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Three Weak Solutions for an Anisotropic Variable Exponent Problem with Neumann Boundary Condition

Tahereh Norouzi Ghara[1] *, Somaye Khademloo[2], Ghasem Alizadeh Afrouzi[3] and Lin Li[4]

Abstract. In the present work, we investigate an interval of real parameters λ for which the problem admits at least one nontrivial solution. Moreover we deal with the existence results of three solutions for anisotropic problems with variable exponents.

1. Introduction

In this paper, we study the existence of three weak solutions for the following Neumann problem

(1.1)
$$\begin{cases} -\sum_{i=1}^{N} \partial_{x_i} a_i(x, \partial_{x_i} u) + h(x) \sum_{i=1}^{N} a_i(x, u) = \lambda f(x, u), & x \in \Omega, \\ \dfrac{\partial u}{\partial \nu} = 0, & x \in \partial\Omega. \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^N$ $(N \geq 3)$ with smooth boundary $\partial\Omega$, ν is the outer unit normal to $\partial\Omega$, λ is a positive parameter, while $f : \Omega \times \mathbb{R} \to \mathbb{R}$ and $a_i : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ are Carathéodory functions and $h(x)$ is a positive function such that $h(.) \in L^\infty(\Omega)$ and

$$h^- = \operatorname{ess} \inf_{x \in \Omega} h\,(x) > 0,$$

and

$$h^+ = \operatorname{ess} \sup_{x \in \Omega} h\,(x) > 0.$$

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The aim of this article is using variational method [1, 2] to prove the existence of one nontrivial solution and then three weak solutions for the problem (1.1) for the appropriate values of the parameter λ belonging to a precise interval. Our approach is variational method and main tools are the Theorem 1.1 and Theorem 1.2 due to Bonanno that we recall here.

Theorem 1.1 ([3, Theorem 5.1])**.** *Let X be a real Banach space and let $\Phi, \Psi : X \to \mathbb{R}$ be two continuously Gâteaux differentiable functions with Φ bounded from below. Assume that there are $r_1, r_2 \in \mathbb{R}$ with $r_1 < r_2$, such that*

$$\beta(r_1, r_2) < \rho(r_1, r_2),$$

where

$$\beta(r_1, r_2) := \inf_{v\in\Phi^{-1}(]r_1,r_2[)} \frac{\sup_{u\in\Phi^{-1}(]r_1,r_2[)} \Psi(u) - \Psi(v)}{r_2 - \Phi(v)},$$

and

$$\rho(r_1, r_2) := \sup_{v\in\Phi^{-1}(]r_1,r_2[)} \frac{\Psi(v) - \sup_{u\in\Phi^{-1}(]r_1,r_2[)} \Psi(u)}{\Phi(v) - r_1},$$

and for each $\lambda \in \left]\frac{1}{\rho(r_1, r_2)}, \frac{1}{\beta(r_1, r_2)}\right[$ the function $I_\lambda = \Phi - \lambda\Psi$ satisfies ${}^{[r_1]}(PS)^{[r_2]}$-condition. Then, for each $\lambda \in \left]\frac{1}{\rho(r_1, r_2)}, \frac{1}{\beta(r_1, r_2)}\right[$, there is $u_{0,\lambda} \in \Phi^{-1}(]r_1, r_2[)$ such that $I_\lambda(u_{0,\lambda}) \le I_\lambda(u)$ for all $u \in \Phi^{-1}(]r_1, r_2[)$ and $I'_\lambda(u_{0,\lambda}) = 0$.

Theorem 1.2 ([3, Theorem 7.1])**.** *Let X be a real Banach space and let $\Phi, \Psi : X \to \mathbb{R}$ be two continuously Gâteaux differentiable functions with Φ bounded from below. Assume that there is $r \in]inf_X\Phi, sup_X\Psi[$ such that*

$$\varphi(r) < \rho(r),$$

where

$$\varphi(r) := \inf_{v\in\Phi^{-1}(]-\infty,r[)} \frac{\sup_{u\in\Phi^{-1}(]-\infty,r[)} \Psi(u) - \Psi(v)}{r - \Phi(v)},$$

and

$$\rho(r) := \sup_{v\in\Phi^{-1}(]r,\infty[)} \frac{\Psi(v) - \sup_{u\in\Phi^{-1}(]-\infty,r[)} \Psi(u)}{\Phi(v) - r}.$$

and for each $\lambda \in \left]\frac{1}{\rho(r)}, \frac{1}{\varphi(r)}\right[$ the function $I_\lambda = \Phi - \lambda\Psi$ is bounded from below and satisfies (PS)-condition.
Then, for each $\lambda \in \left]\frac{1}{\rho(r)}, \frac{1}{\varphi(r)}\right[$ the function I_λ admits at least three critical points.

Remark 1.3 ([3]). If we assume that $\Phi(0) = \Psi(0) = 0$ and there are $r > 0$ and $\overline{u} \in X$, with $\Phi(\overline{u}) > r$, such that

$$\frac{\sup_{u\in\Phi^{-1}(]-\infty,r[)} \Psi(u)}{r} < \frac{\Psi(\overline{u})}{\Phi(\overline{u})},$$

then one has $\varphi(r) < \rho(r)$ and, in addition,

$$\left]\frac{\Phi(\overline{u})}{\Psi(\overline{u})}, \frac{r}{\sup_{u\in\Phi^{-1}(]-\infty,r[)} \Psi(u)}\right[.$$

Proposition 1.4 ([3, Proposition 2.2]). *Let X be a reflexive real Banach space; $\Phi : X \to \mathbb{R}$ be a continuously Gâteaux differentiable function whose Gâteaux derivative admits a continuous inverse on X^*, and $\Psi : X \to \mathbb{R}$ be a continuously Gâteaux differentiable function whose Gâteaux derivative is compact. Assume that the function $\Phi - \Psi$ is coercive.*

Then, for all $r_1, r_2 \in [-\infty, +\infty]$, with $r_1 < r_2$, the function $\Phi - \Psi$ satisfies the $^{[r_1]}(PS)^{[r_2]}$-condition.

