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Abstract. The inherent feature of real-world data is uncertainty. If data is generated in valid experiments or standard collections, probability theory or fuzzy theory is a powerful tool for analyzing them. But data is not always reliable, especially when it is not possible to perform a reliable test or data collection multiple times. In this situations, referring to the beliefs of experts in the field in question is an alternative approach and uncertainty theory is a tool by which the beliefs of experts can be mathematically incorporated into the problem-solving structure. In this paper, we investigate the finding minimum weighted maximal matching with uncertain weights. For this purpose, we offer two methods. In the first method, by introducing the concept of chance constraint, we obtain model with definite coefficients. The second method is based on the concept of uncertain expected value. Finally, a numerical example for these two methods is presented.

1. Introduction

In real-world applications, when it comes to natural phenomena, we encounter a variety of uncertainties that motivate us to study them. Random events are the first case which is analyzed mathematically based on probability theory. Fuzzy theory is another fundamental type of uncertainty initiated by Zadeh and the credibility measure introduced by Liu and Liu [7] for the sake of measure a fuzzy event.

Despite the applicability of fuzzy and probability theories and numerous problems that have been modeled, it may not be possible to express

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and solve some problems using them. These types of issues usually appear when some gathered data is required. In such cases, the amount of information is small, or the available ones are not reliable or some subjects are not fuzzy in nature.

Under these circumstances, the Uncertainty theory seems useful. This theory founded by Liu [8] based on normality, monotonicity, self-duality, countable subadditivity, and product measure axioms. The fundamental concept in this theory is uncertain measure. By using it, useful mathematical tools have been developed to solve uncertainty problems.

In theory, certainty is one of the first presuppositions for the parameters of the optimization problems, while indeterminacy governs in practice. More precisely, uncertainty is seen in most optimization issues which are based on data collection. In such situations, experts opinions on the subject could be advantageous.

A matching in a graph $G = (V, E)$ is a subset M of edges no two of which have a vertex in common. Notice that if M is a matching of a graph G, then every subset $M' \subseteq M$ is also a matching. On the other hand, not every superset $M \subseteq M''$ is necessarily a matching. We will be interested in studying maximal matching in graphs, where a matching M is a maximal if no proper superset $M \subset M''$ be a matching. According to this definition, it is obvious that an edge in a matching, dominates itself and all edges adjacent to it. The problem of finding a maximal matching having minimum cardinality is said to be minimum maximal matching. In figure 1, the set of red edges shows different maximal matching for a graph.

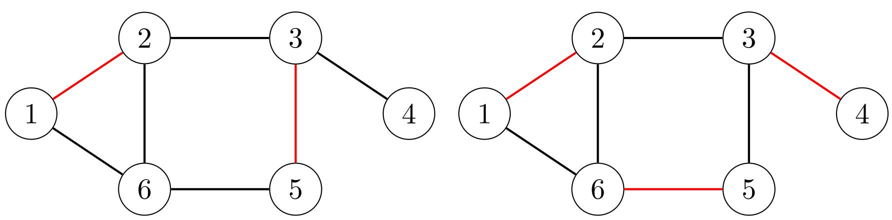

FIGURE 1. Different maximal matchings for a graph

If the edges have weights, then the problem is considered as minimization of the sum of the weights among all maximal matchings of graph G, and the related problem is called minimum weighted maximal matching (MWMM). The minimum maximal matching has been investigated by many researchers. Despite finding a maximum matching in a given graph is a polynomial problem and solves by Edmond's augmenting path algorithm [4], unfortunately, calculating the minimum maximal matching is an NP-hard problem even if G is bipartite or planar graphs with maximum degree 3 [1, 6], planar bipartite graphs and planar cubic graphs [3]. However, there are polynomial algorithms in special cases of graph G. For example, for block graphs [2], trees [10] and bipartite permutation graphs [5] a polynomial time algorithm has been designed. There are also approximation algorithms for obtaining minimum maximal matching for polynomial-expansion and low-density graphs [11].

The structure of paper goes as follows. In Sect. 2, essential notion of uncertainty theory is reviewed, and some properties and theorems are mentioned without proof. In Sect 3, the crisp linear integer program of MWMM is described and in Sect 4 the weights of MWMM model is considred uncertain with a specified distribution for each of them. Then two methods are offered. In the first method, by introducing the concept of chance constraint, we obtain an integer linear programming model with definite coefficients. The second method is based on the concept of uncertain expexted value. In Sect 5 a numerical example for these two methods is presented. Final section contains a review of the approach and some future research directions. To the best of our knowledge, this is the first study on minimum weighted maximal matching set with uncertain weights.

