Study on Some Integral Inequalities for Pseudo-Integrals

Bayaz Daraby

Sahand Communications in Mathematical Analysis

Print ISSN: 2322-5807
Online ISSN: 2423-3900
Volume: 19
Number: 1
Pages: 139-159

Sahand Commun. Math. Anal.
DOI: 10.22130/scma.2022.557671.1148

Sahand Communications in Mathematical Analysis (SCMA) Vol. 19 No. 1 (2022), 139-159
http://scma.maragheh.ac.ir
DOI: 10.22130/scma.2022.557671.1148

Study on Some Integral Inequalities for Pseudo-Integrals

Bayaz Daraby

Abstract. In this paper, we express and prove Stolarsky, Feng Qi and Markov type inequalities for two classes of pseudo-integrals. One of them concerning the pseudo-integrals based on a function reduces on the g-integral where pseudo-operations are defined by a monotone and continuous function g. The other one concerns the pseudo-integrals based on a semiring $([a, b], \max, \odot)$, where $\odot$ is generated. The integral inequalities are appling in multivariate approximation theory and probability theory and etc.

1. Introduction

Pseudo-analysis is a generalization of the classical analysis, where instead of the field of real numbers, a semiring is taken on a real interval $[a, b] \subset [-\infty, \infty]$ endowed with pseudo-addition $\bigoplus$ and with pseudo-multiplication $\odot$ (see [10, 13–17, 20, 37, 42, 43, 47, 57]). Based on the given structure, the concepts of $\oplus$-measure (pseudo-additive measure), pseudo-integral, pseudo-convolution, pseudo-Laplace transform and etc. developed. Pseudo-analysis would be an interesting topic to generalize an inequality from the framework of the classical analysis to that of some integrals which contain the classical analysis as special cases [3, 48].

The integral inequalities are good mathematical tools both in theory and application. Different integral inequalities including Chebyshev, Jensen, Holder and Minkowski inequalities are widely used in various fields of mathematics, such as probability theory, differential equations, decision-making under risk and information sciences.

The study of inequalities for Sugeno integral was initiated by Román-Flores et al. [26, 50, 51, 53–55] and then followed by the authors [1, 2, 5,

2020 *Mathematics Subject Classification.* 03E72, 28E10, 26E50.
Key words and phrases. Carlson-type inequality, Sugeno integral, Fuzzy measure, Comonotone functions, Fuzzy integral inequality.
Received: 25 December 2021, Accepted: 27 January 2022.

8, 9, 12, 18, 21, 22, 28, 30, 31, 38, 40, 52, 59]. The classical Stolarsky's integral inequality holds:

$$\int_0^1 f\left(x^{\frac{1}{a+b}}\right) dx \geq \left(\int_0^1 f\left(x^{\frac{1}{a}}\right) dx\right)\left(\int_0^1 f(x^{\frac{1}{b}})dx\right), \tag{1.1}$$

where $a, b > 0$ and $f : [0,1] \to [0,1]$ is a non-increasing function [56]. A. Flores-Franulič et al. have proved a fuzzy integral inequality of Stolarsky type inequality [27].

In this paper, firstly we generalize the above mentioned inequality and some version of Feng Qi type inequalities for pseudo-integrals of monotone functions. We think that our results will be useful for those areas in which the classical Feng Qi inequality plays a role whenever the environment is non-deterministic. In [49], Feng Qi studied a very interesting integral inequality and proved the following result:

Theorem 1.1. *Let n be a positive integer. Suppose $f(x)$ has continuous derivative of the n-th order on the interval $[a, b]$ such that $f^{(i)}(a) \geq 0$, for $0 \leq i \leq n-1$, and $f^{(n)}(x) \geq n!$, then*

$$\int_a^b [f(x)]^{(n+2)} dx \geq \left(\int_a^b f(x)dx\right)^{(n+1)}. \tag{1.2}$$

In [4] the Feng Qi type inequality for Sugeno integral is presented with several examples given to illustrate the validity of this inequalities.

Theorem 1.2. *Let μ be the Lebesgue measure on R and let $f : [0,1] \to [0,\infty)$ be a real valued function such that $(S)\int_0^1 f d\mu = p$. If f is a continuous and strictly decreasing function, such that $f\left(p^{n+1}\right) \geq p^{\left(\frac{n+1}{n+2}\right)}$, then the inequality:*

$$(S)\int_0^1 f^{n+2} d\mu \geq \left((S)\int_0^1 f d\mu\right)^{n+1} \tag{1.3}$$

holds for all $n \geq 0$.

Theorem 1.3. *Let μ be the Lebesgue measure on R and let $f : [0,1] \to [0,\infty)$ be a real valued function such that $(S)\int_0^1 f d\mu = p$. If f is a continuous and strictly increasing function, such that $f\left(1 - p^{n+1}\right) \geq p^{\left(\frac{n+1}{n+2}\right)}$, then the inequality:*

$$(S)\int_0^1 f^{n+2} d\mu \geq \left((S)\int_0^1 f d\mu\right)^{n+1} \tag{1.4}$$

holds for all $n \geq 0$.

In the continue of this paper, we study on Markov's inequality. The Markov type inequality is a part of the classical mathematical analysis such as the following form:

$$\mu\{x \in A : f(x) \geq c\} \leq \frac{1}{c} \int_A f d\mu, \tag{1.5}$$

where f is a non-negative integrable function and $c > 0$. A. Flores-Franulič et al. have proved Markov type inequalities for fuzzy integrals in [30] as following:

Let $\mu : \sum \to [0, \infty]$ be a fuzzy measure and $c > 0$. Then

$$\fint_A f d\mu \geqq c \wedge \mu\{x \in A : f(x) \geq c\}$$

for all $A \in \sum$. Markov type inequality is applied in multivariate approximation theory, structural engineering and probability theory. In probability theory it gives an upper bound for the probability where a non-negative function of a random variable is greater than or equal to some positive constants.

2. Preliminaries

In this section, it is going to be reviewed some well-known results of pseudo-operations, pseudo-analyses and pseudo-integrals in details, we refer to [11, 23, 24, 34, 37, 43–46, 57, 58].

Let $[a, b]$ be a closed (in some cases can be considered semiclosed) subinterval of $[-\infty, \infty]$. The full order on $[a, b]$ will be denoted by $\preceq$.

Definition 2.1. The operation $\oplus$ (pseudo-addition) is a function $\oplus : [a, b] \times [a, b] \to [a, b]$ which is commutative, nondecreasing (with respect to $\preceq$), associative and with a zero (natural) element denoted by $\mathbf{0}$, i.e., for each $x \in [a, b], \mathbf{0} \oplus x = x$ holds (usually $\mathbf{0}$ is either a or b).

Let $[a, b]_+ = \{x | x \in [a, b], \mathbf{0} \preceq x\}$.

Definition 2.2. The operation $\odot$ (pseudo-multiplication) is a function $\odot : [a, b] \times [a, b] \to [a, b]$ which is commutative, positively non-decreasing, i.e., $x \preceq y$ implies $x \odot z \preceq y \odot z$ for all $z \in [a, b]_+$, associative and for which there exists a unit element $\mathbf{1} \in [a, b]$, i.e., for each $x \in [a, b], \mathbf{1} \odot x = x$.