In fact, we shall study this nonlinear elliptic problem where the weak solutions satisfies a homogeneous Neumann boundary condition, which in recent years, many publications have appeared about it [6, 9, 14, 18]. The investigation of this problem arises from parabolic problems like reaction diffusion systems or evolutionary equations. For instance, such equations and systems are important for the modelling of space-time dependent problems, e.g., problems in physics and biology. In particular, evolutionary equations and systems can be used to model physical processes like heat conduction, diffusion processes or wave propagation. The main interesting aspect here is the nonstandard growth setting. Materials involving nonhomogenities are usually modelled by energetic functionals of the type

$$\int |\nabla u(x)|^{p(x)}\, dx, \tag{1.2}$$

where $p(x) > 1$ is a continuous function. Such kind of functionals are mentioned, for instance, in the work of Ruzicka [26] where they are used to model an electrorheological fluid. They correspond to the so called $p(x)$-Laplace operator which is described by the formula

$$\begin{aligned}\Delta_{\overrightarrow{p}(x)}(u) &= div\left(|\nabla u|^{p(x)-2}\,\nabla u\right)\\ &= \sum_i \partial_{x_i}\left(|\partial_{x_i} u|^{p(x)-2}\partial_{x_i} u\right)\end{aligned}$$

Problems involving operators of type

$$\sum_i \partial_{x_i}\left(|\partial_{x_i} u|^{p_i(x)-2}\partial_{x_i} u\right),$$

will be called anisotropic partial differential equations with variable exponent. This differential operator is a natural generalization of the isotropic p-Laplacian operator $\Delta_p(u) = div(|\nabla u|^{p-2}\nabla u)$ where $p > 1$ denotes a real constant. The main difference between them is that p-Laplacian operator is $p-1$ homogenous, that is, $\Delta_p(tu) = t^{p-1}\Delta_p(u)$ for every $t > 0$, but the $p(x)$-Laplacian operator doesn't have this property. This causes many problems, some classical theories, such as the theory of Sobolev spaces, is not applicable. For the papers involving the $p(x)$-Laplacian operator we refer the readers to [4–6, 8, 9, 13, 14, 16, 18, 24, 25, 27] and references therein. Nonlinear problems involving the $p(x)$-Laplacian operator are extremely attractive because they can be used to model dynamical phenomena which arise from the study of electrorheological fluids or elastic mechanics. Problems with variable exponent growth conditions also appear in the modelling of stationary thermo-rheological viscous flows of non-Newtonian fluids and in the mathematical description of the processes filtration of an ideal barotropic gas through a porous medium. We refer the readers for newer articles on anisotropic variable exponent to [10, 12, 17, 20–22].

The plane of this article is as follows. In section 2 we recall some information that we need. In the last section, we use Theorem 1.1 for the problem (1.1) and then we mention our main results.

2. Preliminaries

Let us present a brief overview on variable exponent spaces.
Assume that $\Omega \subset \mathbb{R}^N$ be an open domain. Set

$$C_+(\overline{\Omega}) := \left\{p : p \in C(\overline{\Omega})\text{and}p(x) > 1, \text{for all}x \in \overline{\Omega}\right\}.$$

For any $p \in C_+(\overline{\Omega})$ we define

$$p^- = \inf_{x\in\Omega} p(x),$$

and

$$p^+ = \sup_{x\in\Omega} p(x).$$

For each $p \in C_+(\overline{\Omega})$, we recall the definition of the variable exponent Lebesgue space,

$$L^{p(x)}(\Omega) = \left\{u : \int_\Omega |u(x)|^{p(x)}dx < \infty\right\},$$

where u is a measurable real-valued function.

This space becomes a Banach space [11, 15] with respect to the Luxemburg norm, that is,

$$|u|_{L^{p(x)}(\Omega)} = |u|_{p(x)}$$

$$= \inf \left\{ \tau > 0 : \int_\Omega \left| \frac{u(x)}{\tau} \right|^{p(x)} dx \le 1 \right\}.$$

Moreover, $L^{p(x)}(\Omega)$ is a reflexive space [11, 15] provided that

$$1 < p^- \le p^+ < \infty.$$

Furthermore, on such kind of spaces a Hölder type inequality is valid [11, 15]. More exactly, denoting by $L^{q(x)}(\Omega)$ the conjugate space of $L^{p(x)}(\Omega)$, where $\frac{1}{p(x)} + \frac{1}{q(x)} = 1$ for any $x \in \overline{\Omega}$, for each $u \in L^{p(x)}(\Omega)$ and each $v \in L^{q(x)}(\Omega)$ the Hölder type inequality reads as follows

$$\left| \int_\Omega u(x)v(x)dx \right| \le \left(\frac{1}{p^-} + \frac{1}{q^-} \right) |u|_{p(x)} |v|_{q(x)}.$$

An immediate consequence of Hölder inequality is connected with some inclusions between various Lebesgue spaces involving variable exponent growth [11, 15]; if $0 < |\Omega| < \infty$ and p_1, p_2 are variable exponents, such that $p_1(x) \le p_2(x)$ almost everywhere in Ω, then there exists the continuous embedding $L^{p_2(x)}(\Omega) \hookrightarrow L^{p_1(x)}(\Omega)$.
We refer to [11, 15, 24, 25] for the elementary properties of these spaces. To recall the definition of the isotropic Sobolev space with variable exponent, $W^{1,p(x)}$, we set

$$W^{1,p(x)}(\Omega) = \left\{ u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega) \right\},$$

endowed with the norm

$$\begin{aligned} \|u\| &= \|u\|_{W^{1,p(x)}(\Omega)} \\ &= |u|_{L^{p(x)}(\Omega)} + |\nabla u|_{L^{p(x)}(\Omega)}. \end{aligned} \tag{2.1}$$