2. Notions from Uncertainty Theory

As already mentioned, uncertainty theory can be a potential tool for expressing experts' beliefs in mathematical language and using them. In this section, we point out some important concepts and features of this theory. For more details, refer the reader to [9].

Let Γ be a nonempty set and $\mathcal{L}$ be a σ-algebra over it . Then $(\Gamma, \mathcal{L})$ is called measurable space and each member $\Lambda \in \mathcal{L}$ is called a measurable set or an *event*. Measurable space $(\Gamma, \mathcal{L})$ with uncertain measure $\mathcal{M}$ (this concept will be introduced later) is saied uncertainty space and is shown by $(\Gamma, \mathcal{L}, \mathcal{M})$. A set function $\mathcal{M}$ over $\mathcal{L}$ is said to be an *uncertain measure* if it satisfies the following four axioms:

Axiom1 : (*Normality*) $\mathcal{M}\{\Gamma\}$=1 for the universal set Γ.
Axiom2 : (*Duality*) $\mathcal{M}\{\Lambda\}+\mathcal{M}\{\Lambda^c\}$=1 for any event Λ.
Axiom3 : (*Subadditivity*) For every countable sequence of events $\Lambda_1, \Lambda_2, \ldots$

$$\mathcal{M}\left\{\bigcup_{i=1}^{\infty}\Lambda_i\right\} \leq \sum_{i=1}^{\infty}\mathcal{M}\{\Lambda_i\}.$$

Axiom4 : (*Product*) Let $(\Gamma_k, \mathcal{L}_k, \mathcal{M}_k)$ be uncertainty spaces for integer $k \geq 1$. The product uncertain measure $\mathcal{M}$ is the one satisfying

$$\mathcal{M}\left\{\prod_{k=1}^{\infty} \Lambda_k\right\} = \bigwedge_{k=1}^{\infty} \mathcal{M}_k\{\Lambda_k\},$$

where Λ_k are arbitrarily chosen events from $\mathcal{L}_k$ for $k = 1, 2, \ldots$ respectively and $\bigwedge$ stands for the minimum operator.

The function $f : (\Gamma, \mathcal{L}, \mathcal{M}) \to \mathbb{R}$ is said to be measurable if for any Borel set B of real numbers, it holds $f^{-1}(B) = \{\gamma | f(\gamma) \in B\} \in \mathcal{L}$. An *uncertain variable* ξ is a measurable function on an uncertainty space. Also, ξ is considered nonegative if $\mathcal{M}\{\xi < 0\} = 0$ and positive if $\mathcal{M}\{\xi \leq 0\} = 0$. The next theorem talks about a fundamental and practical property in uncertainty theory.

Theorem 2.1 ([9]). *Let $\xi_1, \xi_2, \ldots, \xi_n$ be uncertain variables. Further, let f be a real valued measurable function. Then $f(\xi_1, \xi_2, \ldots, \xi_n)$ is an uncertain variable.*

In the following, the uncertain distribution of an uncertain variable is defined. Although the uncertain distribution does not provide a complete description of the uncertain variable, in many cases it is sufficient to know it instead of the variable itself. For an uncertain variable ξ, *uncertainty distribution* Φ is defined as $\Phi(x) = \mathcal{M}\{\xi \leq x\}$. Different type of uncertain variables have been defined in the literature corresponding to different uncertainty distributions. We only mention those which will be used.

Definition 2.2. An uncertain variable ξ is said to be *Linear* if its distribution has the following form

$$\Phi(x) = \begin{cases} 0, & x \leq a, \\ \dfrac{x-a}{b-a}, & a \leq x \leq b, \\ 1, & x \geq b, \end{cases}$$

where a and b are real numbers and $a < b$. The linear uncertain variable will be denoted by $\mathcal{L}(a, b)$.

For example, if a person thinks that the size of a piece is between 10 and 12 cm, then in his opinion, the size of that piece is a linear uncertainty variable $\mathcal{L}(10, 12)$.