We also assume $\mathbf{0} \odot x = \mathbf{0}$ that $\odot$ is a distributive pseudo-multiplication with respect to $\oplus$, i.e., $x \odot (y \oplus z) = (x \odot y) \oplus (x \odot z)$. The structure $([a, b], \oplus, \odot)$ is a semiring (see [34, 46]). We will consider the semiring $([a, b], \oplus, \odot)$ for two important (with completely different behavior) cases.

The first case is when pseudo-operations are generated by a monotone and continuous function $g : [a, b] \to [0, \infty]$, i.e., pseudo-operations are given with:

$x \oplus y = g^{-1}(g(x) + g(y))$ and $x \odot y = g^{-1}(g(x)g(y))$.

For $x \in [a, b]_+$ and $p \in (0, \infty)$, we will introduce the pseudo-power $x_\odot^{(p)}$ as follows: if $p = n$ is a natural number then

$$x_\odot^n = \underbrace{x \odot x \odot \ldots \odot x}_{n}$$

Moreover, $x_\odot^{(\frac{1}{n})} = \sup\left\{y|y_\odot^{(n)} \leq x\right\}$. Then $x_\odot^{(\frac{m}{n})}$ is well defined for any rational $r \in (0, 1)$, independently of representation $r = \dfrac{m}{n} = \dfrac{m_1}{n_1}$; $m; n; m_1; n_1$ being positive integers (the result follows from the continuity and monotonicity of $\odot$). Due to continuity of $\odot$, if p is not rational, then

$$x_\odot^{(p)} = \sup\left\{x_\odot^r | r \in]0, p[, r \in Q\right\}.$$

Evidently, if$x \odot y = g^{-1}(g(x).g(y))$, then $x_\odot^{(r)} = g^{-1}(g^p(x))$. On the other hand, if $\odot$ is idempotent, then $x_\odot^{(p)} = x$ for any $x \in [a, b]$ and $p \in (0, \infty)$.

Then, the pseudo-integral for a function $f : [c, d] \to [a, b]$ reduces on the g-integral [43, 45],

$$\int_{[c,d]}^{\oplus} f(x)dx = g^{-1}\left(\int_c^d g(f(x))dx\right). \tag{2.1}$$

More on this structure as well as on corresponding measures and integrals can be found in [42, 43]. The second class is when $x \oplus y = max(x, y)$ and $x \odot y = g^{-1}(g(x)g(y))$, the pseudo-integral for a function $f : \mathbb{R} \to [a, b]$ is given by

$$\int_{\mathbb{R}}^{\oplus} f \odot dm = \sup\left(f(x) \odot \psi(x)\right), \tag{2.2}$$

where function ψ defines sup-measure m. Any sup-measure generated as essential supremum of a continuous density can be obtained as a limit of pseudo-additive measures with respect to generated pseudo-addition [37]. For any continuous function $f : [0, \infty] \to [0, \infty]$ the integral $\int^{\oplus} f \odot dm$ can be obtained as a limit of g-integrals, [37].

We denote by μ the usual Lebesgue measure on $\mathbb{R}$. We have

$$m(A) = esssup(x|x \in A) = \sup\{a|\mu(\{x|x \in A, x > a\}) > 0\}.$$

Theorem 2.3 ([37]). *Let m be a sup-measure on $([0, \infty], \mathbb{B}([0, \infty]))$, where $\mathbb{B}([0, \infty])$ is the Borel σ-algebra on $[0, \infty]$, $m(A) = esssup_\mu(\psi(x)|x \in A)$, and $\psi : [0, \infty] \to [0, \infty]$ is a continuous density. Then, for any* *pseudo-addition* $\oplus$ *with a generator* g *there exists a family* $\{m_\lambda\}$ *of* $\oplus_\lambda$*–measure on* $([0,\infty),\mathbb{B})$*, where* $\oplus_\lambda$ *is generated by* g^λ *(the function* g *of the power* λ*)*, $\lambda \in (0,\infty)$*, such that* $lim_{\lambda\to\infty} m_\lambda = m$*.*

Theorem 2.4 ([37])**.** *Let* $([0,\infty],\ sup,\ \odot)$ *be a semiring, when* $\odot$ *is generated with* g*, i.e.,we have* $x \odot y = g^{-1}(g(x)g(y))$ *for every* $x, y \in (0,\infty)$*. Let* m *be the same as in Theorem 2.8. Then, there exists a family* $\{m_\lambda\}$ *of* $\oplus_\lambda$*-measures, where* $\oplus_\lambda$ *is generated by* $g^\lambda, \lambda \in (0,\infty)$ *such that for every continuous function* $f : [0,\infty] \to [0,\infty]$*,*

$$\begin{aligned} \int^{sup} f \odot dm &= \lim_{\lambda\to\infty} \int^{\oplus_\lambda} f \odot dm_\lambda \\ &= \lim_{\lambda\to\infty} \left(g^\lambda\right)^{-1} \left(\int g^\lambda(f(x))dx\right). \end{aligned} \tag{2.3}$$

Now easily we can obtain indicate the properties listed in the following proposition.

Proposition 2.5 ([19])**.** *Let* $(X, \mathcal{F}, \mu, \mathbb{R}_+^-, \oplus, \odot)$ *is a pseudo-space and* $f, g \in \mathcal{F}$*, then:*

1. *If* $f = 0$ *on* A *a.e., then* $\int_A^\oplus f d\mu = 0$*.*
2. *If* $\mu(A) = 0$*, then* $\int_A^\oplus f d\mu = 0$*.*
3. $\int_A^\oplus a d\mu \geq a \odot \mu(A)$*.*
4. *If* $f \leq g$ *on* A*, then* $\int_A^\oplus f d\mu \leq \int_A^\oplus g d\mu$*.*
5. *If* $A \subset B$*, then* $\int_A^\oplus f d\mu \leq \int_B^\oplus f d\mu$*.*

3. Stolarsky Type Inequality for Pseudo-Integrals

The classical Stolarsky type inequality firstly is stated by Stolarsky in [56]. A. Flores-Franulič et al. have proved a Stolarsky type inequality for fuzzy integrals in [27]. Our purpose in this section is to prove a Stolarsky type inequality for pseudo-integrals. In this section we restrict the semirings to the semiring $([0,1],\oplus,\odot)$.

Theorem 3.1 (Pseudo Stolarsky type inequality: decreasing case)**.** *Let* $a, b > 0$*,* $f : [0,1] \to [0,1]$ *be a continuous and strictly decreasing function and* μ *be the Lebesgue measure on* $\mathbb{R}$*. If the pseudo-operations are defined by a continuous and increasing function* $g : [0,1] \to [0,1]$*, then the inequality*

$$\int_{[0,1]}^{\oplus} f\left(x^{\frac{1}{a+b}}\right) dx \geq \left(\int_{[0,1]}^{\oplus} f\left(x^{\frac{1}{a}}\right) dx\right) \odot \left(\int_{[0,1]}^{\oplus} f\left(x^{\frac{1}{b}}\right) dx\right) \tag{3.1}$$

holds.