The space $W^{1,p(x)}(\Omega)$, equipped with the norm (2.1) becomes a separable, reflexive and uniformly convex Banach space. See for more details [7].
For $u \in W^{1,p(x)}(\Omega)$, define

$$\|u\|_h = \inf \left\{ \eta > 0 : \int_\Omega \left(\left| \frac{\nabla u}{\eta} \right|^{p(x)} + h(x) \left| \frac{u}{\eta} \right|^{p(x)} \right) dx \le 1 \right\}. \tag{2.2}$$

According to [14], $\|u\|_h$ is a norm on $W^{1,p(x)}(\Omega)$ equivalent to $\|u\|_{W^{1,p(x)}(\Omega)}$.

Proposition 2.1 ([15]). *For $p \in C(\overline{\Omega})$ such that $p^- > N$ for all $x \in \overline{\Omega}$, there is a compact embedding*

$$W^{1,p(x)}(\Omega) \hookrightarrow C^0(\overline{\Omega}).$$

An important role in manipulating the generalized Sobolev spaces is played by the modular of the $W^{1,p(x)}(\Omega)$ space, which is the mapping

$$\kappa_{p(.)}(u) : W^{1,p(x)}(\Omega) \longrightarrow \mathbb{R},$$

defined by

$$\kappa_{p(x)}(u) = \int_\Omega |\nabla u|^{p(x)}\, dx,$$

provided that $p^+ < \infty$.

We point out some relations which can be established between the Luxemburg norm and the modular.

If $u_n, u \in W^{1,p(x)}(\Omega)$, then the following relations hold

(1) $\|u\| < (=;>) 1 \quad \Leftrightarrow \quad \kappa_{p(x)}(u) < (=;>) 1$,
(2) $\|u\| \geq 1 \quad \Rightarrow \quad \|u\|^{p^-} \leq \kappa_{p(x)}(u) \leq \|u\|^{p^+}$,
(3) $\|u\| \leq 1 \quad \Rightarrow \quad \|u\|^{p^+} \leq \kappa_{p(x)}(u) \leq \|u\|^{p^-}$,
(4) $\|u_n\| \to 0 \quad \Leftrightarrow \quad \kappa_{p(x)}(u_n) \to 0$, and
$\|u_n\| \to \infty \quad \Leftrightarrow \quad \kappa_{p(x)}(u_n) \to \infty$.

We assume in the sequel that Ω is a bounded open domain in $\mathbb{R}^N$ and we denote by

$$\overrightarrow{p}(.) : \overline{\Omega} \to \mathbb{R}^N,$$

the vectorial function

$$\overrightarrow{p}(.) = (p_1(.), \ldots, p_N(.)) .$$

We define $W^{1,\overrightarrow{p}(.)}(\Omega)$, the anisotropic variable exponent Sobolev space with respect to the norm

(2.3)

$$\begin{aligned} \|u\|_{\overrightarrow{p}(.)} &= \|u\|_{W^{1,\overrightarrow{p}(.)}(\Omega)} \\ &= \sum_{i=1}^{N} \inf \left\{ \sigma > 0; \int_\Omega \left(\left| \frac{\partial_{x_i} u}{\sigma} \right|^{p_i(x)} + h(x) \left| \frac{u}{\sigma} \right|^{p_i(x)} \right) dx \leq 1 \right\}. \end{aligned}$$

It was argued in [14] that $W^{1,\overrightarrow{p}(.)}(\Omega)$ is a reflexive Banach space and a separable space.

On the other hand, for the convenience of working with the space $W^{1,\overrightarrow{p}(.)}(\Omega)$ we introduce $\overrightarrow{p}_+$, $\overrightarrow{p}_-$ in $\mathbb{R}^N$ as

$$\overrightarrow{p}_+ = \left(p_1^+, \ldots, p_N^+\right), \qquad \overrightarrow{p}_- = \left(p_1^-, \ldots, p_N^-\right),$$

and

$$p_+^+ = \max\left\{p_1^+, \ldots, p_N^+\right\}, \qquad p_-^- = \min\left\{p_1^-, \ldots, p_N^-\right\}.$$

Suppose that

$$\sum_{i=1}^{N} \frac{1}{p_i^+} < 1. \tag{2.4}$$

Then it is proved in [13] that $W^{1,\overrightarrow{p}(.)}(\Omega)$ is compactly embedded in $C^0(\overline{\Omega})$ and there exists a constant $c > 0$ such that

$$\|u\|_\infty \le c\|u\|_{\overrightarrow{p}(.)}, \quad \forall u \in W^{1,\overrightarrow{p}(.)}(\Omega), \tag{2.5}$$

where $\|u\|_\infty := \sup_{x\in\overline{\Omega}} |u(x)|$.

Let us denote by $A_i : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$, $i \in \{1, \dots, N\}$, and by $F : \Omega \times \mathbb{R} \to \mathbb{R}$ the antiderivatives of the Carathéodory functions $a_i : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ and $f : \Omega \times \mathbb{R} \to \mathbb{R}$, respectively;

$$A_i(x, s) = \int_0^s a_i(x, t)dt, \tag{2.6}$$

$$F(x, s) = \int_0^s f(x, t)dt. \tag{2.7}$$

Assume that

(n_1) There exists a positive constant $\overline{c}_i$ such that a_i fulfills

$$|a_i(x, s)| \le \overline{c}_i|s|^{p_i(x)-1}, \tag{2.8}$$

for all $x \in \Omega$ and all $s \in \mathbb{R}$. So

$$|A_i(x, s)| \le \overline{c}_i|s|^{p_i(x)}. \tag{2.9}$$

(n_2) There exists $k_i > 0$ such that

$$k_i|s|^{p_i(x)} \le a_i(x, s)s \le p_i(x)A_i(x, s), \tag{2.10}$$

for all $x \in \Omega$ and all $s \in \mathbb{R}$.