Definition 2.3. The uncertain variable ξ is called *Zigzag* and shown by $\mathcal{Z}(a,b,c)$ when its distribution has the form

$$\Phi(x) = \begin{cases} 0, & x \le a, \\ \dfrac{x-a}{2(b-a)}, & a \le x \le b, \\ \dfrac{x+c-2b}{2(c-b)}, & b \le x \le c, \\ 1. & x \ge c. \end{cases}$$

Definition 2.4. An uncertain variable ξ is said to be *Normal* and denoted by $\mathcal{N}(e,\sigma)$, if it has a normal uncertainty distribution

$$\Phi(x) = \left(1 + \exp\left(\frac{\pi(e-x)}{\sqrt{3}\sigma}\right)\right)^{-1}, \quad x \in \mathbb{R},$$

where e and σ are real numbers with $\sigma > 0$.

When the uncertain parameters are appeared in optimization problems, it becomes more difficult to solve that problem. This difficulty is due to the fact that the optimization algorithms are designed for problems with definite and fixed parameters. So we have to find a way to represent these parameters as a number. In the following, several definitions and theorems for this purpose will be stated.

Definition 2.5. Regular uncertain distribution Φ is the one which is continuous and strictly increasing function with respect to x. As well, $0 < \Phi(x) < 1$, and

$$\lim_{x\to-\infty} \Phi(x) = 0, \qquad \lim_{x\to+\infty} \Phi(x) = 1.$$

For a regular uncertain distribution $\Phi(x)$, the inverse uncertain distribution is defined and denoted by $\Phi^{-1}(\alpha)$. Observe that the inverse uncertainty distribution is well defined on the open interval $(0,1)$. One can extend the domain to $[0,1]$ using

$$\Phi^{-1}(0) = \lim_{\alpha\downarrow 0} \Phi^{-1}(\alpha), \qquad \Phi^{-1}(1) = \lim_{\alpha\uparrow 1} \Phi^{-1}(\alpha).$$

It is easy to verify that the inverse uncertainty distribution of $\mathcal{L}(a,b)$ is $\Phi^{-1}(\alpha) = (1-\alpha)a + \alpha b$, and the inverse distribution of the Zigzag variable $\mathcal{Z}(a,b,c)$ is

$$\Phi^{-1}(\alpha) = \begin{cases} (1-2\alpha)a + 2\alpha b, & \alpha < 0.5, \\ (2-2\alpha)b + (2\alpha-1)c, & \alpha \ge 0.5. \end{cases}$$

Moreover, the inverse uncertainty distributions of Normal uncertain variable is $\Phi^{-1}(\alpha) = e + \frac{\sigma\sqrt{3}}{\pi} \ln \frac{\alpha}{1-\alpha}$.

Definition 2.6. The uncertain variables $\xi_1, \xi_2, \ldots, \xi_n$ are said to be independent if

$$\mathcal{M}\left\{\bigcap_{i=1}^{n}(\xi_i \in B_i)\right\} = \bigwedge_{i=1}^{n} \mathcal{M}\{\xi_i \in B_i\},$$

for any Borel sets $B_1, B_2, \ldots, B_n$.

Recall that for uncertain variables $\xi_1, \xi_2, \ldots, \xi_n$, $\xi = f(\xi_1, \xi_2, \ldots, \xi_n)$ is also an uncertain variable. However, it is not straightforward to calculate the uncertainty distribution Ψ for arbitrary uncertain variables ξ. Next theorem presents practical way to calculate the inverse uncertainty distribution Ψ^{-1} when uncertain variables $\xi_i, i = 1, \ldots, n$ are independent and regular, and the function f is strictly increasing.

Theorem 2.7 ([9])**.** *Let $\xi_1, \xi_2, \ldots, \xi_n$ be independent uncertain variables with regular uncertainty distributions $\Phi_1, \Phi_2, \ldots, \Phi_n$, respectively. If f is a strictly increasing function, then the uncertain variable $\xi = f(\xi_1, \xi_2, \ldots, \xi_n)$ has the inverse uncertainty distribution*

$$\Psi^{-1}(\alpha) = f\left(\Phi_1^{-1}(\alpha), \Phi_2^{-1}(\alpha), \ldots, \Phi_n^{-1}(\alpha)\right).$$

Corollary 2.8. *With hypothesis of Theorem 2.7, the uncertain variable $\xi = \xi_1 + \xi_2 + \cdots + \xi_n$ has inverse distribution*

$$\Psi^{-1}(\alpha) = \Phi_1^{-1}(\alpha) + \Phi_2^{-1}(\alpha) + \cdots + \Phi_n^{-1}(\alpha).$$