Proof. From (2.1), we have:

$$\int_{[0,1]}^{\oplus} f\left(x^{\frac{1}{a+b}}\right) dx = g^{-1}\int_0^1 g\left(f\left(x^{\frac{1}{a+b}}\right)\right) dx = g^{-1}\int_0^1 (g\circ f)\left(x^{\frac{1}{a+b}}\right) dx. \tag{3.2}$$

Since $g\circ f$ is continuous and non-increasing function, so from (1.1) and (2.1) we have:

$$\int_0^1 (g\circ f)\left(x^{\frac{1}{a+b}}\right) dx \geq \int_0^1 (g\circ f)\left(x^{\frac{1}{a}}\right) dx \int_0^1 (g\circ f)\left(x^{\frac{1}{b}}\right) dx. \tag{3.3}$$

Since g is an increasing function, so its inverse function also is increasing function. It follows from (3.2) and (3.3) that

$$\begin{aligned}
\int_{[0,1]}^{\oplus} f\left(x^{\frac{1}{a+b}}\right) dx &\geq g^{-1}\left(\left(\int_0^1 (g\circ f)\left(x^{\frac{1}{a}}\right) dx\right)\left(\int_0^1 (g\circ f)(x^{\frac{1}{b}})dx\right)\right)\\
&= g^{-1}\left(\left(\int_0^1 g\left(f\left(x^{\frac{1}{a}}\right)\right) dx\right)\left(\int_0^1 g\left(f\left(x^{\frac{1}{b}}\right)\right) dx\right)\right)\\
&= g^{-1}\left(\left(gg^{-1}\int_0^1 g\left(f\left(x^{\frac{1}{a}}\right)\right) dx\right)\left(gg^{-1}\int_0^1 g\left(f\left(x^{\frac{1}{b}}\right) dx\right)\right)\right)\\
&= g^{-1}\left(\left(g\int_{[0,1]}^{\oplus} f\left(x^{\frac{1}{a}}\right) dx\right)\left(g\int_{[0,1]}^{\oplus} f\left(x\frac{1}{b}\right) dx\right)\right)\\
&= \left(\int_{[0,1]}^{\oplus} f\left(x^{\frac{1}{a}}\right) dx\right)\odot\left(\int_{[0,1]}^{\oplus} f\left(x^{\frac{1}{b}}\right) dx\right).
\end{aligned}$$

The proof is now complete. □

Example 3.2. (i) Let $g(x) = x^{\alpha}$ for some $\alpha\in[1,\infty)$, so $x\oplus y = \sqrt[\alpha]{x^{\alpha}+y^{\alpha}}$ and $x\odot y = xy$. Then (5) reduces on the following inequality

$$\sqrt[\alpha]{\int_0^1 f\left(x^{\frac{1}{a+b}}\right)^{\alpha} dx} \geq \left(\sqrt[\alpha]{\int_0^1 f\left(x^{\frac{1}{a}}\right)^{\alpha} dx}\right)\left(\sqrt[\alpha]{\int_0^1 f\left(x^{\frac{1}{b}}\right)^{\alpha} dx}\right).$$

(ii) Let $g(x) = e^x$. The corresponding pseudo-operations are $x\oplus y = \ln(e^x+e^y)$ and $x\odot y = x+y$. Then (3.1) reduces on the following inequality

$$\ln\int_0^1 e^{f\left(x^{\frac{1}{a+b}}\right)} dx \geq \ln\int_0^1 e^{f\left(x^{\frac{1}{a}}\right)} dx + \ln\int_0^1 e^{f\left(x^{\frac{1}{b}}\right)} dx.$$

Unfortunately, this inequality is not valid when the pseudo-operations are generated by a decreasing function g as it is showed in the following example:

Example 3.3. Let f and g be two strictly decreasing functions from $[0,1]$ into $[0,1]$ like: $f(x) = g(x) = 1 - x$.

The corresponding pseudo-operations are $x \oplus y = x + y - 1$ and $x \odot y = x + y - xy$. If we suppose that

$$g^{-1}\left(\int_0^1 \left(1 - f\left(x^{\frac{1}{a}}\right)\right) dx\right) = \alpha, \qquad g^{-1}\left(\int_0^1 \left(1 - f\left(x^{\frac{1}{b}}\right)\right) dx\right) = \beta.$$

Then, (3.3) reduces on the following inequality:

$$g^{-1}\left(\int_0^1 \left(1 - f\left(x^{\frac{1}{a+b}}\right)\right) dx\right)$$
$$\geq \alpha + \beta - g^{-1}\left(\int_0^1 \left(1 - f\left(x^{\frac{1}{a}}\right)\right) dx\right) g^{-1}\left(\int_0^1 \left(1 - f\left(x^{\frac{1}{b}}\right)\right) dx\right).$$

It follows that:

$$1 - \int_0^1 \left(1 - f\left(x^{\frac{1}{a+b}}\right)\right) dx$$
$$\geq \alpha + \beta - g^{-1}\left(\int_0^1 \left(1 - f\left(x^{\frac{1}{a}}\right)\right) dx\right) g^{-1}\left(\int_0^1 \left(1 - f\left(x^{\frac{1}{b}}\right)\right) dx\right),$$

hence we have:

$$\int_0^1 f\left(x^{\frac{1}{a+b}}\right) dx$$
$$\geq \int_0^1 f\left(x^{\frac{1}{a}}\right) dx + \int_0^1 f\left(x^{\frac{1}{b}}\right) dx - \left(\int_0^1 f\left(x^{\frac{1}{a}}\right) dx \int_0^1 f(x^{\frac{1}{b}}) dx\right).$$

Now we can suppose that $a = b = \frac{1}{2}$, and with replacing in the above mentioned inequality we have:

$$\int_0^1 f(x)dx \geq \int_0^1 f\left(x^2\right) dx + \int_0^1 f\left(x^2\right) dx - \left(\int_0^1 f\left(x^2\right) dx\right)^2.$$

It follows that:

$$\frac{1}{2} \geq \frac{8}{9}.$$

This is a contradiction.

Theorem 3.4 (Pseudo Stolarsky type inequality: increasing case)**.** *Let $a, b > 0$, $f : [0,1] \to [0,1]$ be a continuous and strictly increasing function and μ be the Lebsgue measure on $\mathbb{R}$. If the pseudo-operations are defined by a continuous and decreasing function $g : [0,1] \to [0,1]$. Then, the inequality*

$$\int_{[0,1]}^{\oplus} f\left(x^{\frac{1}{a+b}}\right) dx \geq \left(\int_{[0,1]}^{\oplus} f\left(x^{\frac{1}{a}}\right) dx\right) \odot \left(\int_{[0,1]}^{\oplus} f\left(x^{\frac{1}{b}}\right) dx\right)$$

holds.

Proof. The proof is similar with the Theorem 3.1. □

Example 3.5. Let $g(x) = 1-x$, so $x \oplus y = x+y-1$ and $x \odot y = x+y-xy$. Then, (3.1) reduces on the following inequality:

$$\int_0^1 f\left(x^{\frac{1}{a+b}}\right) dx \geq \int_0^1 f\left(x^{\frac{1}{a}}\right) dx + \int_0^1 f\left(x^{\frac{1}{b}}\right) dx - \int_0^1 f\left(x^{\frac{1}{a}}\right) dx \int_0^1 f\left(x^{\frac{1}{b}}\right) dx.$$

Now we generalize the Stolarsky type inequality by the semiring $([0,1], \max, \odot)$, where $\odot$ is generated.