(n_3) $a_i(x, 0) = 0$ for all $x \in \partial\Omega$.

(n_4) a_i fulfills

$$(a_i(x, s) - a_i(x, t)) (s - t) > 0, \tag{2.11}$$

for all $x \in \Omega$ and $s, t \in \mathbb{R}$ with $s \ne t$.

(n_5) There exist $k > 0$ and $q \in C_+(\overline{\Omega})$ with $p_+^+ < q^- < q^+ < \overline{p}^*(x)$ for all $x \in \overline{\Omega}$, such that

$$|f(x, s)| \le k \left(1 + |s|^{q(x)-1}\right), \tag{2.12}$$

for all $x \in \Omega$ and all $s \in \mathbb{R}$ where

$$\overline{p}(x) = \frac{N}{\sum\limits_{i=1}^{N} \frac{1}{p_i(x)}},$$

and

$$r^*(x) = \begin{cases} \frac{Nr(x)}{N-r(x)}, & r(x) < N, \\ \infty, & r(x) \geq N. \end{cases}$$

(n_6) There exist $\gamma > p_+^+$ and $s_0 > 0$ such that the Ambrosetti-Rabinowitz condition

$$0 < \gamma F(x,s) < sf(x,s),$$

holds for all $x \in \Omega$ and for all $s \in \mathbb{R}$ with $|s| > s_0$.

3. Main Results

We say that $u \in W^{1,\overrightarrow{p}(.)}(\Omega)$ is a weak solution of the problem (1.1) if

$$\int_\Omega \sum_{i=1}^N a_i(x, \partial_{x_i} u)\partial_{x_i} v dx + \int_\Omega \sum_{i=1}^N h(x) a_i(x,u) v dx = \lambda \int_\Omega f(x,u) v dx,$$

for all $v \in W^{1,\overrightarrow{p}(.)}(\Omega)$. For each $u \in W^{1,\overrightarrow{p}(.)}(\Omega)$, let the functionals $\Phi, \Psi : W^{1,\overrightarrow{p}(.)}(\Omega) \to \mathbb{R}$ be defined by

$$\Phi(u) = \int_\Omega \sum_{i=1}^N A_i(x, \partial_{x_i} u) dx + \int_\Omega \sum_{i=1}^N h(x) A_i(x,u) dx, \tag{3.1}$$

and

$$\Psi(u) = \int_\Omega F\left(x, u(x)\right) dx. \tag{3.2}$$

By standard arguments, it follows that the functionals Φ and Ψ are well defined and of class C^1 with the derivatives given by

$$\left\langle \Phi'(u), v \right\rangle = \int_\Omega \sum_{i=1}^N a_i(x, \partial_{x_i} u)\partial_{x_i} v dx + \int_\Omega \sum_{i=1}^N h(x) a_i(x,u) v dx, \tag{3.3}$$

and

$$\left\langle \Psi'(u), v \right\rangle = \int_\Omega f\left(x, u(x)\right) v(x) dx, \tag{3.4}$$

for any $u, v \in W^{1,\overrightarrow{p}(.)}(\Omega)$.

Lemma 3.1. *For all $u \in W^{1,\overrightarrow{p}(.)}(\Omega)$ we have*

(i) if $\|u\|_{\overrightarrow{p}(.)} \geq 1$ then

$$\frac{\min\{k_1, \dots, k_N\}}{p_+^+} \|u\|_{\overrightarrow{p}(.)}^{p_-^-} \leq \Phi(u) \leq \max\{\overline{c}_1, \dots, \overline{c}_N\} \|u\|_{\overrightarrow{p}(.)}^{p_+^+},$$

(ii) if $\|u\|_{\overrightarrow{p}(.)} \leq 1$ then

$$\frac{\min\{k_1, \dots, k_N\}}{p_+^+} \|u\|_{\overrightarrow{p}(.)}^{p_+^+} \leq \Phi(u) \leq \max\{\overline{c}_1, \dots, \overline{c}_N\} \|u\|_{\overrightarrow{p}(.)}^{p_-^-}.$$

Proof. It is an immediate result of (2.8) and (2.10) in the case where k_i corresponds to p_i. □

Lemma 3.2. *The functional $\Phi - \lambda\Psi$ is coercive.*

Proof. It follows from assumptions (n_2), (n_5), that

$$A_i\,(x,s) \geq k_i \frac{|s|^{p_i(x)}}{p_i(x)},$$

and

$$F(x,s) \leq k\left(|s| + |s|^{q(x)}\right).$$

Thus, for each $u \in W^{1,\overrightarrow{p}(\cdot)}(\Omega)$

$$\begin{aligned}
&\Phi(u) - \lambda\Psi(u)\\
&= \int_\Omega \sum_{i=1}^N A_i(x, \partial_{x_i} u)dx + \int_\Omega \sum_{i=1}^N A_i(x,u)dx - \lambda \int_\Omega F(x,u(x))dx\\
&\geq \int_\Omega \sum_{i=1}^N k_i \frac{|\partial_{x_i} u|^{p_i(x)}}{p_i(x)} dx + \int_\Omega \sum_{i=1}^N k_i \frac{|u|^{p_i(x)}}{p_i(x)} dx - \lambda \int_\Omega k(|u| + |u|^{q(x)})dx.
\end{aligned}$$

So, one has

$$\lim_{\|u\|\to+\infty} \left(\Phi(u) - \lambda\Psi(u)\right) = +\infty,$$

which means the functional $\Phi - \lambda\Psi$ is coercive for every $\lambda > 0$. □

The following theorem guarantees the coercivity and continuity of the Gâteaux derivative of the functional Φ.

