (2016), pp. 147-155.
11. K. Ullah and M. Arshad, *Numerical reckoning fixed points for Suzuki's generalized nonexpansive mappings via new iteration process*, Filomat, 32 (2018), pp. 187-196.

12. Z. Opial, *Weak convergence of the sequence of successive approximations for nonexpansive mappings*, Bull. Amer. Math. Soc., 73 (1967), pp. 591-597.
13. W. Takahashi, *Nonlinear Functional Analysis*, Yokohoma Publishers, Yokohoma, (2000).
14. R.P. Agarwal, D.O. Regan and D.R. Sahu, *Fixed point theory for Lipschitzian-type mappings with applications*, Series: Topological Fixed Point Theory and Its Applications, 6, Springer, New York, (2009).
15. J. Schu, *Weak and strong convergence to fixed points of asymptotically nonexpansive mappings*, Bull. Austral. Math. Soc., 43 (1991), pp. 153-15.
16. V. Berinde, *Picard iteration converges faster than Mann iteration for a class of quasi contractive operators*, Fixed Point Theory Appl., 2 (2004), pp. 97-105.
17. F. Shahin, G. Adrian, P. Mihai and R. Shahram, *A comparative study on the convergence rate of some iteration methods involving contractive mappings*, J. Fixed Point Theory Appl., (2015), 24 page.

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