Theorem 3.6. *Let $a, b > 0$, if $f : [0,1] \to [0,1]$ is a continuous and strictly decreasing function and let m be the same as in Theorem 2.3. If $\odot$ is represented by an increasing multiplicative generator g, then the inequality*

$$\int_{[0,1]}^{\sup} f\left(x^{\frac{1}{a+b}}\right) dm \geq \left(\int_{[0,1]}^{\sup} f\left(x^{\frac{1}{a}}\right) dm\right)\left(\int_{[0,1]}^{\sup} f\left(x^{\frac{1}{b}}\right) dm\right),$$

holds.

Proof. Theorem 2.4 implies that

$$\begin{aligned}\int_{[0,1]}^{\sup} f\left(x^{\frac{1}{a+b}}\right) dm &= \lim_{\lambda\to\infty} \int_{[0,1]}^{\oplus_\lambda} f\left(x^{\frac{1}{a+b}}\right) dm_\lambda \\ &= \lim_{\lambda\to\infty} \left(g^\lambda\right)^{-1} \int_0^1 g^\lambda \left(f\left(x^{\frac{1}{a+b}}\right)\right) dx \\ &= \lim_{\lambda\to\infty} \left(g^\lambda\right)^{-1} \int_0^1 \left(g^\lambda f\right)\left(x^{\frac{1}{a+b}}\right) dx.\end{aligned} \tag{3.4}$$

Since g is increasing, so g^λ and $\left(g^\lambda\right)^{-1}$ are increasing functions and since $g^\lambda f$ is continuous and non-increasing function, if we get

$$\lim_{\lambda\to\infty} \left(g^\lambda\right)^{-1} \int_0^1 \left(g^\lambda f\right)\left(x^{\frac{1}{a+b}}\right) dx = a,$$

then from (1.1) and (3.4) we have:

$$\begin{aligned} a &\geq \lim_{\lambda\to\infty} \left(g^\lambda\right)^{-1} \left(\int_0^1 \left(g^\lambda f\right)\left(x^{\frac{1}{a}}\right) dx\right)\left(\int_0^1 \left(g^\lambda f\right)(x^{\frac{1}{b}}) dx\right) \\ &= \lim_{\lambda\to\infty} \left(\left(g^\lambda\right)^{-1} \left(\int_0^1 \left(g^\lambda f\right)\left(x^{\frac{1}{a}}\right) dx\right) \left(g^\lambda\right)^{-1} \left(\int_0^1 \left(g^\lambda f\right)(x^{\frac{1}{b}}) dx\right)\right) \\ &= \left(\lim_{\lambda\to\infty} \left(g^\lambda\right)^{-1} \left(\int_0^1 g^\lambda \left(f\left(x^{\frac{1}{a}}\right)\right) dx\right)\right)\left(\lim_{\lambda\to\infty} \left(g^\lambda\right)^{-1} \left(\int_0^1 g^\lambda \left(f\left(x^{\frac{1}{b}}\right)\right) dx\right)\right).\end{aligned}$$

So we have

$$\begin{aligned} a &\geq \left(\lim_{\lambda\to\infty}\left(\int_{[0,1]}^{\oplus_\lambda} f\left(x^{\frac{1}{a}}\right) dm_\lambda\right)\right)\left(\lim_{\lambda\to\infty}\left(\int_{[0,1]}^{\oplus_\lambda} f\left(x^{\frac{1}{b}}\right) dm_\lambda\right)\right) \\ &= \left(\int_{[0,1]}^{\sup} f\left(x^{\frac{1}{a}}\right) dm\right)\left(\int_{[0,1]}^{\sup} f\left(x^{\frac{1}{b}}\right) dm\right), \end{aligned}$$

and the proof is now complete. □

Example 3.7. Using Example 3.2 we have: (ii) we have that $g^\lambda(x) = e^{\lambda x}$. It follows that

$$\begin{aligned} x \oplus y &= \lim_{\lambda\to\infty} \frac{1}{\lambda} \ln\left(e^{\lambda x} + e^{\lambda y}\right) \\ &= \max(x, y), \end{aligned}$$

and

$$\begin{aligned} x \odot y &= \lim_{\lambda\to\infty} \frac{1}{\lambda} \ln\left(e^{\lambda x} e^{\lambda y}\right) \\ &= x + y. \end{aligned}$$

Therefore (3.1) reduces on the following inequality

$$\sup\left(f\left(x^{\frac{1}{a+b}}\right) + \psi(x)\right) \geq \sup(f\left(x^{\frac{1}{a}}\right) + \psi(x)) + \sup\left(f\left(x^{\frac{1}{b}}\right) + \psi(x)\right),$$

where ψ is from Theorem 2.4.

Theorem 3.8. *Let $a, b > 0$, if $f : [0,1] \to [0,1]$ is a continuous and strictly increasing function and let m be the same as in the Theorem 2.3. If $\odot$ is represented by a decreasing multiplicative generator g, then the inequality*

$$\int_{[0,1]}^{\sup} f\left(x^{\frac{1}{a+b}}\right) dm \geq \left(\int_{[0,1]}^{\sup} f\left(x^{\frac{1}{a}}\right) dm\right)\left(\int_{[0,1]}^{\sup} f\left(x^{\frac{1}{b}}\right) dm\right) \tag{3.5}$$

holds.

Proof. The proof is similar with the Theorem 3.6. □

Example 3.9. Using Example 3.5. We have $g^\lambda(x) = x^{-\lambda}$. So

$$x \oplus y = (x^{-\lambda} + y^{-\lambda})^{-\frac{1}{\lambda}}$$

and

$$x \odot y = xy.$$

Therefore 3.5 reduces on the following inequality:

$$\sup\left(f\left(x^{\frac{1}{a+b}}\right) + \psi(x)\right) \geq \sup\left(f\left(x^{\frac{1}{a}}\right) + \psi(x)\right) \sup\left(f\left(x^{\frac{1}{b}}\right) + \psi(x)\right),$$

where ψ is from Theorem 2.4.

Note that the third important case $\oplus = max$ and $\odot = min$ has been studied in [27] and the pseudo-integral in such a case yields the Sugeno integral.

4. Feng Qi Type Inequalities for Pseudo-Integrals

The aim of this section is to show Feng Qi type inequality derived from [4] for the pseudo-integral.

Now we peresent generalations of two inequalities mentioned in Relations 1.3 and 1.4 for pseudo-integrals.

Theorem 4.1. *For a given measurable space* (X, A), *let* $f : [0,1] \to [0,1]$ *be a real valued function such that* $(S)\int_0^1 f d\mu = p$. *If* f *is a continuous and strictly decreasing function, such that* $f\left(p^{n+1}\right) \geq p^{\left(\frac{n+1}{n+2}\right)}$ *and let a generator* $g : [0,1] \to [0,\infty)$ *of pseudo-addition* $\oplus$ *and psudo-multipication* $\odot$ *be decreasing function. Then the inequality:*

$$\int_{[0,1]}^{\oplus} f_{\odot}^{n+2} \odot dm \geq \left(\int_{[0,1]}^{\oplus} f_{\odot} \odot dm\right)_{\odot}^{n+1}$$

holds for all $n \geq 0$ *and* $\sigma - \oplus$*-measure* m.