Theorem 3.3 ([19, Theorem 6.2.1])**.** *Let X be a reflexive Banach space, and let*
$f : M \subseteq X \to \mathbb{R}$ be Gâteaux differentiable over the closed, convex set M. Then the following conditions are equivalent:

(*i*) *f is convex over M.*

(*ii*) *We have*

$$f(u) - f(v) \geq \left\langle f'(v), u - v \right\rangle_{X^*\times X}, \quad \forall u, v \in M,$$

where X^* denotes the dual of the space X.

(*iii*) *The first Gâteaux derivative is monotone, that is,*

$$\left\langle f'(u) - f'(v), u - v \right\rangle_{X^*\times X} \geq 0, \quad \forall u, v \in M.$$

(*iv*) *The second Gâteaux derivative of f exists and it is positive, that is,*

$$\left\langle f''(u) \circ v, v \right\rangle_{X^*\times X} \geq 0, \quad \forall v \in M.$$

Now, we want to use Theorem 1.1 and find interval of real parameter λ for which the problem (1.1) admits at least one nontrivial solution.

Theorem 3.4. *Assume that there exist $r > 0$ and $c_1 > 0$ such that*

$$\frac{\int_\Omega \sup_{|t|\le c_1} F(x,t)dx - \int_\Omega F(x,\delta)dx}{\zeta(c_1^{p_+^+}-1)} \le \frac{\int_\Omega F(x,\delta)dx}{\zeta}, \tag{3.5}$$

where

$$\zeta := h^+ \max\{\overline{c}_1,\dots,\overline{c}_N\} N\delta^{p_+^+} \operatorname{meas}(\Omega).$$

Then, for each λ belonging to

$$\left]\frac{\zeta}{\int_\Omega F(x,\delta)dx}, \frac{\zeta(c_1^{p_+^+}-1)}{\int_\Omega \sup_{|t|\le c_1} F(x,t)dx - \int_\Omega F(x,\delta)dx}\right[,$$

the problem (1.1) admits at least one non-trivial solution $u^* \in W^{1,\overrightarrow{p}(.)}(\Omega)$, such that $0 < \Phi(u^*) < r$.

Proof. Our aim is to apply Theorem 1.1 to our problem. To this end, let Φ, Ψ be the functionals defined in (3.1), (3.2).

Then $\Psi' : W^{1,\overrightarrow{p}(.)}(\Omega) \to W^{1,\overrightarrow{p}(.)}(\Omega)^*$ is a compact operator. On the other hand the fact that $W^{1,\overrightarrow{p}(.)}(\Omega)$ is compactly embedded into $C^0(\overline{\Omega})$ implies that the operator $\Psi' : W^{1,\overrightarrow{p}(.)}(\Omega) \to (W^{1,\overrightarrow{p}(.)}(\Omega))^*$ is compact. Furthermore, according to Lemma 3.1 and 3.2, Φ is bounded from below and $\Phi - \lambda\Psi$ is coercive.
So the functionals Φ, Ψ satisfy in all regularity assumptions requested in Theorem 1.1, (we apply Proposition 1.4 and do not require (PS)-condition).

In the following we choose $\overline{v} := \delta \in W^{1,\overrightarrow{p}(.)}(\Omega)$ with $\delta > 1$. In this case

$$\begin{aligned}
\Phi(\overline{v}) &= \int_\Omega \sum_{i=1}^N h(x) A_i(x,\delta)dx \\
&\le \int_\Omega \sum_{i=1}^N h(x)\overline{c}_i|\delta|^{p_i(x)}dx \\
&\le \max\{\overline{c}_1,\dots,\overline{c}_N\} \int_\Omega \sum_{i=1}^N h(x)|v_\delta|^{p_i(x)}dx \\
&\le h^+ \max\{\overline{c}_1,\dots,\overline{c}_N\} \\
&\quad \times \int_\Omega \sum_{i=1}^N |v_\delta|^{p_i(x)}dx \le h^+ \max\{\overline{c}_1,\dots,\overline{c}_N\} N\delta^{p_+^+} \operatorname{meas}(\Omega).
\end{aligned}$$

On the other hand

$$\begin{aligned}\Phi(\overline{v}) &= \int_\Omega \sum_{i=1}^N h(x)A_i(x,\delta)dx \\ &\geq \int_\Omega \sum_{i=1}^N h(x)\frac{k_i|\delta|^{p_i(x)}}{p_i(x)} \\ &\geq h^- \min\{k_1,\ldots,k_N\}\frac{\delta^{p_-^-}}{p_+^+}\operatorname{meas}(\Omega).\end{aligned}$$

Therefore

$$h^- \min\{k_1,\ldots,k_N\}\frac{\delta^{p_-^-}}{p_+^+}\operatorname{meas}(\Omega) \leq \Phi(\overline{v}) \leq h^+ \max\{\overline{c}_1,\ldots,\overline{c}_N\} N\delta^{p_+^+}\operatorname{meas}(\Omega).$$