Corollary 2.9. *Suppose that $\xi_1, \xi_2, \ldots, \xi_n$ are independent and positive uncertain variables with regular uncertainty distributions $\Phi_1, \Phi_2, \cdots, \Phi_n$, respectively. The inverse uncertainty distribution of $\xi = \xi_1 \times \xi_2 \times \ldots \times \xi_n$ is*

$$\Psi^{-1}(\alpha) = \Phi_1^{-1}(\alpha) \times \Phi_2^{-1}(\alpha) \times \cdots \times \Phi_n^{-1}(\alpha).$$

In some cases, validity of an equality is not determined and α-chance model can be a useful interpretation for such situations. It is said that an equality $g(x, \xi) \leq 0$ holds with the belief degree α when $\mathcal{M}\{g(x, \xi) \leq 0\} \geq \alpha$. Determining the feasible region associated to such constraints in higher dimensional spaces is not straightforward. Next theorem presents an equivalent crisp constraint in specific circumstances.

Theorem 2.10 ([9])**.** *Let $g(x, \xi_1, \xi_2, \ldots, \xi_n)$ be a strictly increasing function with respect to $\xi_1, \ldots, \xi_k$, and strictly decreasing with respect to $\xi_{k+1}, \ldots, \xi_n$. Further, let $\xi_1, \ldots, \xi_n$ be independent uncertain variables with uncertainty distributions $\Phi_1, \ldots, \Phi_n$, respectively. Then the relation $\mathcal{M}\{g(x, \xi_1, \xi_2, \ldots, \xi_n) \leq 0\} \geq \alpha$ holds if and only if*

$$g\left(x, \Phi_1^{-1}(\alpha), \ldots, \Phi_k^{-1}(\alpha), \Phi_{k+1}^{-1}(1-\alpha), \ldots, \Phi_n^{-1}(1-\alpha)\right) \leq 0.$$

One of the important concepts in uncertainty theory is the expected value of an uncertain variable. In most situations, working with the expected value is much easier than working with the uncertain variables themselves despite of different interpretation. The *expected value* of an uncertain variable ξ is defined by

$$\mathrm{E}[\xi] = \int_0^{+\infty} \mathcal{M}\{\xi \geq x\}dx - \int_{-\infty}^0 \mathcal{M}\{\xi \leq x\}dx.$$

Assume that the objective function of an uncertain programming model is in the form $f(x, \xi)$, where x and ξ are decision and uncertain vectors, respectively. Recall that there is no direct process to optimize a mathematical program with such objective function. In this case, dealing with the expected value $E[f(x, \xi)]$ instead of $f(x, \xi)$ is reasonable. Determining the value of $E[f(x, \xi)]$ is easy in special situations. Next theorem speaks of such conditions.

Theorem 2.11 ([9]). *Let f satisfy the hypothesis of Theorem 2.10, the expected objective function $E_U[f(x, \xi_1, \xi_2, \ldots, \xi_n)]$ equals to*

$$\int_0^1 f(x, \phi_1^{-1}(\alpha), \phi_2^{-1}(\alpha), \ldots, \phi_k^{-1}(\alpha), \phi_{k+1}^{-1}(1-\alpha), \ldots, \phi_n^{-1}(1-\alpha))d\alpha.$$

3. Problem Description and Formulation

In this section, we first explain the integer linear programming model for the MWMM problem. Let $G = (V, E)$ be an undirected graph with the sets of vertices and edges $V = \{1, 2, \ldots, n\}$ and E respectively. To formulate the problem, first, for each edge $\{i, j\} \in E$ we define the variable x_{ij} according to the membership or non-membership in an arbitary set $S \subseteq E$ as follows:

$$x_{ij} = \begin{cases} 1, & \{i,j\} \in S, \\ 0. & \{i,j\} \notin S. \end{cases}$$

According to the definition of the maximal matching of a graph G, for each $i \in V$ only one of the edges connected to the vertex i can be in that set. The, we have

$$\sum_{j \in N(i)} x_{ij} \leq 1, \quad \forall i \in V. \tag{3.1}$$

Hence, these inequalities guarantee that the solution is a matching set. In addition, to ensure that an arbitrary subset of edges be a maximal matching set, the following inequalities must be hold.