Proof. We apply the classical Feng Qi inequality and we obtain:

$$\int_0^1 (g \circ f)^{n+2} d(g \circ m) \geq \left(\int_0^1 (g \circ f) d(g \circ m)\right)^{n+1}.$$

since function g is decreasing function, then g^{-1} is also decreasing function and we obtain:

$$g^{-1}\left(\int_0^1 (g \circ f)^{n+2} d(g \circ m)\right) \geq g^{-1}\left(\int_0^1 (g \circ f) d(g \circ m)\right)^{n+1}$$

for left side of inequality we have:

$$\begin{aligned}
g^{-1}\left(\int_0^1 (g \circ f)^{n+2} d(g \circ m)\right) &= g^{-1}\left(\int_0^1 g\left(g^{-1}(g \circ f)^{n+2}\right) d(g \circ m)\right) \\
&= g^{-1}\left(\int_0^1 g\left(f_{\odot}^{n+2}\right) d(g \circ m)\right) \\
&= g^{-1}\left(g\left(g^{-1}\left(\int_0^1 g\left(f_{\odot}^{n+2}\right) d(g \circ m)\right)\right)\right) \\
&= g^{-1}\left(g\left(\int_{[0,1]}^{\oplus} f_{\odot}^{n+2} \odot dm\right)\right) \\
&= \int_{[0,1]}^{\oplus} f_{\odot}^{n+2} \odot dm.
\end{aligned}$$

For right side of the inequality we have:

$$\begin{aligned} g^{-1}\left(\int_0^1 (g\circ f)d(g\circ m)\right)^{n+1} &= g^{-1}\left(\int_0^1 g\left(g^{-1}(g\circ f)\right)d(g\circ m)\right)^{n+1} \\ &= g^{-1}\left(\int_0^1 g\left(f_\odot\right)d(g\circ m)\right)^{n+1} \\ &= g^{-1}\left(g\left(g^{-1}\left(\int_0^1 g\left(f_\odot\right)d(g\circ m)\right)\right)\right)^{n+1} \\ &= g^{-1}\left(g\int_{[0,1]}^{\oplus} f_\odot \odot dm\right)^{n+1} \\ &= \left(\int_{[0,1]}^{\oplus} f_\odot \odot dm\right)^{n+1}. \end{aligned}$$

Hence we have:

$$\int_{[0,1]}^{\oplus} f_\odot^{n+2}\odot dm \geq \left(\int_{[0,1]}^{\oplus} f_\odot \odot dm\right)_\odot^{n+1}$$

In this generalation we have term $f\left(p^{n+1}\right) \geq p^{\left(\frac{n+1}{n+2}\right)}$, because this theorem is ture in fuzzy case. □

Theorem 4.2. *For a given measurable space (X, A), let $f : [0,1] \to [0,1]$ be a ral valued function such that $(S)\int_0^1 f d\mu = p$. If f is a continuous and strictly increasing function, such that $f\left(1-p^{n+1}\right) \geq p^{\left(\frac{n+1}{n+2}\right)}$ and let a generator $g : [0,1] \to [0,\infty)$ of pseudo-addition $\oplus$ and psudo-multipication $\odot$ be an increasing function. Then the inequality:*

$$\int_{[0,1]}^{\oplus} f_\odot^{n+2}\odot dm \geq \left(\int_{[0,1]}^{\oplus} f_\odot \odot dm\right)_\odot^{n+1}$$

holds for all $n \geq 0$ and $\sigma - \oplus$-measure m.

Proof. The proof is similar whit Theorem 4.1. □

Example 4.3. Let $g(x) = \ln(x)$, then

$$x \oplus y = xy, \qquad x \odot y = e^{\ln x . \ln y}.$$

by Theorem 4.1, the following inequality

$$\ln\int_0^1 e^{(\ln f(x))^{n+2}} \geq \left(\ln\int_0^1 e^{(\ln f(x))}\right)^{n+1}$$

holds.

In the sequel, we generalize the Feng Qi inequality by the semiring $([a,b], \max, \odot)$, where $\odot$ is generated .

Theorem 4.4. *Let $f : [0,1] \to [0,1]$ be a real valued, continuous and strictly increasing function such that $(S)\int_0^1 f d\mu = p$. If $\odot$ is represented by a increasing generator g and m is complet* sup*-measure same as in Theorem 2.4, then whit condition $f\left(1 - p^{n+1}\right) \geq p^{\left(\frac{n+1}{n+2}\right)}$*

$$\int_{[0,1]}^{\sup} f_{\odot}^{n+2} \odot dm \geq \left(\int_{[0,1]}^{\sup} f \odot dm\right)_{\odot}^{n+1}$$

holds for all $n \geq 0$ and $\sigma - \oplus$-measure m.

Proof. Since $\left(g^{\lambda}(x)\right)^{-1} = g^{-1}\left(x^{\frac{1}{\lambda}}\right)$ we have:

$$\begin{aligned} x \odot y &= g^{-1}\left(g(x)g(y)\right) \\ &= \left(g^{\lambda}\right)^{-1}\left(g^{\lambda}(x)g^{\lambda}(y)\right) \\ &= x \odot_{\lambda} y. \end{aligned}$$

I other words, g^{λ} is a generator of $\odot$. By Theorem 2.4 we have:

$$\begin{aligned} \int^{\sup} f \odot dm &= \lim_{\lambda\to\infty} \int^{\oplus_{\lambda}} f \odot dm_{\lambda} \\ &= \lim_{\lambda\to\infty} \left(g^{\lambda}\right)^{-1}\left(\int g^{\lambda}(f(x))dx\right). \end{aligned}$$

As g is decreasing so $g^{-1}, g^{\lambda}, \left(g^{\lambda}\right)^{-1}$ are also decreasing functions. Then:

$$\begin{aligned} \left(\int_{[0,1]}^{\sup} f \odot dm\right)_{\odot}^{n+1} &= \left(\int_{[0,1]}^{\sup} \left(g^{\lambda}\right)^{-1}\left(g^{\lambda}(f(x)\right) dm\right)_{\odot}^{n+1} \\ &= \left(\lim_{\lambda\to\infty} \left(g^{\lambda}\right)^{-1} \int_0^1 g^{\lambda}\left(\left(g^{\lambda}\right)^{-1}\left(g^{\lambda}(f(x)\right)\right) dx\right)_{\odot}^{n+1} \\ &= \left(g^{\lambda}\right)^{-1}\left(g^{\lambda}\left(\lim_{\lambda\to\infty} \left(g^{\lambda}\right)^{-1} \int_0^1 g^{\lambda}\left(f(x)dx\right)\right)\right)^{n+1} \\ &= \lim_{\lambda\to\infty} \left(g^{\lambda}\right)^{-1}\left(\int_0^1 g^{\lambda}\left(f(x)\right)dx\right)^{n+1}. \end{aligned}$$