We have

$$\Psi(\overline{v}) = \int_\Omega F(x,\delta)dx.$$

Choosing

$$r = h^+ \max\{\overline{c}_1,...,\overline{c}_N\}N\delta^{p_+^+}\operatorname{meas}(\Omega)c_1^{p_+^+}$$

one has $0 < \Phi(\overline{v}) < r$. Moreover, for all $u \in W^{1,\overrightarrow{p}(.)}(\Omega)$ such that $u \in \Phi^{-1}(]-\infty,r[)$, taking (2.5) into account, one has $|u(x)| < r$ for all $x \in \Omega$, from which it follows

$$\begin{aligned}\sup_{u\in\Phi^{-1}(]-\infty,r[)} \Psi(u) &= \sup_{u\in\Phi^{-1}(]-\infty,r[)} \int_\Omega F(x,u(x))dx \\ &\leq \int_\Omega \sup_{|t|\leq c_1} F(x,t)dx.\end{aligned}$$

Therefore, one has

$$\begin{aligned}\beta(0,r) &= \inf_{v\in\Phi^{-1}(]0,r[)} \frac{\sup_{u\in\Phi^{-1}(]0,r[)}\Psi(u)-\Psi(v)}{r-\Phi(v)} \\ &\leq \frac{\sup_{u\in\Phi^{-1}(]-\infty,r[)}\Psi(u)-\Psi(\overline{v})}{r-\Phi(\overline{v})} \\ &\leq \frac{\int_\Omega \sup_{|t|\leq c_1}F(x,t)dx-\Psi(\overline{v})}{r-\Phi(\overline{v})} \\ &\leq \frac{\int_\Omega \sup_{|t|\leq c_1}F(x,t)dx-\int_\Omega F(x,\delta)dx}{r-h^+\max\{\overline{c}_1,\ldots,\overline{c}_N\}N\delta^{p_+^+}\operatorname{meas}(\Omega)} \\ &= \frac{\int_\Omega \sup_{|t|\leq c_1}F(x,t)dx-\int_\Omega F(x,\delta)dx}{h^+\max\{\overline{c}_1,\ldots,\overline{c}_N\}N\delta^{p_+^+}\operatorname{meas}(\Omega)\left(c_1^{p_+^+}-1\right)},\end{aligned}$$

and

$$\begin{aligned}
\rho(0,r) &= \sup_{v\in\Phi^{-1}(]0,r[)} \frac{\Psi(v) - \sup_{u\in\Phi^{-1}(]-\infty,0[)} \Psi(u)}{\Phi(v)} \\
&\geq \frac{\Psi(\overline{v}) - \sup_{u\in\Phi^{-1}(]-\infty,0[)} \Psi(u)}{\Phi(\overline{v})} \\
&\geq \frac{\Psi(\overline{v}) - \sup_{u\in\Phi^{-1}(]-\infty,r[)} \Psi(u)}{\Phi(\overline{v})} \\
&\geq \frac{\int_\Omega F(x,\delta)dx - \int_\Omega \sup_{|t|\leq c_1} F(x,t)dx}{h^+ \max\{\overline{c}_1,\dots,\overline{c}_N\} N\delta^{p_+^+} \operatorname{meas}(\Omega)} \\
&\geq \frac{\int_\Omega F(x,\delta)dx}{h^+ \max\{\overline{c}_1,\dots,\overline{c}_N\} N\delta^{p_+^+} \operatorname{meas}(\Omega)}.
\end{aligned}$$

Hence, by using (3.5) one has $\beta(0,r) < \rho(0,r)$.
Therefore, from Theorem 1.1, for each

$$\begin{aligned}
\lambda \in &\left] \frac{\zeta}{\int_\Omega F(x,\delta)dx}, \frac{\zeta(c_1^{p_+^+} - 1)}{\int_\Omega \sup_{|t|\leq c_1} F(x,t)dx - \int_\Omega F(x,\delta)dx} \right[\\
&\subseteq \left] \frac{1}{\beta(0,r)}, \frac{1}{\rho(0,r)} \right[,
\end{aligned}$$

the functional I_λ admits at least one critical point u^* such that
$0 < \Phi(u^*) < r$. □

Theorem 3.5. *Let there exist $c_2, c_3 > 0$ such that*

$$W_\delta(x) \leq \Gamma_\delta(x), \tag{3.6}$$

where

$$W_\delta(x) := \frac{\int_\Omega \sup_{|t|\leq c_3} F(x,t)dx - \int_\Omega F(x,\delta)dx}{\zeta(c_3^{p_+^+} - 1)},$$

and

$$\Gamma_\delta(x) := \frac{\int_\Omega F(x,\delta)dx - \int_\Omega \sup_{|t|\leq c_2} F(x,t)}{\zeta - h^- \min\{k_1,\dots,k_N\} \dfrac{\delta^{p_-^-}}{p_+^+} \operatorname{meas}(\Omega) c_2^{p_-^-}}.$$

Then, for each $\lambda \in \left] \frac{1}{\Gamma_\delta(x)}, \frac{1}{W_\delta(x)} \right[$, the problem (1.1) admits at least one non-trivial solution $u^{**} \in W^{1,\overrightarrow{p}(.)}(\Omega)$, such that $r_1 < \Phi(u^{**}) < r_2$.