$$\sum_{\substack{k \in N(i) \\ k \neq j}} x_{ik} + \sum_{\substack{k \in N(j) \\ k \neq i}} x_{kj} + x_{ij} \geq 1, \quad \forall \{i,j\} \in E. \tag{3.2}$$

Morever, if each edge $\{i,j\}$ has a weight w_{ij}, it is clear that $\sum_{\{i,j\}\in E} w_{ij}x_{ij}$ shows the wheghit of matching set. Now, using (3.1) and (3.2) an integer programming model is written as follows:

$$\min \sum_{\{i,j\}\in E} w_{ij}x_{ij} \tag{3.3}$$

$$\text{s.t.} \sum_{j\in N(i)} x_{ij} \leq 1, \quad \forall i \in V,$$

$$\sum_{\substack{k\in N(i)\\ k\neq j}} x_{ik} + \sum_{\substack{k\in N(j)\\ k\neq i}} x_{kj} + x_{ij} \geq 1, \quad \forall \{i,j\} \in E,$$

$$x_{ij} \in \{0,1\}, \quad \forall \{i,j\} \in E,$$

where $N(i) = \{j \in V | \{i,j\} \in E\}$ is the open neighborhood of the vertex i.

4. The Minimum Maximal Matching Problem with Uncertain Weights

In the sequel, it is assumed that the weight of edge $\{i,j\}$ is an uncertain variable ξ_{ij}, and analogous problem is referred to as uncertain minimum weighted maximal matching problem (UMWMM). The following model is a resemblance of problem (3.3) in indeterministic environment

$$\min \sum_{\{i,j\}\in E} \xi_{ij}x_{ij} \tag{4.1}$$

$$\text{s.t.} \sum_{j\in N(i)} x_{ij} \leq 1, \quad \forall i \in V,$$

$$\sum_{\substack{k\in N(i)\\ k\neq j}} x_{ik} + \sum_{\substack{k\in N(j)\\ k\neq i}} x_{kj} + x_{ij} \geq 1, \quad \forall \{i,j\} \in E,$$

$$x_{ij} \in \{0,1\}, \quad \forall \{i,j\} \in E.$$

The problem (4.1) is not solvable due to the uncertain form of the objective function. To overcome this challenge, two methods will be suggested below. In the first case, we use the trick of introducing an auxiliary variable that acts as the upper bound for the objective function. Then, using the concept of chance constraint, we present a determinate model. In another method, using the concept of uncertain expected value, we will obtain a model with certain coefficients.

4.1. **Uncertain α-chance Model.** In this method, we use the concept of chance constraint and create a model with definite coefficients.

Theorem 4.1. *By using the concept of chance constraint, an equivalet integer linear programming model with definite weights for model* (4.1) *is obtained.*

Proof. We first define the auxiliary variable t as an upper bound for the objective function in(4.1). Therefore, by minimizing the value of t, the value of the objective function of problem (4.1) will also decrease.

$$\begin{aligned}
&\min t \\
&\text{s.t.} \sum_{\{i,j\}\in E} \xi_{ij} x_{ij} \leq t \\
&\sum_{j\in N(i)} x_{ij} \leq 1, \quad \forall i \in V, \\
&\sum_{\substack{k\in N(i)\\ k\neq j}} x_{ik} + \sum_{\substack{k\in N(j)\\ k\neq i}} x_{kj} + x_{ij} \geq 1, \quad \forall \{i,j\} \in E, \\
&x_{ij} \in \{0,1\}, \quad \forall \{i,j\} \in E.
\end{aligned} \tag{4.2}$$

The first constraint in (4.2) does not describe a Specified region. Hence, the indeterminate constraint is then substituted by the α-chance one where $\alpha \in (0,1)$ is a prespecified confidence level. Thus, problem (4.2) converts to

$$\begin{aligned}
&\min t \\
&\text{s.t. } \mathcal{M}\{\sum_{\{i,j\}\in E} \xi_{ij} x_{ij} \leq t\} \geq \alpha
\end{aligned} \tag{4.3}$$

$$\sum_{j\in N(i)} x_{ij} \le 1, \quad \forall i \in V,$$

$$\sum_{\substack{k\in N(i)\\ k\neq j}} x_{ik} + \sum_{\substack{k\in N(j)\\ k\neq i}} x_{kj} + x_{ij} \ge 1, \quad \forall \{i,j\} \in E,$$