By classical Feng Qi inequality we have:

$$\left(\int_{[0,1]}^{\sup} f \odot dm\right)_{\odot}^{n+1} \leq \lim_{\lambda\to\infty} \left(g^{\lambda}\right)^{-1}\left(\int_0^1 \left(g^{\lambda}\left(f(x)\right)\right)^{n+2} dx\right)$$

$$= \lim_{\lambda\to\infty} \left(g^{\lambda}\right)^{-1} \left(\int_0^1 g^{\lambda}\left(\left(\left(g^{\lambda}\right)^{-1}\left(g^{\lambda}\left(f(x)\right)\right)\right)^{n+2} dx\right)$$

$$= \lim_{\lambda\to\infty} \left(g^{\lambda}\right)^{-1} \left(\int_0^1 \left(g^{\lambda}\left(f_{\odot}^{n+2}\right)^{n+2} dx\right)\right)$$

$$= \int_{[0,1]}^{\sup} f_{\odot}^{n+2} \odot dm. \qquad \square$$

Example 4.5. Let $g^{\lambda}(x) = e^{\lambda x}$, the corresponding pseudo-operations are:

$$x \oplus y = \lim_{\lambda\to\infty} \frac{1}{\lambda}\left(e^{\lambda x} + e^{\lambda y}\right)$$
$$= \max(x, y)$$

and

$$x \odot y = \lim_{\lambda\to\infty} \frac{1}{\lambda}\left(e^{\lambda x} e^{\lambda y}\right)$$
$$= x + y.$$

By Theorem 4.1 the following inequality holds:

$$\sup((n+2)f(x) + \psi(x)) \geq (n+1)(\sup(f(x) + \psi(x)).$$

Theorem 4.6. *Let $f : [0,1] \to [0,1]$ be a real valued, continuous and strictly decreasing function such that $(S)\int_0^1 f d\mu = p$. If $\odot$ is represented by a decreasing generator g and m is complet* sup*-measure same as in Theorem 2.4, then whit condition $f\left(p^{n+1}\right) \geq p^{\left(\frac{n+1}{n+2}\right)}$ the inequality*

$$\int_{[0,1]}^{\sup} f_{\odot}^{n+2} \odot dm \geq \left(\int_{[0,1]}^{\sup} f \odot dm\right)_{\odot}^{n+1}$$

holds for all $n \geq 0$ and $\sigma - \oplus$-measure m.

Proof. The proof is similar whit e Theorem 3.4. $\square$

Remark 4.7. Typical example for the two above cases are operation $\oplus = \vee$ and $\odot = \wedge$ that already proved in fuzzy case.

5. Markov Type Inequality for Pseudo-Integrals

In the following, we assume the semiring with the pseudo-operations that are defined by a monotone and continuous function $g : [a, b] \to [0, \infty]$, i.e., pseudo operations are given with $x \oplus y = g^{-1}(g(x) + g(x))$ and $x \odot y = g^{-1}(g(x)g(x))$.
If the generator function g is increasing, then zero element for the $\oplus$ is a, i.e. $g(a) = \mathbf{0}$ and $g(b) = \mathbf{1}$. If the zero element for the pseudo-addition is b, we will consider decreasing generators. Then $g(b) = \mathbf{0}$ and

$g(a) = \mathbf{1}$. If the generator g is increasing (respectively decreasing), then the operation $\oplus$ induces the usual order (respectively opposite to the usual order) on the interval $[a, b]$, i.e.: $x \preceq y$ if and only if $g(x) \leq g(y)$.

The following Theorem is a generalization of the Markov type inequality for the Sugeno-integrals, where the operations $\oplus$ and $\odot$ are constructed by a monotone continuous generator function g.

Theorem 5.1. *Let $g : [a, b] \to [0, \infty]$ be a continuous and increasing function, then for any non-negative integrable function $f : [c, d] \to [a, b]$ the inequality*

$$\mu(\{x \in A : f(x) \geq e\}) \leq \frac{1}{e} \int_A^{\oplus} f^2 d\mu \tag{5.1}$$

holds where $A = [c, d]$ and $e \in [a, b]$.

Proof. Let us consider $A^* = \{x \in A : f(x) \geq e\}$, we must to show that:

$$\int_A^{\oplus} f d\mu \geq e.\mu(A^*).$$

As $A^* \subseteq A$, then by (5) of Proposition 2.5 we have

$$\int_A^{\oplus} f d\mu \geq \int_{A*}^{\oplus} f d\mu. \tag{5.2}$$

Since $f(x) \geq e$ for all $x \in A^*$, so

$$(f) \geq (e).$$

Since g is an increasing function, then $g(f) \geq g(e)$. Therefore by (4) of Proposition 2.5 we have

$$\int_{A^*} g(f) d\mu \geq \int_{A^*} g(e) d\mu.$$

Since inverse of increasing function is increasing, so g^{-1} is also increasing. It follows that

$$\begin{aligned} g^{-1}\left(\int_{A^*} g(f) d\mu\right) &\geq g^{-1}\left(\int_{A^*} g(e) d\mu\right) \\ &= g^{-1} g(e).\mu(A^*) \\ &= e.\mu(A^*) \end{aligned}$$

i.e.

$$\begin{aligned} \int_{A^*}^{\oplus} f d\mu &= g^{-1}\left(\int_{A^*} g(f) d\mu\right) \\ &\geq e.\mu(A^*). \end{aligned}$$

From the inequality (5.2) we have

$$\begin{aligned}\int_A^{\oplus} f d\mu &\geq \int_{A^*}^{\oplus} f d\mu \\ &\geq e.\mu(A^*).\end{aligned}$$

Consequently

$$\mu(\{x \in A : f(x) \geq e\}) \leq \frac{1}{e} \int_A^{\oplus} f d\mu,$$

which completes the proof. □

In the following example, we illustrate the Theorem 5.1.

Example 5.2. Let $f(x) = x$, for all $x \in [1,2]$ and $g : [1,2] \to [0,\infty]$ be defined as $g(x) = e^x$. Taking $A = [1,2]$ and $e = \frac{3}{2}$, we have

$$\begin{aligned}\mu(\{x \in A : f(x) \geq e\}) &= \mu\left(\{x \in [1,2] : x \geq \frac{3}{2}\}\right) \\ &= \mu\left(\left[\frac{3}{2}, 2\right]\right) \\ &= \frac{1}{2}\end{aligned}$$

and

$$\begin{aligned}\int_A^{\oplus} f d\mu &= \int_A^{\oplus} x d\mu \\ &= g^{-1}\left(\int_1^2 g(x)dx\right) \\ &= g^{-1}\left(e^2 - e\right) \\ &= \ln\left(e^2 - e\right).\end{aligned}$$

Therefore

$$\begin{aligned}\mu(\{x \in A : f(x) \geq e\}) = \frac{1}{2} &\leq \ln\left(e^2 - e\right) \\ &= \frac{1}{e} \int_A^{\oplus} f d\mu.\end{aligned}$$

In the continue, we express the Markov's inequality when the pseudo-addition is idempotent operation and the pseudo-multiplication is not. Also pseudo-multiplication is constructed by a monotone and continuous function g. If we assume that $x \oplus y = \sup(x,y)$ and $\odot$ and is arbitrary and is not idempotent on the interval $[a,b]$, then we have $\mathbf{0} = a$ and the idempotent operation sup induces a full order in the following way:
$x \preceq y$ if and only if $\sup(x,y) = y$.