Proof. The proof method is similar to the Theorem 3.4. Let $\overline{v} := \delta \in W^{1,\overrightarrow{p}(.)}(\Omega)$ with $\delta > 1$, then

$$h^{-} \min\{k_1,\dots,k_N\} \frac{\delta^{p_{-}^{-}}}{p_{+}^{+}} \operatorname{meas}(\Omega) \leq \Phi(\overline{v}) \leq h^{+} \max\{\overline{c}_1,\dots,\overline{c}_N\} N\delta^{p_{+}^{+}} \operatorname{meas}(\Omega),$$

and

$$\Psi(\overline{v}) = \int_{\Omega} F(x,\delta)dx.$$

Put

$$r_1 := h^{-} \min\{k_1,\dots,k_N\} \frac{\delta^{p_{-}^{-}}}{p_{+}^{+}} \operatorname{meas}(\Omega) c_2^{p_{-}^{-}},$$

and

$$r_2 := h^{+} \max\{\overline{c}_1,\dots,\overline{c}_N\} N\delta^{p_{+}^{+}} \operatorname{meas}(\Omega) c_3^{p_{+}^{+}}.$$

We obtain $r_1 < \Phi(\overline{v}) < r_2$. Moreover, for all $u \in W^{1,\overrightarrow{p}(.)}(\Omega)$ such that $u \in \Phi^{-1}(]-\infty, r_2[)$, taking (2.5) into account, one has $|u(x)| < r_2$ for all $x \in \Omega$, from which it follows

$$\begin{aligned}\sup_{u\in\Phi^{-1}(]-\infty,r_2[)} \Psi(u) &= \sup_{u\in\Phi^{-1}(]-\infty,r_2[)} \int_{\Omega} F(x,u(x))dx \\ &\leq \int_{\Omega} \sup_{|t|\leq c_3} F(x,t)dx.\end{aligned}$$

Therefore, one has

$$\begin{aligned}\beta(r_1,r_2) &= \inf_{v\in\Phi^{-1}(]r_1,r_2[)} \frac{\sup_{u\in\Phi^{-1}(]r_1,r_2[)} \Psi(u) - \Psi(v)}{r_2 - \Phi(v)} \\ &\leq \frac{\sup_{u\in\Phi^{-1}(]-\infty,r_2[)} \Psi(u) - \Psi(\overline{v})}{r_2 - \Phi(\overline{v})} \\ &\leq \frac{\int_{\Omega} \sup_{|t|\leq c_3} F(x,t)dx - \Psi(\overline{v})}{r_2 - \Phi(\overline{v})} \\ &\leq \frac{\int_{\Omega} \sup_{|t|\leq c_3} F(x,t)dx - \int_{\Omega} F(x,\delta)dx}{r_2 - h^{+} \max\{\overline{c}_1,\dots,\overline{c}_N\} N\delta^{p_{+}^{+}} \operatorname{meas}(\Omega)} \\ &\leq \frac{\int_{\Omega} \sup_{|t|\leq c_3} F(x,t)dx - \int_{\Omega} F(x,\delta)dx}{\zeta(c_3^{p_{+}^{+}} - 1)},\end{aligned}$$

and

$$\begin{aligned}\rho(r_1,r_2) &= \sup_{v\in\Phi^{-1}(]r_1,r_2[)} \frac{\Psi(v) - \sup_{u\in\Phi^{-1}(]-\infty,r_1[)} \Psi(u)}{\Phi(v) - r_1} \\ &\geq \frac{\Psi(\overline{v}) - \sup_{u\in\Phi^{-1}(]-\infty,r_1[)} \Psi(u)}{\Phi(\overline{v}) - r_1}\end{aligned}$$

$$\geq \frac{\Psi(\overline{v}) - \int_\Omega \sup_{|t|\leq c_2} F(x,t)}{\Phi(\overline{v}) - r_1}$$
$$\geq \frac{\int_\Omega F(x,\delta)dx - \int_\Omega \sup_{|t|\leq c_2} F(x,t)}{h^+ \max\{\overline{c}_1,\dots,\overline{c}_N\} N\delta^{p_+^+} \operatorname{meas}(\Omega) - r_1}$$
$$\geq \frac{\int_\Omega F(x,\delta)dx - \int_\Omega \sup_{|t|\leq c_2} F(x,t)}{\zeta - h^- \min\{k_1,\dots,k_N\} \dfrac{\delta^{p_-^-}}{p_+^+} \operatorname{meas}(\Omega) c_2^{p_-^-}}.$$

Due to (3.6)
$$\beta(r_1,r_2) \leq \rho(r_1,r_2).$$
Therefore, from Theorem 1.1, for each
$$\lambda \in \left] \frac{1}{\Gamma_\delta(x)}, \frac{1}{W_\delta(x)} \right[,$$
the functional I_λ admits at least one critical point u^{**} such that $r_1 < \Phi(u^{**}) < r_2$. □

In this article, the relationship between two nontrivial solutions is not investigated. If these two solutions are distinct, the existence of a third ones can be guaranteed by using [3, Theorem 6.2].
Now, we state second main result to find three weak solution for the problem (1.1).

Theorem 3.6. *Assume that c be a positive constants with*

$$\frac{\int_\Omega \sup_{|t|\leq c} F(x,t)dx}{r} < \frac{\int_\Omega F(x,\delta)dx}{\zeta c^{p_+^+}}, \tag{3.7}$$

and

$$0 < r < h^- \min\{k_1,\dots,k_N\} \frac{\delta^{p_-^-}}{p_+^+} \operatorname{meas}(\Omega) \tag{3.8}$$

Then, for each parameter λ belonging to

$$\Lambda_{(r,\delta)} := \left] \frac{\zeta c^{p_+^+}}{\int_\Omega F(x,\delta)dx}, \frac{r}{\int_\Omega \sup_{|t|\leq c} F(x,t)dt} \right[, \tag{3.9}$$

the problem (1.1) *possesses at least three distinct weak solutions in* $W^{1,\overrightarrow{p}(.)}(\Omega)$.

Proof. Our aim is to apply Theorem 1.2 to our problem. To this end, let Φ, Ψ be the functionals defined in (3.1), (3.2).

Then, $\Psi' : W^{1,\overrightarrow{p}(.)}(\Omega) \to \left(W^{1,\overrightarrow{p}(.)}(\Omega)\right)^*$ is a compact operator. On the other hand the fact that $W^{1,\overrightarrow{p}(.)}(\Omega)$ is compactly embedded into

$C^0(\overline{\Omega})$ implies that the operator $\Psi' : W^{1,\overrightarrow{p}(.)}(\Omega) \to \left(W^{1,\overrightarrow{p}(.)}(\Omega)\right)^*$ is compact. Furthermore, according to Lemma 3.1 and 3.2, Φ is bounded from below and $\Phi - \lambda\Psi$ is coercive.