$$x_{ij} \in \{0,1\}, \quad \forall \{i,j\} \in E.$$

The first constraint in (4.3) reads as the believe degree of the domain expert which means that in their opinion, the possibility of imposing that constraint is at least as great as alpha. Because, ξ_i's is considered independent positive uncertain variables then $\sum_{\{i,j\}\in E} \xi_{ij}x_{ij} - t$ is increasing function. So based on theorem 2.10 we can replace first constraint in (4.3) with $\sum_{\{i,j\}\in E} \Phi_{ij}^{-1}(\alpha)x_{ij} - t \le 0$. Finally, instead of t we minimize $\sum_{\{i,j\}\in E} \Phi_{ij}^{-1}(\alpha)x_{ij}$. Therefore, the final model which its coefficients are real numbers is obtained as follows:

$$\tag{4.4}\begin{aligned}
\min \quad & \sum_{\{i,j\}\in E} \Phi_{ij}^{-1}(\alpha)x_{ij} \\
\text{s.t.} \quad & \sum_{j\in N(i)} x_{ij} \le 1 && \forall i \in V, \\
& \sum_{\substack{k\in N(i)\\ k\neq j}} x_{ik} + \sum_{\substack{k\in N(j)\\ k\neq i}} x_{kj} + x_{ij} \ge 1 && \forall \{i,j\} \in E, \\
& x_{ij} \in \{0,1\}, && \forall \{i,j\} \in E.
\end{aligned}$$

□

Model (4.4) is an integer linear programming problem and, it can be solved, although it's hard and time-consuming in general because of its NP-completeness. In the sequel, using the concept of uncertain expected value, another integer linear programming will be extracted.

4.2. **Uncertain Expected Value Model.**

Theorem 4.2. *By using the concept of uncertain expected value, an equivalet integer linear programming model with definite weights for model* (4.1) *is obtained.*

Proof. We use the uncertain expected value instead of the objective function of (4.1). Then, we have:

$$\min \mathrm{E}\left[\sum_{\{i,j\}\in E} \xi_{ij} x_{ij}\right] \tag{4.5}$$

$$\text{s.t.} \sum_{j\in N(i)} x_{ij} \le 1, \quad \forall i \in V,$$

$$\sum_{\substack{k\in N(i)\\ k\neq j}} x_{ik} + \sum_{\substack{k\in N(j)\\ k\neq i}} x_{kj} + x_{ij} \ge 1, \quad \forall \{i,j\} \in E,$$

$$x_{ij} \in \{0,1\}, \quad \forall \{i,j\} \in E.$$

It can be seen that $\sum_{\{i,j\}\in E} \xi_{ij} x_{ij}$ is strictly increasing with respect to each of the ξ_i's. Therefore, according to the theorem 2.11 the expected value of $\sum_{\{i,j\}\in E} \xi_{ij} x_{ij}$ can be calculated as follows:

$$\begin{aligned} \mathrm{E}\left[\sum_{\{i,j\}\in E} \xi_{ij} x_{ij}\right] &= \int_0^1 \sum_{\{i,j\}\in E} \Phi_{ij}^{-1}(\alpha) x_{ij}\, d\alpha \\ &= \sum_{\{i,j\}\in E} x_{ij} \int_0^1 \Phi_{ij}^{-1}(\alpha) d\alpha. \end{aligned} \tag{4.6}$$

Now, using (4.5) and (4.6) the next integer linear programming model is obtained.

$$\min \sum_{\{i,j\}\in E} x_{ij} \int_0^1 \Phi_{ij}^{-1}(\alpha) d\alpha. \tag{4.7}$$

$$\texttt{s.t.} \sum_{j\in N(i)} x_{ij} \le 1, \quad \forall i \in V,$$

$$\sum_{\substack{k\in N(i)\\ k\neq j}} x_{ik} + \sum_{\substack{k\in N(j)\\ k\neq i}} x_{kj} + x_{ij} \ge 1, \quad \forall \{i,j\} \in E,$$

$$x_{ij} \in \{0,1\}, \quad \forall \{i,j\} \in E.$$

□

It is clear that (4.7) is a model with determinate coefficients and can be solved by existing methods. In the next section, these two methods will be used to solve a numerical example.