Theorem 5.3. *Let $f:[c,d]\to[a,b]$ be a non-negative integrable function. If $\odot$ is represented by an increasing multiplicative generator g and m be the same as in Theorem 2.4 Then the inequality*

$$m(\{x\in A: f(x)\geq e\})\leq \frac{1}{e}\int_A^{\sup} f\odot dm$$

holds, where $A=[c,d]$ and $e\in[a,b]$.

Proof. Suppose that $A^*=\{x\in A: f(x)\geq e\}$. It following that

$$\begin{aligned}
\int_{[c,d]}^{\sup} f\odot dm &= \lim_{\lambda\to\infty}\int_{[c,d]}^{\oplus_\lambda} f\odot dm_\lambda\\
&= \lim_{\lambda\to\infty}\left(g^\lambda\right)^{-1}\left(\int_c^d g^\lambda(f(x))dx\right)\\
&\geq \lim_{\lambda\to\infty}\left(g^\lambda\right)^{-1}\left(\int_{A^*} g^\lambda(f(x))dx\right)\\
&\geq \lim_{\lambda\to\infty}\left(g^\lambda\right)^{-1}\left(\int_{A^*} g^\lambda(e)dx\right)\\
&= \lim_{\lambda\to\infty}\left(g^\lambda\right)^{-1} g^\lambda(e).m(A^*)\\
&= e.m(A^*),
\end{aligned}$$

therefore

$$m(A^*)\leq\frac{1}{e}\int_{[c,d]}^{\sup} f\odot dm.$$

This completes the proof. □

Example 5.4. Let $f:[0,2]\to[a,b]$ be defined as $f(x)=x$ and $g(x)=e^x$ on $[a,b]$. Let function Ψ be define *sup*-measure m and also suppose that $c=1$. Then

$$\begin{aligned}
x\odot y &= \ln(e^x.e^y)\\
&= \ln(e^{x+y})\\
&= x+y
\end{aligned}$$

and

$$x\oplus y=\max(x,y).$$

So, we have

$$\begin{aligned}
\mu(\{x\in[0,2]|f(x)=x\geq 1\} &\leq \left(\int_{[0,2]}^{\sup} x\odot dm\right)\\
&= \sup_{x\in[0,2]}\left(x+\psi(x)\right).
\end{aligned}$$

Note that in the third important case, we assume that $\oplus = sup$ and $\odot = inf$ on the interval $[a, b]$. So we have $\mathbf{0} = a$ and $\mathbf{1} = b$. The idempotent operation sup induces the usual order ($x \preceq y$ if and only if $sup(x, y) = y$). The Theorem 5.1 has been studied in [29] for this case that the pseudo-integral in such a case yields the Markov type inequality for the Sugeno integral.

6. Conclusion

We have proved the Stolarsky, Feng Qi and Markov type inequalities for pseudo-integrals when we consider the semiring $([0, 1], \oplus, \odot)$. We concentrate for two classes of pseudo-integrals: The first class includes the pseudo-integral based on a function reduces on the g-integral, where $\oplus$ and $\odot$ are defined by a monotone and continuous function g. The second class includes the pseudo-integral based on the semiring $([0, 1], max, \odot)$ is given by sup-measure where $x \odot y$ is generated by $g^{-1}(g(x)g(y))$.

References

1. H. Agahi, R. Mesiar and Y. Ouyang, *New general extensions of Chebyshev type inequalities for Sugeno integrals*, Int. J. Approx. Reasoning, 51 (2009), pp. 135-140.
2. H. Agahi, R. Mesiar and Y. Ouyang, *General Minkowski type inequalities for Sugeno integrals*, Fuzzy Sets Syst., 161 (2010), pp. 708-715.
3. H. Agahi, R. Mesiar and Y. Ouyang, *Chebyshev type inequalities for pseudo-integrals*, Nonlinear Analysis, 72 (2010), pp. 2737-2743.
4. H. Agahi and M.A. Yaghoobi, *A Feng Qi type inequality for Sugeno integral*, Fuzzy Inf. Eng., 3 (2010), pp. 293-304.
5. G. Anastassiou, *Chebyshev Grüss type inequalities via Euler type and Fink identities*, Math. Comput. Modelling, 45 (2007), pp. 1189-1200.
6. L. Bougoffa, *On Minkowski and Hardy integral inequalities*, JIPAM, J. Inequal. Pure Appl. Math., 7 (2) (2006), Article 60.
7. P.S. Bullen, *A Dictionary of Inequalities*, Addison Wesley Longman Limited, 1998.
8. J. Caballero and K. Sadaragani, *Chebyshev inequality for Sugeno integrals*, Fuzzy Sets Syst., 161 (2010), pp. 1480-1487.
9. T.Y. Chen, H.L. Chang and G.H. Tzeng, *Using fuzzy measures and habitual domains to analyze the public attitude and apply to the gas taxi policy*, Eur. J. Oper. Res., 137 (2002), pp. 145-161.

10. B. Daraby, *Investigation of a Stolarsky type inequality for integrals in pseudo-analysis*, Fractional Calculus & Applied Analysis, 13 (5) (2010), pp. 467-473.
11. B. Daraby, *Generalization of the Stolarsky type inequality for pseudo-integrals*, Fuzzy Sets Syst., 194 (2012), pp. 90-96.
12. B. Daraby and L. Arabi, *Related Fritz Carlson type inequality for sugeno integrals*, Soft Comput., 17 (10) (2013), pp. 1745-1750.
13. B. Daraby, *Markov type integral inequality for pseudo-integrals*, Casp. J. Appl. Math. Ecol. Econ., 1 (1) (2013), pp. 13-23.
14. B. Daraby, A. Shafiloo and A. Rahimi, *Geberalizations of the Feng Qi type inequality for pseudo-integral*, Gazi University Journal of Science, 28 (4) (2015), pp. 695-702.
15. B. Daraby, F. Rostampour and A. Rahimi, *Hardy's type tnequality for pseudo-integrals*, Acta Univ. Apulensis, Math. Inform., 42 (2015), pp. 53-65.
16. B. Daraby, *A convolution type inequality for pseudo-integrals*, Acta Univ. Apulensis, Math. Inform., 48 (2016), pp. 27-35.
17. B. Daraby, *Results Of The Chebyshev Type Inequality For Pseudo-Integral*, Sahand Commun. Math. Anal., 4 (1) (2016), pp. 91-100.
18. B. Daraby, H.G. Asll and I. Sadeqi, *General related inequalities to Carlson-type inequality for the Sugeno integral*, Appl. Math. Comput., 305 (2017), pp. 323-329.
19. B. Daraby, *Generalizations of the Well-Known Chebyshev type inequalities for pseudo-integrals*, Gen. Math. Notes, 38 (1) (2017), pp. 32-45.
20. B. Daraby, F. Rostampour and A. Rahimi, *Minkowski type inequality for fuzzy and pseudo-integrals*, Tbil. Math. J., 10 (4) (2017), pp. 159-174.
21. B. Daraby, A, Shafiloo and A. rahimi, *Carlson type inequality for Choquet-like expectation*, Acta Univ. Apulensis, Math. Inform., 49 (2017), pp. 23-36.
22. B. Daraby, H. Ghazanfary Asll and I. Sadeqi, *General related inequalities to Carlson-type inequality for the Sugeno integral*, Appl. Math. Comput., 305 (15) (2017), pp. 323-329.
23. B. Daraby, H. Ghazanfary Asll and I. Sadeqi, *Favard's inequality for pseudo-integral*, Asian-Eur. J. Math., 11 (1) 2018, 1850015.
24. B. Daraby, F. Rostampour, A.R. Khodadadi and A. Rahimi, *Related Gauss-Winkler type inequality for fuzzy and pseudo-integrals*, Thai J. Math., 19 (2) (2021), 713-724.
25. B. Daraby, R. Mesiar, F. Rostampour and A. Rahimi *Related Thunsdorff type and Frank-Pick type inequalities for Sugeno integral*, Appl. Math. Comput., 414 (2022), 126683.