So, the functionals Φ, Ψ satisfy in all regularity assumptions requested in Theorem 1.2, (we apply Proposition 1.4 and do not require (PS)-condition).

Here and in the sequel we have $\Phi(0) = \Psi(0) = 0$ and $\Phi(u) \geq 0$ for every $u \in X$. In the following, our aim is to verify condition (3.7). Put $\overline{v} := \delta \in W^{1,\overrightarrow{p}(.)}(\Omega)$ with $\delta > 1$, we have

$$h^- \min\{k_1, \dots, k_N\} \frac{\delta^{p_-^-}}{p_+^+} \operatorname{meas}(\Omega) \leq \Phi(\overline{v}) \leq \zeta,$$

and

$$\Psi(\overline{v}) = \int_\Omega F(x, \delta)dx.$$

So, $r < \Phi(\overline{v})$. Moreover, for all $u \in W^{1,\overrightarrow{p}(.)}(\Omega)$ such that $u \in \Phi^{-1}(]-\infty, r[)$, taking (2.5) into account, one has $|u(x)| < c$ for all $x \in \Omega$, from which it follows

$$\sup_{u\in\Phi^{-1}(]-\infty,r[)} \Psi(u) = \sup_{u\in\Phi^{-1}(]-\infty,r[)} \int_\Omega F(x, u(x))dx \leq \int_\Omega \sup_{|t|\leq c} F(x,t)dx,$$

and

$$\frac{\sup_{u\in\Phi^{-1}(]-\infty,r[)} \Psi(u)}{r} \leq \frac{\int_\Omega \sup_{|t|\leq c} F(x,t)dx}{r}. \tag{3.10}$$

Moreover, one has

$$\frac{\Psi(\overline{v})}{\Phi(\overline{v})} \geq \frac{\int_\Omega F(x,\delta)dx}{\zeta c^{p_+^+}}. \tag{3.11}$$

From (3.7) it follows that

$$\frac{\sup_{u\in\Phi^{-1}(]-\infty,r[)} \Psi(u)}{r} < \frac{\Psi(\overline{v})}{\Phi(\overline{v})}. \tag{3.12}$$

Now, we observe that

$$\varphi(r) = \inf_{v\in\Phi^{-1}(]-\infty,r[)} \frac{\sup_{u\in\Phi^{-1}(]-\infty,r[)} \Psi(u) - \Psi(v)}{r - \Phi(v)} \leq \frac{\sup_{u\in\Phi^{-1}(]-\infty,r[)} \Psi(u)}{r},$$

and

$$\begin{aligned}\rho(r) &= \sup_{v\in\Phi^{-1}(]r,\infty[)} \frac{\Psi(v) - \sup_{u\in\Phi^{-1}(]-\infty,r])}\Psi(u)}{\Phi(v)-r}\\ &\geq \frac{\Psi(\overline{v}) - \sup_{u\in\Phi^{-1}(]-\infty,r])}\Psi(u)}{\Phi(\overline{v})-r}\\ &\geq \frac{\Psi(\overline{v}) - r\dfrac{\Psi(\overline{v})}{\Phi(\overline{v})}}{\Phi(\overline{v})-r}\\ &= \frac{\Psi(\overline{v})}{\Phi(\overline{v})}.\end{aligned}$$

Hence,

$$\begin{aligned}\varphi(r) &\leq \frac{\sup_{u\in\Phi^{-1}(]-\infty,r])}\Psi(u)}{r}\\ &< \frac{\Psi(\overline{v})}{\Phi(\overline{v})}\\ &\leq \rho(r).\end{aligned}$$

So, all conditions that we need is verified. Since all the assumptions of Theorem 1.2 are satisfied, for each

$$\lambda \in \Lambda_{(r,\delta)} = \left]\frac{\zeta c^{p_+^+}}{\int_\Omega F(x,\delta)dx}, \frac{r}{\int_\Omega \sup_{|t|\leq c} F(x,t)dt}\right[,$$

the functional I_λ has at least three distinct critical points that are weak solutions of the problem(1.1). The proof is complete. □

We conclude this paper with an example.

Example 3.7. Define $g:\mathbb{R}\longrightarrow\mathbb{R}$ as follows

$$r^*(x) = \begin{cases} 0, & t<0,\\ t^{p_+^+}, & 0\leq t\leq 1,\\ t^{\xi(x)}, & t>1,\end{cases}$$

where $\xi(x) \in \left]0, p_-^- - 1\right[$. Further, let $m:\Omega\longrightarrow\mathbb{R}$ be a bounded measurable and positive function. From Theorem 3.6, for each

$$\lambda > \frac{\text{meas}(\Omega)}{\|m\|_{L^1(\Omega)}} \inf_{\delta>0, G(\delta)>0} \frac{|\delta|^{p_+^+}}{p_-^- G(\delta)},$$

where $G(\delta) := \int_0^\delta g(t)dt$, the following problem

$$\begin{cases} -\sum_{i=1}^{N} \partial_{x_i} \left(|\partial_{x_i} u|^{p_i(x)-2} \partial_{x_i} u \right) + \sum_{i=1}^{N} |u|^{p_i(x)-2} u = \lambda m(x) g(u), & x \in \Omega, \\ \dfrac{\partial u}{\partial \nu} = 0, & x \in \partial\Omega. \end{cases}$$

possesses at least three weak solutions in $W^{1,\overrightarrow{p}(.)}(\Omega)$.

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