5. Illustrative Example

Consider the graph presented in figure 2, with uncertain weights on vertices as $\xi_{12} = \mathcal{Z}(1,6,7)$, $\xi_{23} = \mathcal{L}(3,8)$, $\xi_{34} = \mathcal{N}(2,6)$, $\xi_{35} = \mathcal{Z}(4,6,9)$, $\xi_{56} = \mathcal{L}(5,10)$, $\xi_{16} = \mathcal{N}(8,12)$, and $\xi_{26} = \mathcal{Z}(2,3,9)$. Corresponding to $\alpha = 0.3, 0.5, 0.7, 0.9$ and 0.99, inverse uncertainty distributions are listed in Table 1.

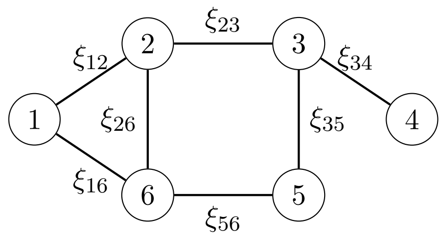

Figure 2. Different maximal matchings for a graph

Table 1. $\phi^{-1}(\alpha)$ for $\alpha = 0.3, 0.5, 0.7, 0.9$ and 0.99.

ξ_{ij}	ξ_{12}	ξ_{23}	ξ_{34}	ξ_{35}	ξ_{56}	ξ_{16}	ξ_{26}
$\phi_i^{-1}(0.3)$	4.0000	4.5000	-0.8028	5.2000	6.5000	2.3943	2.6000
$\phi_i^{-1}(0.5)$	6.0000	5.5000	2.0000	6.0000	7.5000	8.0000	3.0000
$\phi_i^{-1}(0.7)$	6.4000	6.5000	4.8028	7.2000	8.5000	13.6057	5.4000
$\phi_i^{-1}(0.9)$	6.8000	7.5000	9.2648	8.4000	9.5000	22.5367	7.8000
$\phi_i^{-1}(0.99)$	6.9800	7.9500	17.2005	8.9400	9.9500	38.4011	8.8800

Based on these data, the optimal solutions and the objective values of the corresponding problem (4.4) for different values of α are depicted in Table 2

In the uncertain expected value model (4.7), it is important to pay attention that $\int_0^1 \Phi_{ij}^{-1}(\alpha)d\alpha$ is a number and is not depending on the value of α. In Table 3, $\int_0^1 \Phi_{ij}^{-1}(\alpha)d\alpha$ for each ξ_{ij} is obtained.

The model (4.7) is solved by using data in Table 3 and the solution $(0,0,1,0,0,0,1)$ is computed which equivalent with the edges $\{3,4\}, \{2,6\}$.

TABLE 2. $\phi^{-1}(\alpha)$ for $\alpha = 0.3, 0.5, 0.7, 0.9$ and 0.99.

α	Optimal Solution	UMWMM	Objective Value
0.3	$(0,0,1,0,0,1,0)$	{3, 4}, {1, 6}	24.3915
0.5	$(0,0,1,0,0,0,1)$	{3, 4}, {2, 6}	38.0000
0.7	$(0,0,1,0,0,0,1)$	{3, 4}, {2, 6}	52.4085
0.9	$(1,0,0,1,0,0,0)$	{1, 2}, {3, 5}	71.8051
0.99	$(1,0,0,1,0,0,0)$	{1, 2}, {3, 5}	98.3016

TABLE 3. $\phi^{-1}(\alpha)$ for $\alpha = 0.3, 0.5, 0.7, 0.9$ and 0.99.

ξ_{ij}	ξ_{12}	ξ_{23}	ξ_{34}	ξ_{35}	ξ_{56}	ξ_{16}	ξ_{26}
$\int_0^1 \Phi_{ij}^{-1}(\alpha)d\alpha$	5.0000	5.5000	2.0000	6.2500	7.5000	8.0000	4.2500

6. CONCLUSIONS

In this paper, we considered the minimum weighted maximal matching problem with indeterminate weights on edges. The concepts of chance constriant and uncertain expected value is utilitized on linear integer programming model of uncertain minimum weighted maximal matching problem and two model with definite cofficients are exracred. There are many different versions of matching sets, and uncertainty may exists on their data as well. Analogous approach can be considered for them when they possess uncertainty theory implications. We acknowledge that there is no real case example available with us to clarify the efficiency of the models and we examined the two obtained integer linear programming models on simple graph.

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