26. A. Flores-Franulič and H. Román-Flores, *A Chebyshev type inequality for fuzzy integrals*, Appl. Math. Comput., 190 (2007), pp. 1178-1184.
27. A. Flores-Franulič, H. Román-Flores and Y. Chalco-Cano, *A note on fuzzy integral inequality of Stolarsky type*, Appl. Math. Comput., 196 (2008), pp. 55-59.
28. A. Flores-Franulič, H. Román-Flores and Y. Chalco-Cano, *A convolution type inequality for fuzzy integrals*, Appl. Math. Comput., 195 (2008),94-99.
29. A. Flores-Franulic, H. Roman-Flores and Y. Chalco-Cano, *Markov type inequalities for fuzzy integrals*, Appl. Math. Comput., 207 (2009), pp. 242-247.
30. A. Flores-Franulič, H. Román-Flores and Y. Chalco-Cano, *Markov type inequalities for fuzzy integrals*, Appl. Math. Comput., 207 (2009), pp. 242-247.
31. D.H. Hong, *A sharp Hardy-type inequality of Sugeno integrals*, Appl. Math. Comput., 217 (2010), 437-440.
32. A. Kolmogorov and S. Fomin, *Elements of the Theory of Functions and Functional Analysis*, Dover, New York, 1999.
33. S.G. Krantz, *Jensen's Inequality, ♯ 9.1.3 in Handbook of Complex Variables*, Boston, MA: Birkhäuser, 119, 1999.
34. W. Kuich, *Semirings, Automata, Languages*, Springer-Verlag, Berlin, 1986.
35. Y. Liu and M. Luo, *Fuzzy topology*, Adv. Fuzzy Syst., Appl. Theory, 9, World Scientific, Singapore, 1997.
36. J.-Y. Lu, K.-S. Wu and J.-C. Lin, *Fast full search in motion estimation by hierarchical use of Minkowski's inequality*, Pattern Recognition, 31 (1998), pp. 945-952.
37. R. Mesiar and E. Pap, *Idempotent integral as limit of g-integrals*, Fuzzy Sets Syst., 102 (1999), pp. 385-392.
38. R. Mesiar and Y. Ouyang, *General Chebyshev type inequalities for Sugeno integrals*, Fuzzy Sets Syst., 160 (2009), pp. 58-64.
39. H. Minkowski, *Geometrie der Zahlen*, Teubner, Leipzig, 1910.
40. Y. Ouyang, J. Fang and L. Wang, *Fuzzy Chebyshev type inequality*, Int. J. Approx. Reasoning, 48 (2008), pp. 829-835.
41. U. M, Özkan, M.Z. Sarikaya and H. Yildirim, *Extensions of certain integral inequalities on time scales*, Appl. Math. Lett., 21 (2008), pp. 993-1000.
42. E. Pap, *An integral generated by decomposable measure*, Univ. Novom Sadu Zb. Rad. Prirod. -Mat. Fak. Ser. Mat., 20 (1) (1990), pp. 135-144.

43. E. Pap, *g-calculus*, Univ. Novom Sadu Zb. Rad. Prirod.-Mat. Fak. Ser. Mat., 23 (1) (1993), pp. 145-156.
44. E. Pap, *Null-additive Set Functions*, Kluwer, Dordrecht, 1995.
45. E. Pap, *N. Ralević, Pseudo-Laplace transform*, Nonlinear Analysis, 33 (1998), pp. 553-560.
46. E. Pap, *Pseudo-additive measures and their applications*, in: E. Pap (Ed.), Handbook of Measure Theory, Elsevier, Amsterdam, 2002, pp. 1403-1465.
47. E. Pap, *Generalized real analysis and its applications*, Int. J. Approx. Reasoning, 47 (2008), pp. 368-386.
48. E. Pap and M. Štrboja, *Generalization of the Jensen inequality for pseudo-integral*, Inf. Sci., 180 (2010), pp. 543-548.
49. F. Qi, *Several integral inequalities*, J. Inequal. Pure Appl. Math., 1 (2) (2000), Article 19.
50. H. Román-Flores, A. Flores-Franulič and Y. Chalco-Cano, *The fuzzy integral for monotone functions*, Appl. Math. Comput., 185 (2007), pp. 492-498.
51. H. Román-Flores, A. Flores-Franulič and Y. Chalco-Cano, *A Jensen type inequality for fuzzy integrals*, Inf. Sci., 177 (2007), pp. 3192-3201.
52. H. Román-Flores, H.Y. Chalco-Cano, *Sugeno integral and geometric inequalities*, International Journal of Uncertainity, Fuzziness and Knowledge-Based Systtem, 15 (2007), pp. 1-11.
53. H. Román-Flores, A. Flores-Franulič and Y. Chalco-cano, *A convolution type inequality for fuzzy integrals*, Appl. Math. Comput., 195 (2008), pp. 94-99.
54. H. Román-Flores, A. Flores-Franulič and Y. Chalco-cano, *A note on fuzzy integral inequality of Stolarsky type*, Appl. Math. Comput., 196 (2008), pp. 55-59.
55. H. Román-Flores, A. Flores-Franulič and Y. Chalco-cano, *A Hardy type inequality for fuzzy integrals*, Appl. Math. Comput.„ 204 (2008), pp. 178-183.
56. K.B. Stolarsky, *From Wythoff,s Nim to Chebyshev,s inequality*, Am. Math. Mon., 98 (1991), pp. 889-900.
57. M. Sugeno and T. Murofushi, *Pseudo-additive measures and integrals*, J. Appl. Math. Anal. Appl., 122 (1987), pp. 197-222.
58. Z. Wang and G.J. Klir, *Fuzzy Measure Theory*, Plenum Press, New York, 1992.
59. L. Wu, J. Sun, X. Ye and L. Zhu, *Hölder type inequality for Sugeno integral*, Fuzzy Sets Syst., 161 (2010), pp. 2337-2347.
60. K.W. Yu. and F. Qi, *A short note on an integral inequality*, RGMIA Res. Rep. Coll., 4 (1) (2001), pp. 23-25

61. L.A. Zadeh, Is there a need for fuzzy logic?, Inf. Sci., 178 (2008), 2751-2779.
62. L.A. Zadeh, Fuzzy sets, Inf. Control, 8 (1965), pp. 338-353.

Department of Mathematics, University of Maragheh, P. O. Box 55181-83111, Maragheh, Iran.

Email address: bdaraby@maragheh.ac.ir